Superstring Amplitudes From Threshold Effects to Soft Theorems and Back?

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Plan of the talk

Back to ……

1994. The beginning: Threshold corrections in string theories.

2015. Soft Theorems in Field and String Theories

The beginning: String Thresholds

- In *grand unification* theories, the low energy gauge interactions are unified in a single gauge theory *SU(5)*, *SO(10)* or E_6 that is spontaneously broken at the energy scale $M_{GUT} \sim 10^{14} - 10^{19}$ GeV.
- The low-energy coupling constants are related to the gauge coupling g_a $a = 1{,}2{,}3$ of the unifying group g_{GUT} by the Georgi-Quinn- Weinberg equation

$$\frac{16\,\pi^2}{g_a^2(\mu)} = k_a\,\frac{16\,\pi^2}{g_{GUT}^2} + \frac{16\,\pi^2}{g_a^3}\,\beta_a\,\log\frac{M_{GUT}^2}{\mu^2} + \mathcal{O}(1) \qquad \text{for } \mu \leq M_{GUT}$$

β_a is the one-loop β-function, k_a are the tree-level relation between the couplings (e.g. $k_1 = \frac{5}{3}$, $k_2 = k_3 = 1$ in $SU(5)$)

The beginning: String Thresholds

- In *grand unification* theories, the low energy gauge interactions are unified in a single gauge theory *SU(5)*, *SO(10)* or E_6 that is spontaneously broken at the energy scale $M_{GUT} \sim 10^{14} - 10^{19}$ GeV.
- The low-energy coupling constants are related to the gauge coupling g_a $a = 1,2,3$ of the unifying group g_{GUT} by the Georgi-Quinn- Weinberg equation

To improve the accuracy, the effect of the heavy massive particles should be taken in acount. These are collected by the threshold corrections Δ_a.

$$\frac{16\,\pi^2}{g_a^2(\mu)} = k_a\,\frac{16\,\pi^2}{g_{GUT}^2} + \frac{16\pi^2}{g_a^3}\,\beta_a\,\log\frac{M_{GUT}^2}{\mu^2} + \Delta_a$$

- String theories unify all the known interactions, and the three-level predictions of string unification are similar to GUT predictions

$$\frac{1}{g_a^2} = \frac{k_a}{g_{\rm string}^2} \quad ; \quad G_N = \alpha' \, \frac{g_{string}^2}{8\pi} \quad ; \quad M_{GUT}^2 \sim \mathcal{O}\left(\frac{1}{\alpha'}\right)$$

- The threshold corrections in string theory take in account the effects of massive particles on the renormalization group equation. These are obtained by considering the string partition function in the presence of a constant background gauge field $A_\mu = -\frac{1}{2} F_{\mu\upsilon} X^\upsilon$

$$\left.\frac{Z(F)}{Z_0}\right|_{F^2} \simeq -\frac{1}{4} F^2 \, (2\alpha')^2 \int dz_1 \, dz_2 \theta(z_1 - z_2) \partial_{z_1} \mathcal{G}_h(z_1, z_2) \, \partial_{z_2} \mathcal{G}_h(z_1, z_2) \rangle \quad ;$$

- ➢ $\mathcal{G}_h(z_1, z_2)$ multiloop bosonic Green’s function (*h* is the genus of the world-sheet)

- At 1-loop in closed (heterotic) string

$$\int d^2 z_1 \frac{1}{\bar{z}_1} \langle \partial_{z_1} X(z_1)\, X(z_2)\rangle \langle X(z_1)\, \partial_{z_2} X(z_2)\rangle = -2\pi i\, \tau_2\, \omega_2 \frac{\partial}{d\tau} Z_4(\tau)$$

- ω_2 is the prime form, τ the modular parameter of the world-sheet torus and Z_4 the four-dimensional partition function.
- The extension at multiloop of this identity was very difficult to achieve and we started a 1-loop analysis of the field theory limit of string amplitudes.

Alberto Lerda’s talk.

- Vadim S. Kaplunovsky, One loop threshold effects in string unification, hep-th/9205070.
- H. Georgi, H. R. Quinn and S. Weinberg, Phys. Rev. Lett. 33 (1974), 451.

a spatial slice of space-time
D-brane
D-brane
BACK TO THE FUTURE
11-D Supergravity
E8 X E8 Heterotic
SO(32) Heterotic
Type I
"M-Theory"
Type IIA
Type IIB
Black hole
Fuzzball
Black Hole
AdS Boundary (z=0)
5-Dimensional Anti-de Sitter Spacetime
4-Dimensional Flat Spacetime (hologram)

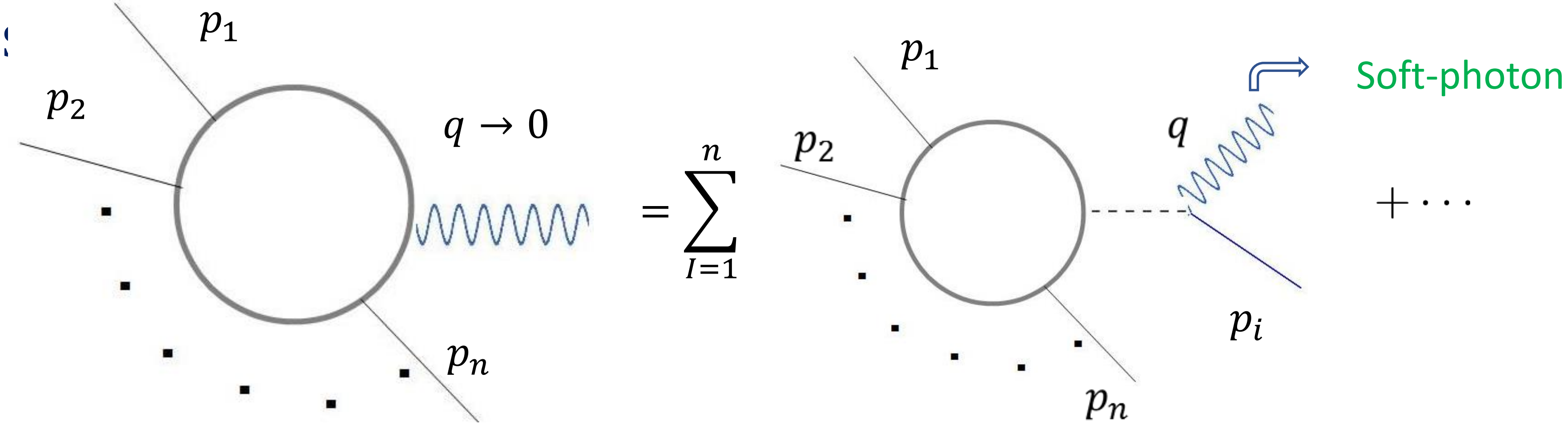
Soft Theorems
in
Field and String Theories

- ✓ P. Di Vecchia, R.M. , M. Mojaza JHEP 05 (2015) 137.
- ✓ P. Di Vecchia, R. M., M. Mojaza, JHEP 12 (2015) 150
- ✓ P. Di Vecchia, R. M., M. Mojaza, Fortsch.Phys. 64 (2016) 389-393
- ✓ P. Di Vecchia, R.M. , M. Mojaza, J. Nohle, Phys. Rev D93 (2016) n.8, 080515.
- ✓ P. Di Vecchia, R. M., M. Mojaza,JHEP 06 (2016) 054.
- ✓ P. Di Vecchia, R. M., M. Mojaza, JHEP 12 (2016), 020.
- ✓ P. Di Vecchia, R. M., M. Mojaza,JHEP 09 (2017),001
- ✓ P. Di Vecchia, R. M., M. Mojaza, JHEP 10(2017)017
- ✓ P. Di Vecchia, R. M., M. Mojaza, JHEP 01 (2019) 038
- ✓ P. Di Vecchia, R.M., M. Mojaza, Phys. Rev. D 100 (2019) 4, 041902.

What are Soft theorems?

- It is well know that scattering amplitudes in the deep infrared region (or soft limit) satisfy interesting relations.

Low's theorem: Amplitudes with a soft photon are determined from the corresponding amplitude without the



Soft-photon polarization

$$\mathcal{M}_{n+1}(q,\, p_1,\, \ldots p_n) \sim \sum_{i=1}^{n} e_i \frac{\epsilon_q \cdot p_i}{q \cdot p_i} \mathcal{M}_n(p_1, \ldots p_n) + N_{n+1}(q,\, p_1, \ldots p_n)$$

e_i charge of the particle i

smooth in the soft limit

- Weinberg: Amplitudes involving gravitons and matter particles show and universal behavior when one graviton becomes soft.

$$\mathcal{M}^{\mu\nu}_{n+1}(q,\, p_1,\, \ldots p_n) \sim \sum_{i=1}^{n} \frac{p_i^{\mu} \cdot p_i^{\nu}}{q \cdot p_i} \mathcal{M}_n(p_1, \ldots p_n) + N_{n+1}(q,\, p_1 \ldots p_n)$$

$$\Downarrow$$

They were recognized to be a consequence of the gauge invariance

$$q^{\mu} M^{\mu}_{n+1}(q,\, p_1, \ldots p_n) = 0 \quad ; \quad q_{\mu} M^{\mu\nu}_{n+1}(q,\, p_1, \ldots p_n) = q_{\nu} M^{\mu\nu}_{n+1}(q,\, p_1, \ldots p_n) = 0$$

$\Downarrow$ Soft-Photon

$\Downarrow$ Soft-graviton

- The interest in the argument was renewed by a proposal of A. Strominger (*JHEP* 07 (2014) 152) and T. He, V. Lysov, P. Mitra and A. Strominger (*JHEP* 05 (2015) 151) asserting that soft theorems are nothing but the Ward identities of the BMS symmetry of asymptotic flat metrics.
- A. Strominger and F. Cachazo proposed that 4D three level graviton amplitudes have a universal behavior through the second subleading order in the soft-graviton momentum. (F. Cachazo, A. Strominger:arXiv:1404.4091)
- These considerations were extended to gluons through the subleading order. The soft-gluon theorem arises as the Wad identity of a two dimensional Kac-Moody type symmetry. (T. He, V. Lysonv, P. Mitra, A. Strominger, *JHEP* 05 (2015) 151)
- These theorems to subleading order for gluons and sub-subleading order for gravitons have been proved in arbitrary dimensions by using Poincarré and on-shell gauge invariance of the amplitudes. (J. Broedel, M. de Leeuw, J. Plefka, M. Bosso , *Phys.Rev.D* 90 (2014) 6, 065024 and Z. Bern, S. Davies, P. Di Vecchia, and J. Nohle, *Phys.Rev.D* 90 (2014) 8, 084035)

- In these new soft theorems $n+1$-point amplitudes with a soft graviton or gluon are obtained acting on n-point amplitude with universal soft operators.

Soft-particle momentum

Only for soft- gravitons

$$A_{n+1}(q,\, p_1, \ldots p_n) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) A_n(p_1, \ldots p_n)$$

Graviton soft operators

$$S^{(0)} = \epsilon_{\mu\nu} \sum_{i=1}^{n} \frac{k_i^\mu \cdot k_i^\nu}{k_i \cdot q} \qquad S^{(1)} = -i\epsilon_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho\, k_i^\mu\, J_i^{\nu\rho}}{k_i \cdot q} \qquad S^{(2)} = -\epsilon_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho\, J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{2k_i \cdot q}$$

Gluon soft operators

$$S^{(0)} = \frac{k_1 \cdot \epsilon}{\sqrt{2}k_1 \cdot q} - \frac{k_n \cdot \epsilon}{\sqrt{2}k_n \cdot q} \qquad S^{(1)} = -i\epsilon_\mu p_\sigma \left(\frac{J_1^{\mu\sigma}}{\sqrt{2}k_1 \cdot q} - \frac{J_n^{\mu\sigma}}{\sqrt{2}k_n \cdot q}\right)$$

ϵ is the polarization of the soft particle and J_i total angular momentum (orbital+spin) of the matter particles.

- String theory is also a powerful tool to get field theory amplitudes. There are few diagrams at each order of the perturbative expansions that are represented as complex integrals on the string moduli space.

$$\Downarrow$$

- Single and double soft limit can be “easily” studied on amplitudes involving n arbitrary external states.

Amplitudes in Bosonic, Heterotic and Superstring theories

- In closed bosonic and superstring theory amplitudes with a graviton or a dilaton with soft momentum q and n hard particles with momentum k_i are obtained from the same two index tensor

$$M^{\mu\nu}_{n+1}(q,\, k_1, \ldots k_n)$$

z_l are complex coordinates parametrizing the insertion on the string world-sheet of the hard state vertex operators

$F^{\mu\nu}$ is a function of all Koba-Nielsen variables having pole for $z \sim z_i$

$$M^{\mu\nu}_{n+1}(q,\, k_1, \ldots k_n) = M_n(\epsilon_i,\, k_i) * \int dz \prod_{l=1}^{n} |z - z_l|^{\alpha' q \cdot k_l} \mathcal{F}^{\mu\nu}(q,\, \{\epsilon_i\},\, \{k_i\};\, z,\, \{z_i\})$$

Model dependent n points amplitude

z Koba-Nielsen variable of the soft particle

One soft-graviton and n-hard particles

- Soft-graviton amplitude is obtained by saturating the n+1-amplitude with the polarization:

$$\epsilon^g_{\mu\nu} = \frac{1}{2}\left(\epsilon_{\mu\nu} + \epsilon_{\nu\mu}\right) \quad ; \quad \eta^{\mu\nu}\epsilon^g_{\mu\nu} = 0$$

$$\epsilon^g_{\mu\nu} M^{\mu\nu}_{n+1}(q,\, k_1 \ldots k_n) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) M_n(k_1\ \ldots k_n)$$

  - $S^{(0)}$ is the standard Weinberg leading soft behavior.

  - $S^{(1)}$ in bosonic and superstring theory is:

$$S^{(1)} = -i\epsilon^g_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho\, k_i^\mu\, J_i^{\nu\rho}}{k_i \cdot q} \qquad\qquad J_i^{\nu\rho} = L_i^{\mu\rho} + \mathcal{S}_i^{\mu\rho} + \bar{\mathcal{S}}_i^{\mu\rho}$$

$$L_i^{\mu\rho} = k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}} \quad ; \quad \mathcal{S}_i^{\mu\rho} = \epsilon_i^\mu \frac{\partial}{\partial \epsilon_{i\rho}} - \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\mu}}$$

$$\boxed{\epsilon_{\mu\nu} = \epsilon_\mu\ \bar{\epsilon}_\nu}$$

Subsubleading order is different in bosonic, heterotic and superstring amplitudes.

$$S^{(2)} = -\frac{\epsilon^{g}_{\mu\nu}}{2}\left[\frac{q_\rho : J_i^{\mu\rho}\, q_\sigma J_i^{\nu\sigma} :}{k_i \cdot q} - \alpha'\left(q_\sigma k_{i\nu}\eta_{\rho\mu} + q_\rho k_{i\mu}\eta_{\sigma\nu} - \eta_{\rho\mu}\eta_{\sigma\nu}(k_i\cdot q) - q_\rho q_\sigma \frac{k_{i\nu}k_{i\nu}}{k_i\cdot q}\right)\right]\Pi^{\rho\sigma}$$

The colons denotes that the action of one operator on the other is excluded.

String correction.

$$\Pi^{\rho\sigma} = \epsilon_i^{\rho}\frac{\partial}{\partial \epsilon_{i\sigma}} + \bar{\epsilon}_i^{\rho}\frac{\partial}{\partial \bar{\epsilon}_{i\sigma}}$$

- The expression is the same both for soft graviton and dilaton.

- String corrections are present only in the bosonic and heterotic amplitudes. They are due to coupling between the dilaton and the Gauss-Bonnet term which is present in the bosonic string effective action but not in its supersymmetric extension.

Kalb-Ramond soft theorem

Soft-Kalb-Ramond amplitude is obtained by saturating the n+1-amplitude with the polarization:

$$\epsilon^{B}_{\mu\nu} = \frac{1}{2}\left(\epsilon_{\mu\nu} - \epsilon_{\nu\mu}\right)$$

The leading divergent term doesn't contribute because is symmetric in the two free indices and one gets:

$$\mathcal{M}_{n+1} \simeq -i\epsilon^{q}_{\mu\nu} \sum_{i=1}^{n} \frac{k_i^{\nu}\, q_{\rho}}{q \cdot k_i} \left[(L_i + S_i)^{\mu\rho} - (\bar{L}_i + \bar{S}_i)^{\mu\rho} \right] M_n(k_i, \epsilon_i, \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}} + \mathcal{O}(q^1)$$

Gauge invariance $q_{\mu} M^{\mu\nu}_{n+1} = q_{\nu} M^{\mu\nu}_{n+1} = 0$ follows from the following identity:

$$\sum_{i=1}^{n} (L_i + S_i)^{\mu\rho} M_n(k_i, \epsilon_i, \bar{k}_i \bar{\epsilon}_i) \Big|_{k=\bar{k}} = \sum_{i=1}^{n} (\bar{L}_i + \bar{S}_i)^{\mu\rho} M_n(k_i, \epsilon_i, \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}} + \mathcal{O}(q^1)$$

Once used in the subleading expression of the amplitude, gives:

$$\mathcal{M}_{n+1} \simeq -2i\epsilon^{B}_{\mu\nu} \sum_{i=1}^{n} \frac{k_i^{\nu}\, q_{\rho}}{q \cdot k_i} (S_i - \bar{S}_i)^{\mu\rho} \mathcal{M}_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q^1)$$

⇩

Sub-subleading soft theorem for the antisymmetric tensor is still an open problem!

Soft-theorem for the gravity dilaton.

- Soft-dilaton amplitude is obtained by saturating the n+1- string amplitude with a soft graviton/dilaton with the dilaton projector:

$$\epsilon_d^{\mu\nu} = \frac{1}{\sqrt{D-2}}\left(\eta^{\mu\nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu\right) \quad ; \quad \bar{q} \text{ lightlike vector } q\cdot\bar{q} = 1$$

D is the space-time dimension

The soft behavior with n-hard Tachyons is:

$$\boxed{m_i^2 = -\frac{4}{\alpha'}}$$

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}}\Big\{ -\sum_{i=1}^{n} \frac{m_i^2}{k_i\cdot q}\left(1 + q^\rho \frac{\partial}{\partial k_i^\rho} + \frac{q^\rho q^\sigma}{2}\frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma}\right)$$

$$+2 - \sum_{i=1}^{n} k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2}\sum_{i=1}^{n}\left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho}\frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}}\right)\Big\}\mathcal{M}_n + \mathcal{O}(q^2)$$

- n-tachyon amplitude: $\mathcal{M}_n = \frac{8\pi}{\alpha'}\left(\frac{\kappa_D}{2\pi}\right)^{n-2}\int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}}\prod_{i\neq j}|z_i - z_j|^{\frac{\alpha'}{2}k_i k_j}$

- Similarly, the soft-behavior with n-hard massless particles, is:

$$
\begin{aligned}
\epsilon_d^{\mu\nu} M_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}}\Big\{ & 2 - \sum_{i=1}^{n-1} k_i^\mu \frac{\partial}{\partial k_i^\mu} \\
& + \frac{q^\rho}{2}\sum_{i=1}^{n-1}\left[\left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho}\frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}}\right) - iS_{\mu\rho}^{(i)}\frac{\partial}{\partial k_{i\mu}}\right] \\
& + \sum_{i=1}^{n-1}\frac{q^\rho q^\sigma}{2k_i\cdot q}\left((S^{(i)}_{\rho\mu})\eta^{\mu\nu}(S^{(i)}_{\nu\sigma}) + D\epsilon_{i\rho}\frac{\partial}{\partial\epsilon_i^\sigma}\right)\Big\} M_n + \mathcal{O}(q^2)\,.
\end{aligned}
$$

Scale transformation generator

Special conformal transformation generator

- It can be obtained via gauge invariance from a string-inspired three-point vertex describing graviton/dilaton interactions.
- No-string corrections in the soft-dilaton behavior.
- Universal, it is the same in bosonic, heterotic and superstring theories.
- It depends on the scale and special conformal generators in D-dimensions. Why?

Multiloop extension

- The h-loop amplitude involving one graviton/dilaton and n-tachyons in bosonic string is:

$$\mathcal{A}^{(h)}{}_{N+1} = C_h \left(N_0\right)^{h+1} \int dV_N \prod_{i<j=1}^{N} e^{\frac{\alpha'}{2} k_i k_j \mathcal{G}_h(z_i, z_j)} \epsilon_q^{\mu\nu} \star \int d^2 z \prod_{l=1}^{N} e^{\frac{\alpha'}{2} k_i q \mathcal{G}_h(z, z_l)}$$

$$\times \left[\frac{\alpha'}{2} \sum_{i,j} k_i^M \, k_j^N \, \partial_z \mathcal{G}_h(z,\, z_i) \partial_{\bar{z}} \mathcal{G}_h(z,\, z_j) + \frac{1}{2} \eta^{MN} \omega^{\rho}(z) (2\pi Im\tau)^{-1}_{\rho\sigma} \bar{\omega}^{\sigma}(z) \right]$$

- It has been obtained with the formalism of the N-string vertex that has the advantage of not requiring the external states to be on the mass shell.

[Di Vecchia, Pezzella, Frau, Hornfeck, Lerda, Sciuto, Nucl. Phys. B322(1989),317; Petersen and Sidenius Nucl Phys. B301 (1988) 247; Mandelstam, (1985)]

- The h-loops string Green's function:

$$\mathcal{G}_h(z_i,z_j)=\log\frac{|E(z_i,z_j)|^2}{|V_i'(0)V_j'(0)|}+\Re\left(\int_{z_j}^{z_i}\omega_I\right)(2\pi\mathrm{Im}\tau)^{-1}_{IJ}\Re\left(\int_{z_i}^{z_j}\omega_J\right)$$

- $\omega^\mu(z)$, $E(z,w)$ and τ_{IJ} are the abelian differentials, the Prime-form and the period matrix, respectively.
- $V_i^{-1}(z)=w_i$ are local coordinates defined around the punctures w_i. On-shell the amplitude will not depend on these local coordinates.
- Under world-sheet reparametrization $|V'_i(0)|^2$ transform as the inverse of the metric. In the conformal gauge $\rho\, dz\, d\bar{z}$ $\Longrightarrow$ $|V'_i(0)|^2=\frac{1}{\rho(z_i)}$
- The string Green-function satisfies:

$$\partial_z\partial_{\bar{z}}\log\mathcal{G}_h(z,w)=\pi\delta^{(2)}(z-w)-\frac{1}{4\pi}\omega_I(z)(\mathrm{Im}\tau)^{-1}_{IJ}\bar{\omega}_J(\bar{z})+\frac{1}{2}\partial_z\partial_{\bar{z}}\log\rho(z)$$

Genus

- The Gauss-Bonnet theorem : $\int d^2z\sqrt{g}R(g)=-2\int d^2z\partial_z\partial_{\bar{z}}\log\rho=8\pi(1-h)$

The Jacobian variety is the torus $\mathbb{C}^h/\Lambda$ with:

$$\Lambda := \left\{ \sum_{I=1}^{h} \left[n_I \int_{a_I} \vec{\omega} + n_{I+h} \int_{b_I} \vec{\omega} \right], \, n_I \in \mathbb{Z} \right\} \equiv \mathbb{Z}^h + \tau \mathbb{Z}^h$$

- $(a_I, b_I), I = 1, \ldots h,$ are the homology cycles.
- The Jacobian variety is a Kähler manifold and its Kähler form is: [Jost, Geometry and Physics; D'Hoker, Green , Pioline, Comm. Math Phys. 366, Isuue 3, (2019) 927]

$$\kappa_h = \frac{1}{4\pi h} \sum_{IJ} (2\pi \mathrm{Im}\tau)^{-1\,IJ} \, d\xi_I \wedge d\xi_J$$

- The pull-back, under the embedding (Abelian map) $\vec{\xi} = \int_{z_0}^{z} \vec{\omega}$, of the Kähler form, defines the metric:

$$\kappa_h = \tfrac{1}{4\pi h} \sum_{IJ} (2\pi \mathrm{Im}\tau)^{-1\,IJ} \, \omega_I(z) \, \bar{\omega}_J(\bar{z}) \, dz \wedge d\bar{z} \equiv g_{z\bar{z}} \, dz \wedge d\bar{z}$$

$$\int_{\Sigma_h} d^2 z \, \kappa_h = 1 \Longrightarrow \partial_z \partial_{\bar{z}} \log \rho(z) = 4\pi (h-1) \kappa_h$$

- This identity fixes ρ only up to multiplication by holomorphic functions and a constants i.e. $\rho = \tilde{\rho} e^{-f(z)-\bar{f}(\bar{z})+c}$. It also implies

$$\partial_z \partial_{\bar{z}} \mathcal{G}_h(z, w) = \pi \delta^2(z-w) - 2\pi \kappa_h(z) \Longleftrightarrow \text{Arakelov-type Green's function}$$

- Since $f, \bar{f}$ and c are arbitrary, we can fix them by imposing

$$\int d^2 z k_h(z) \, \mathcal{G}_h(z, w) = 0$$

- $\mathcal{G}_h(z, w)$ becomes invariant around the homology cycles and it coincides with the Arakelov Green's function.
- By using the properties of the Arakelov Green's function, all the integrals are calculable and one can see that the graviton soft theorem is the same as at three level.

- The dilaton soft-theorem becomes:

$$M^{(h)}_{N;\phi}(k_i;q) = \frac{\kappa_D}{\sqrt{D-2}}\left[-\sum_{i=1}^{N}\frac{m^2}{k_i q}\mathrm{e}^{q\partial_{k_i}} + 2 - \sum_{i=1}^{N}\hat{D}_i \right.$$
$$\left. + h(D-2) + q_\mu \sum_{i=1}^{N}\hat{K}_i^\mu\right] M_N^{(h)} + O(q^2)$$

$$\hat{D}_i = k_i \cdot \partial_{k_i} \qquad \hat{K}_i^\mu = \frac{1}{2}k_i^\mu \left(\partial_{k_i}\cdot\partial_{k_i}\right) - \left(k_i\cdot\partial_{k_i}\right)\partial_{k_i}^\mu$$

- Because of the dependence of h, we cannot immediately write the all-loop soft behaviour.

- However from the scaling properties:

$$M_N^{(h)} = \frac{\kappa_D^{2(h-1)+N}}{\sqrt{\alpha'}^{-(2-D)h+2}} F\left(\sqrt{\alpha'}k_i, R/\sqrt{\alpha'}\right) \quad ; \quad \kappa_D = \frac{(2\pi)^{\frac{D-3}{2}}}{\sqrt{2^{-9}}}\, g_s\, \sqrt{\alpha'}^{\frac{D-2}{2}} \left(\frac{\sqrt{\alpha'}}{R}\right)^{\frac{26-D}{2}}$$

we deduce,

$$\left[2 - \sum_{i=1}^{N} k_i \cdot \frac{\partial}{\partial k_i} + (D-2)h\right] M_N^{(h)} = \left[\frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} - R\frac{\partial}{\partial R}\right] M_N^{(h)}$$

- Getting the loop independent expression:

$$\mathcal{M}_{N;\phi}(k_i;q) = \frac{\kappa_D}{\sqrt{D-2}} \left[-\sum_{i=1}^{N} \frac{m^2}{k_i q} \mathrm{e}^{q\partial_{k_i}} + \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} - R\frac{\partial}{\partial R} + q_\mu \sum_{i=1}^{N} \hat{K}_i^\mu\right] \mathcal{M}_N(k_i)$$
$$+O(q^2)$$

Spontaneosly breaking of the Conformal Symmetry

- A conformal transformation of the coordinates is an invertible mapping, $x \to x'$, leaving the metric invariant up to a local scale factor:

$$g_{\mu\nu} \to \Lambda(x) g_{\mu\nu}$$

✓The group is an extension, with dilatations, $\mathcal{D}$, and special conformal transformations, $\mathcal{K}_\mu$, of the Poincaré group which belong to $\Lambda = 1$.

✓Infinitesimally, the group transforms the space-time coordinates as follows:

D : space-time dimension

$$x^\mu \to x'^\mu \quad = x^\mu + \epsilon^{MN} f^\mu_{MN}(x) \quad ; \quad \partial^\mu f^\nu_{MN} + \partial^\nu f^\mu_{MN} = \frac{2}{D} g^{\mu\nu} \partial_\rho f^\rho_{MN}$$

- For D>2 and in flat space, the generators are:

$$\mathcal{D} = i(d_\Phi + x_\mu \partial^\mu) \,, \quad \mathcal{J}^{\mu\nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu) - \mathcal{S}^{\mu\nu} \,,$$
$$\mathcal{P}^\mu = i\partial^\mu \,, \qquad \mathcal{K}^\mu = i(2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu + 2d_\Phi x^\mu) + 2x_\nu \mathcal{S}^{\mu\nu}$$

$\mathcal{S}^{\mu\nu}$: Spin angular momentum operator.

- The Nöther currents associated to the dilatations and special conformal transformations are conserved, because of the traceless of the improved energy momentum tensor:

$$\partial_\mu J^\mu_D = \partial_\mu [x^\nu T^{\mu\nu}] = T^\mu_\mu \,, \qquad \partial_\mu J^\mu_{K,\rho} = \partial_\mu [(2x_\nu x_\rho - \eta_{\rho\nu} x^2) T^{\mu\nu}] = 2x_\rho T^\mu_\mu$$

- Let's consider now a situation where the conformal symmetry is spontaneously broken due to a scalar field getting a nonzero vev:

$$\langle 0|\phi|0\rangle = v^{d_\phi}$$

d_ϕ is the scaling dimension of the field

- v is the only scale mass of the theory and the vacuum remains invariant under the Poincaré group.
- When the conformal group is spontaneously broken to the Poincaré group, although the broken generators are the dilatations and the special conformal transformations, only one massless mode, the Dilaton $\xi(x)$, is needed.

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- The dilaton couples linearly to the trace of the energy momentum tensor:

$$T_{\mu\nu} = -d_\xi v^{d_\xi} \left(\frac{\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu}{D-1} \right) \xi(x) \ldots \qquad T^\mu_\mu = -d_\xi v^{d_\xi} \partial^2 \xi(x)$$

Space-time dimension

- ✓ Soft-Theorems follows from the Ward-identities of the spontaneously broken symmetry.

- ➢ In a theory invariants under some global transformation, the starting point is the derivative of the matrix element of Nöther currents $J_i^{\mu_i}(y_i)$ and scalar fields $\phi(x_i)$.

$$\frac{\partial}{\partial y^{\mu_1}} \cdots \frac{\partial}{\partial y^{\mu_m}} T^* \langle J_1^{\mu_1}(y_1) \ldots J_m^{\mu_m}(y_m) \phi(x_1) \ldots \phi(x_n) \rangle$$

- T^* denotes the T-product with the derivatives placed outside of the time-ordering symbols.
- Single soft theorem is obtained by considering only one current:

$$-\partial_\mu T^* \langle 0 | j^\mu(x) \phi(x_1) \ldots \phi(x_n) | 0 \rangle + T^* \langle 0 | \partial_\mu j^\mu(x) \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$
$$= -i \sum_{i=1}^{n} \delta^D(x - x_i) \, T^* \langle 0 | \phi(x_1) \ldots \delta\phi(x_i) \ldots \phi(x_n) | 0 \rangle \, ,$$

- If the currents is unbroken, $\partial_\mu J^\mu = 0$, and one gets the usual Ward-identity of conserved symmetries.

- For spontaneously broken scale

$$\partial_\mu J^\mu_{\mathcal{D}}(y) = T^\mu_\mu = -d_\xi v^{d_\xi}\partial^2\xi(y) \qquad \delta\phi(x) = [\mathcal{D},\phi(x)] = i\,(d + x^\mu\partial_\mu)\,\phi(x)\,,$$

- and special conformal transformations

$$j^\mu{}_{(\lambda)} \;= T^{\mu\nu}(2x_\nu x_\lambda - \eta_{\nu\lambda}x^2) \;\;;\;\; \partial_\mu\, j^\mu{}_{(\lambda)} \;= 2\,x_\lambda\, T^\mu{}_\mu = 2\,d_\xi v^{d_\xi}\, x_\lambda(-\partial^2)\,\xi(x)$$

$$\delta_{(\lambda)}\phi(x) = [\mathcal{K}_\lambda,\phi(x)] = i\left((2x_\lambda x_\nu - \eta_{\lambda\nu}x^2)\partial^\nu + 2\,d\,x_\lambda\right)\phi(x)\,.$$

- Through the LSZ reduction formula that translate the relation between correlation functions in a relation between amplitudes

$$d_\xi v^{d_\xi}\,\mathcal{T}_{n+1}(q;k_1,\ldots,k_{n-1}) = \Bigg\{-\sum_{i=1}^{n-1}\frac{m_i^2}{k_i\cdot q}\left(1 + q^\mu\frac{\partial}{\partial k_i^\mu} + \frac{1}{2}q^\mu q^\nu\frac{\partial^2}{\partial k_i^\mu\partial k_i^\nu}\right) + D - nd - \sum_{i=1}^{n-1}k_i^\mu\frac{\partial}{\partial k_i^\mu}$$

$$-q^\lambda\sum_{i=1}^{n-1}\left[\frac{1}{2}\left(2\,k_i^\mu\frac{\partial^2}{\partial k_i^\mu\partial k_i^\lambda} - k_{i\,\lambda}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu}\right) + d\,\frac{\partial}{\partial k_i^\lambda}\right]\Bigg\}\mathcal{T}_n(k_1,\ldots,k_{n-1}) + \mathcal{O}(q^2)$$

- The Gravity dilaton and the Nambu-Goldstone boson of the broken conformal symmetry satisfy similar soft theorems. Why?
- The Ward identities satisfied by the perturbative multiloop string amplitudes are similar to those CFT correlators in momentum space. Maybe there are more general solutions of these Ward identities valid not only for small values of the string coupling constant g_s

Conclusions

- We can go back to 1994 and compute the threshold corrections in string theory.

- We can go back to the future, starting new journeys.

Thank you for this long friendship and happy birthday!