Inelastic Exponentiation and Gravitational Waveforms

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Di Vecchia-80 Fest, May 16th, 2023

Based on

P. Di Vecchia, C.H., R. Russo, G. Veneziano [2104.03256,2210.12118+to appear]

A. Georgoudis, C.H., I. Vazquez–Holm [2303.07006]

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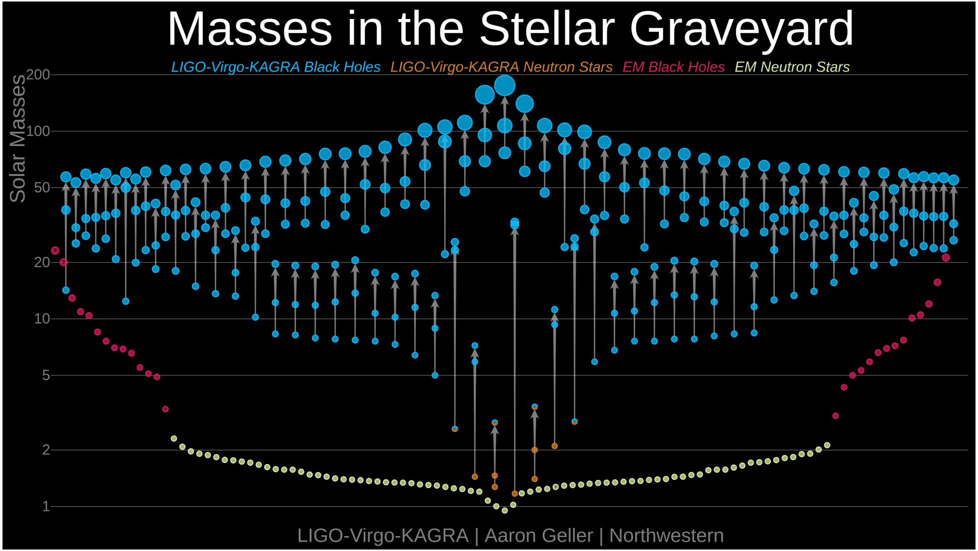
Outline

Outline

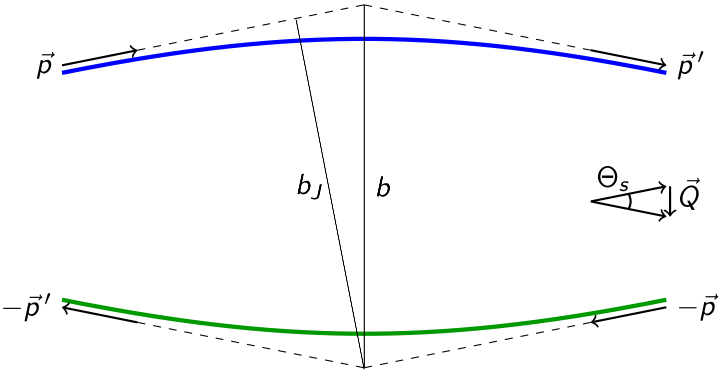
Gravitational Wave Astronomy



Post-Minkowskian (PM) Scattering

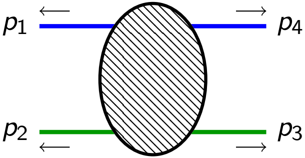
[see e.g. Kosower, Maybee, O'Connell '18; Bern et al. '19; Di Vecchia, C.H., Russo, Veneziano '21; Bellazzini, Isabella, Riva '22]

$$\boxed{\frac{\hbar}{m} \ll Gm \ll b}$$



$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\text{PM}}{\ll} 1\,, \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic).}$$

The Elastic Eikonal



$$s = -(p_1 + p_2)^2 = E^2$$
$$= m_1^2 + 2 m_1 m_2 \, \sigma + m_2^2 \, ,$$
$$t = -(p_1 + p_4)^2 = -q^2 \, .$$

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}(b) = \frac{1}{4Ep} \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \, e^{ib\cdot q} \mathcal{A}(s, q) \, , \qquad \boxed{1 + i\tilde{\mathcal{A}}(b) = e^{2i\delta(b)}}$$

with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar} \left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots \right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

$$\int d^{D-2}b \, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial \, \mathrm{Re} \, 2\delta}{\partial b^\mu}$$

The 3PM Eikonal in General Relativity

[Di Vecchia, C.H., Russo, Veneziano '20, '21]

[Related work at 3PM: Bern al.'19; Damour '20; Herrmann et al. '21, Bjerrum-Bohr et al.'21; Brandhuber et al.'21]

- Eikonal phase:

$$\mathrm{Re}\, 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}}-\frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}}-\frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right] + \mathrm{Re}\, 2\delta_2^{\mathrm{RR}}$$

with

$$\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} = \frac{G}{2} Q_{1\mathrm{PM}}^2 \mathcal{I}(\sigma)\,, \quad \mathcal{I}(\sigma) \equiv \frac{8-5\sigma^2}{3(\sigma^2-1)}+\frac{\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\, 2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} + \cdots$$

Elastic Final State

- Final state (schematically):

$$\boxed{|\text{out}\rangle = e^{2i\delta(b)}|\text{in}\rangle}$$

- Impulse:

$$Q_\mu = \Big(- i\langle\text{out}|\frac{\overleftrightarrow{\partial}}{\partial b^\mu}|\text{out}\rangle\Big)/\langle\text{out}|\text{out}\rangle = \frac{\partial \operatorname{Re} 2\delta}{\partial b^\mu}\,.$$

Problems:

1. How do we restore (nonperturbative) unitarity?

$$\langle\text{out}|\text{out}\rangle = e^{-\operatorname{Im} 2\delta}\langle\text{in}|\text{in}\rangle \to 0 \qquad \text{as } D \to 4$$

2. How do we calculate the emitted waveform?

Outline

Operator Exponential

- **Option 1**: Look for an exponential structure in the full S-matrix [Damgaard, Planté, Vanhove '21]

$$S = 1 + iT = e^{iN} \qquad \boxed{|\text{out}\rangle = e^{iN}|\text{in}\rangle}$$

The "N-matrix" elements are better behaved in the classical limit.

- **Option 2**: Look for an exponential structure in the classical limit,

$$\boxed{|\text{out}\rangle \simeq e^{2i\hat{\delta}}|\text{in}\rangle}$$

with $2\hat{\delta}$ a suitable eikonal operator [Cristofoli et al. '21] [Di Vecchia, CH, Russo, Veneziano '22].

N-Operator, T-Operator and Unitarity

- N-operator: $S = e^{iN}$

$$N = -i\log(1+iT) = T - \frac{i}{2}T^2 + \cdots$$

  up to one loop.

- Unitarity: $S^\dagger S = 1$,

$$\frac{1}{2}(T - T^\dagger) = +\frac{i}{2}T^\dagger T$$

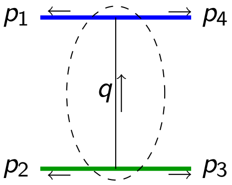
- We shall denote by $\mathcal{B}$ the N-matrix elements, just like $\mathcal{A}$ denotes the the usual amplitudes (T-matrix elements). Then, by unitarity,

$$\mathcal{B}_0 = \mathcal{A}_0\,, \qquad \mathcal{B}_1 = \operatorname{Re}\mathcal{A}_1\,.$$

The $2 \to 2$ Amplitude

[Parra-Martinez, Ruf, Zeng '20]

$$\mathcal{A}^{(4)}(q) =$$



$$\begin{aligned}
p_1^\mu &= -\bar{m}_1 u_1^\mu + q^\mu/2 \\
p_4^\mu &= +\bar{m}_1 u_1^\mu + q^\mu/2 \\
p_2^\mu &= -\bar{m}_2 u_2^\mu - q^\mu/2 \\
p_3^\mu &= +\bar{m}_2 u_2^\mu - q^\mu/2 \\
q &\sim \hbar/b
\end{aligned}$$

- The "average velocities" $u_{1,2}^\mu$ obey

$$u_1^2 = -1 = u_2^2 \qquad \& \qquad u_1 \cdot q = 0 = u_2 \cdot q$$

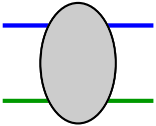
- We consider the physical variables

$$y = -u_1 \cdot u_2 \geq 1 \,, \qquad q^2 \,.$$

Note that $y = \sigma + \mathcal{O}(\hbar^2)$ and $\bar{m}_{1,2} = m_{1,2} + \mathcal{O}(\hbar^2)$.

Tree-Level $2 \to 2$ Amplitude

- $2 \to 2$ amplitude in momentum space

$$\mathcal{A}_0^{(4)} = \mathcal{B}_0^{(4)} =$$  $$= \frac{32\pi G \bar{m}_1^2 \bar{m}_2^2 \left(y^2 - \frac{1}{2(1-\epsilon)}\right)}{q^2}$$

$$+ \left[\frac{4\pi G \left(\bar{m}_1^2 + \bar{m}_2^2\right)}{1-\epsilon} + \frac{\pi G(3-2\epsilon)}{\epsilon - 1} q^2 \right]$$

analytic in q^2

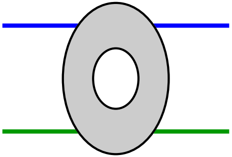
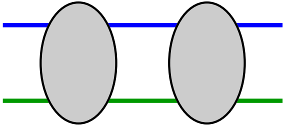
- In impact-parameter space $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{2i\delta_0}}$

$$\tilde{\mathcal{A}}_0^{(4)} = \tilde{\mathcal{B}}_0^{(4)} = 2\delta_0 = \frac{4Gm_1 m_2 \left(\sigma^2 - \frac{1}{2(1-\epsilon)}\right)}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b)^{-\epsilon}}$$

$+$ short-range

[Cheung, Rothstein, Solon '18; Collado, Di Vecchia, Russo '19]

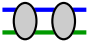
Elastic subtraction:

$\mathcal{A}_1^{(4)} =$  $= \frac{i}{2}$  $+ \mathcal{B}_1^{(4)}$

with

$$\mathcal{B}_1^{(4)} = \frac{6\pi^2 G^2 \bar{m}_1^2 \bar{m}_2^2 \left(5y^2 - 1\right)\left(\bar{m}_1 + \bar{m}_2\right)}{\sqrt{q^2}}$$

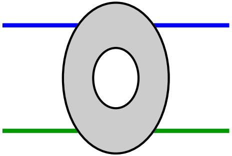
in $D = 4$

- The subtracted term $\frac{i}{2}$  is imaginary and infrared divergent
- It is also "superclassical" $\mathcal{O}(\frac{1}{\hbar^2})$
- $\mathcal{B}_1^{(4)}$ is real, finite and classical $\mathcal{O}(\frac{1}{\hbar})$ [and tr. weight 2].

Elastic Exponentiation at One Loop

[Collado, Di Vecchia, Russo '19]

Inelastic subtraction in b-space:

$\tilde{\mathcal{A}}_1^{(4)} = \text{FT}$  $= \frac{i}{2}(2\delta_0)^2 + \tilde{\mathcal{B}}_1^{(4)}$

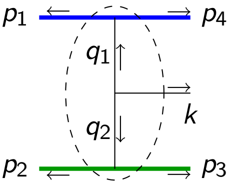
- This matches the exponential $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{i(2\delta_0 + 2\delta_1)}}$
- And it identifies

$$\tilde{\mathcal{B}}_1^{(4)} = 2\delta_1 = \frac{3\pi G^2 m_1 m_2 \left(5\sigma^2 - 1\right)(m_1 + m_2)}{4b\sqrt{\sigma^2 - 1}}$$

$D = 4$, which determines 2PM correction to the deflection angle.

The $2 \to 3$ Amplitude

[Brandhuber et al. '23; Herderschee, Teng, Roiban '23] [Georgoudis, CH, Vazquez-Holm '23]

$\mathcal{A}^{(5)\mu\nu}(q_1, q_2, k) =$ 

$$p_1^\mu = -\bar{m}_1 u_1^\mu + q_1^\mu/2$$
$$p_4^\mu = +\bar{m}_1 u_1^\mu + q_1^\mu/2$$
$$p_2^\mu = -\bar{m}_2 u_2^\mu + q_2^\mu/2$$
$$p_3^\mu = +\bar{m}_2 u_2^\mu + q_2^\mu/2$$
$$q_1 \sim q_2 \sim k \sim \hbar/b$$

- The "average velocities" $u_{1,2}^\mu$ obey

$$u_1^2 = -1 = u_2^2 \qquad \& \qquad u_1 \cdot q_1 = 0 = u_2 \cdot q_2$$

- We consider the physical variables

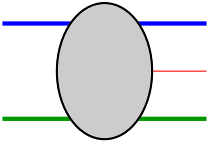
$$y = -u_1 \cdot u_2 \geq 1\,, \qquad \omega_{1,2} = -u_{1,2} \cdot k \geq 0\,, \qquad q_{1,2}^2\,.$$

Note that $y = \sigma + \mathcal{O}(\hbar)$ and $\bar{m}_{1,2} = m_{1,2} + \mathcal{O}(\hbar^2)$.

Tree-Level $2 \to 3$ Amplitude

[2 → 3 amplitude: Goldberger, Ridgway '17; Luna, Nicholson, O'Connell, White '17] [Di Vecchia, CH, Russo, Veneziano, '21]

Gauge-invariant amplitude $k_\mu \mathcal{A}_0^{(5)\mu\nu} = 0$,

$\mathcal{A}_0^{(5)\mu\nu} = \mathcal{B}_0^{(5)\mu\nu} =$ 

$$= (8\pi G)^{\frac{3}{2}} \Bigg\{ 4m_1^2 m_2^2 (\sigma^2 - \tfrac{1}{D-2}) \Bigg[-\frac{p_1^\mu p_1^\nu (k\cdot q_1)}{(p_1\cdot k)^2 q_2^2} - \frac{p_2^\mu p_2^\nu (k\cdot q_2)}{(p_2\cdot k)^2 q_1^2} + \frac{p_1^{(\mu}(q_1-q_2)^{\nu)}}{2(p_1\cdot k) q_2^2} - \frac{p_2^{(\mu}(q_1-q_2)^{\nu)}}{2(p_2\cdot k) q_1^2}$$

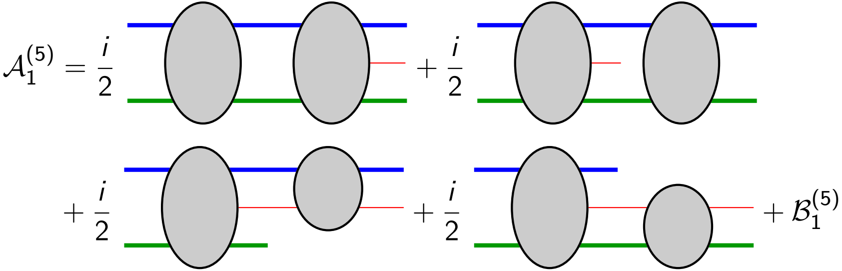
$$+ \frac{(q_1-q_2)^\mu (q_1-q_2)^\nu}{2q_1^2 q_2^2} \Bigg] + 8\frac{((p_1\cdot k)p_2^\mu - (p_2\cdot k)p_1^\mu)((p_1\cdot k)p_2^\nu - (p_2\cdot k)p_1^\nu)}{q_1^2 q_2^2}$$

$$+ (2p_1\cdot p_2) \left(\frac{4p_1^\mu p_1^\nu \frac{k\cdot p_2}{k\cdot p_1} - 2p_1^{(\mu}p_2^{\nu)}}{q_2^2} + \frac{4p_2^\mu p_2^\nu \frac{k\cdot p_1}{k\cdot p_2} - 2p_1^{(\mu}p_2^{\nu)}}{q_1^2} + \frac{(q_1-q_2)^{(\mu}\left(-2(p_1\cdot k)p_2 + 2(p_2\cdot k)p_1\right)^{\nu)}}{q_1^2 q_2^2} \right) \Bigg\}.$$

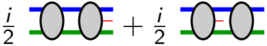
One-Loop $2 \to 3$ Amplitude

[Brandhuber et al. '23; Herderschee, Roiban, Teng '23] [Georgoudis, C.H., Vazquez-Holm '23]

Inelastic subtraction:



with $\mathcal{B}_1^{(5)}$ a rational function of ($\sqrt{\ }$ of) y, ω_1, ω_2, q_1^2, q_2^2.

- The subtracted terms are all imaginary and infrared divergent
-  are also "superclassical" $\mathcal{O}(\frac{1}{\hbar^2})$
- The leftover $\mathcal{B}_1^{(5)}$ is real, finite and classical $\mathcal{O}(\frac{1}{\hbar})$ [and tr. weight 2].

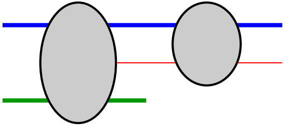
Inelastic Exponentiation at One Loop

[Brandhuber et al. '23; Herderschee, Roiban, Teng '23] [Georgoudis, C.H., Vazquez-Holm '23]

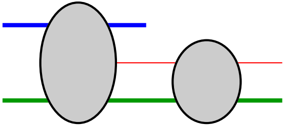
Inelastic subtraction in b-space:

$$\tilde{\mathcal{A}}_1^{(5)} = 2i\delta_0 \, \tilde{\mathcal{B}}_0^{(5)}$$

$$+ \frac{i}{2} \, \mathrm{FT}$$



$$+ \frac{i}{2} \, \mathrm{FT}$$



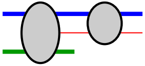
$$+ \tilde{\mathcal{B}}_1^{(5)}$$

- This matches the exponential (here $\int_k \simeq \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)$)

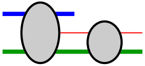
$$\boxed{i\tilde{\mathcal{A}}_1^{(5)} = \langle \mathrm{in} | \, a(k) \, e^{i(2\delta_0 + 2\delta_1)} e^{i\int_k \left[\tilde{\mathcal{W}}(k) a^\dagger(k) + \tilde{\mathcal{W}}^*(k) a(k)\right]} \, |\mathrm{in}\rangle}$$

- And it identifies

$$\mathcal{W}_0 = \mathcal{B}_0^{(5)}, \quad \mathcal{W}_1 = \mathcal{B}_1^{(5)} + \frac{i}{2}$$



$$+ \frac{i}{2}$$



Properties of $\mathcal{B}_0^{(5)\mu\nu}$, $\mathcal{B}_1^{(5)\mu\nu}$

- Under "time reversal"

$$\omega_{1,2} \mapsto -\omega_{1,2}$$

- The leading contribution $\mathcal{B}_0^{(5)\mu\nu}$ is *even*,
- The subleading contribution has both *odd* and *even* parts

$$\mathcal{B}_1 = \mathcal{B}_{1E} + \mathcal{B}_{1O}\,.$$

- The odd one is equal to

$$\mathcal{B}_{1O} = \left(1 - \frac{y(y^2 - \frac{3}{2})}{(y^2-1)^{\frac{3}{2}}}\right) \lim_{\epsilon\to 0} \left[-\pi\epsilon\, \mathrm{Im}\, \mathcal{W}_1\right]$$

Radiation-reaction? [Di Vecchia, CH, Russo, Veneziano '21]

Inelastic Final State

[Di Vecchia, C.H., Russo, Veneziano 2210.12118]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)}\, e^{i\int_k \left[\tilde{\mathcal{W}}(k)a^\dagger(k)+\tilde{\mathcal{W}}^*(k)a(k)\right]}\,.$$

- Final state (again, schematically):

$$\boxed{|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle}$$

- Unitarity:

$$\langle \mathrm{out}|\mathrm{out}\rangle = \langle \mathrm{in}|\mathrm{in}\rangle = 1$$

- Consistency with the elastic exponentiation: by the BCH formula,

$$\langle \mathrm{in}|\mathrm{out}\rangle = e^{i\,\mathrm{Re}\,2\delta(b)}\, e^{-\,\mathrm{Im}\,2\delta(b)} = e^{2i\delta(b)}$$

because, by unitarity, $\mathrm{Im}\,2\delta_2 = \frac{1}{2}\int_k \tilde{\mathcal{A}}_0^{(5)}\tilde{\mathcal{A}}_0^{(5)*}$.

Outline

Metric Fluctuation in the Final State

- Free graviton field (recall $\int_k = \int 2\pi\delta(k^2)\theta(k^0)\frac{d^Dk}{(2\pi)^D}$)

$$H_{\mu\nu}(x) = \int_k \left[e^{ik\cdot x} a_{\mu\nu}(k) + e^{-ik\cdot x} a^\dagger_{\mu\nu}(k)\right].$$

- Expectation in the final state

$$h_{\mu\nu}(x) = \langle \text{out}|H_{\mu\nu}(x)|\text{out}\rangle = \int_k e^{ik\cdot x} i\tilde{\mathcal{W}}_{\mu\nu}(k) + (\text{c.c.})$$

with

$$\boxed{i\tilde{\mathcal{W}}_{\mu\nu}(k) = \langle \text{out}|a_{\mu\nu}(k)|\text{out}\rangle}$$

- This is the canonically normalized metric fluctuation

$$g_{\mu\nu}(x) - \eta_{\mu\nu} = \sqrt{32\pi G}\, h_{\mu\nu}(x).$$

The Asymptotic Waveform

- Asymptotic limit: We consider a detector with four-velocity t^μ and retarded time u placed at a distance $r \to \infty$ in a null direction n^μ, i.e. near future null infinity $\mathcal{I}^+$,

$$x^\mu = u\, t^\mu + r\, n^\mu \,, \qquad n \cdot n = 0 \,, \qquad n \cdot t = -1 \,.$$

- E.g. the detector's rest frame,

$$t^\mu = (1, 0, 0, 0) \,, \qquad n^\mu = (1, \hat{x}) \,, \qquad \hat{x} \cdot \hat{x} = 1 \,.$$

- Then

$$h^{\mu\nu} \sim \frac{1}{4\pi r} \int_0^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \tilde{\mathcal{W}}^{\mu\nu}(k = \omega n) + (\text{c.c.})$$

with ω the frequency measured by the detector.

L.O. Gravitational Waveform

[Di Vecchia, C.H., Russo, Veneziano (to appear)]

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22]

- Convenient variables: $(x + y = 1)$

$$\Omega_j = \frac{b\omega_j}{\sqrt{\sigma^2 - 1}}\,, \qquad b^\mu(x) = x b_1^\mu + y b_2^\mu\,, \qquad \Omega_1(x) = \sqrt{\Omega_1^2 x^2 + 2\Omega_1\Omega_2 xy + \Omega_2^2 y^2}$$

- The covariant waveform in frequency domain is $\tilde{\mathcal{W}}_0^{\mu\nu} = \tilde{\mathcal{W}}_{0,12}^{\mu\nu} + \tilde{\mathcal{W}}_{0,\text{irr}}^{\mu\nu}$,

$$\frac{\tilde{\mathcal{W}}_{0,12}^{\mu\nu}}{\sqrt{8\pi G}} = \frac{G m_1 m_2}{\omega_1\omega_2(\sigma^2 - 1)} \sum_{j=1,2} \left[A_j^{\mu\nu} e^{-ib_j\cdot k} \frac{K_0(\Omega_j)}{\sqrt{\sigma^2 - 1}} + B_j^{\mu\nu} e^{-ib_j\cdot k} K_1(\Omega_j) \right]$$

$$\frac{\tilde{\mathcal{W}}_{0,\text{irr}}^{\mu\nu}}{\sqrt{8\pi G}} = \frac{G m_1 m_2}{(\sigma^2 - 1)^{5/2}} \int_0^1 e^{-ib(x)\cdot k} dx \left[C^{\mu\nu} K_0(\Omega(x)) + D^{\mu\nu} \frac{K_1(\Omega(x))}{\Omega(x)} \right]$$

with A_j, B_j, C, D polynomials in u_1, u_2, k, b.

$\times$ Polarization

[Di Vecchia, C.H., Russo, Veneziano (to appear)]

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22]

For the $\times$ polarization,

$$\frac{\tilde{\mathcal{W}}_{12,\times}}{\sqrt{8\pi G}} = -\frac{2iGm_1 m_2 c_0}{b(\sigma^2 - 1)}\, b\cdot e_\phi \left(e^{-ib_1\cdot k} K_1\left(\Omega_1\right) u_1 \cdot \tilde{e}_\theta - e^{-ib_2\cdot k} K_1\left(\Omega_2\right) u_2 \cdot \tilde{e}_\theta \right)$$

and

$$\frac{\tilde{\mathcal{W}}_{\text{irr},\times}}{\sqrt{8\pi G}} = \frac{2iGm_1 m_2}{\sqrt{\sigma^2 - 1}} \left(\frac{c_0 \omega_1 \omega_2}{\sqrt{\mathcal{P}}} - 2\sigma\sqrt{\mathcal{P}} \right) b\cdot e_\phi \int_0^1 e^{-ib(x)\cdot k} K_0(\Omega(x)) dx$$

with $\mathcal{P} = -\omega_1^2 + 2\sigma\omega_1\omega_2 - \omega_2^2$ and $c_0 = 2\sigma^2 - 1$.

+ Polarization

[Di Vecchia, C.H., Russo, Veneziano (to appear)]

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22]

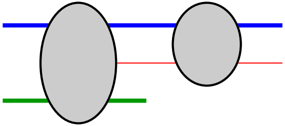
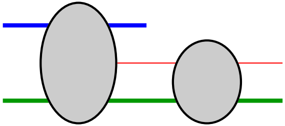
For the + polarization,

$$\frac{\tilde{\mathcal{W}}_{12,+}}{\sqrt{8\pi G}} = \frac{G m_1 m_2}{\omega_1\omega_2\left(\sigma^2-1\right)}\Big[i\frac{b\cdot k}{b}\, c_0 \left(e^{-ib_2\cdot k} K_1\left(\Omega_2\right)\left(u_2\cdot\tilde{e}_\theta\right)^2\omega_1 - e^{-ib_1\cdot k}K_1\left(\Omega_1\right)\left(u_1\cdot\tilde{e}_\theta\right)^2\omega_2\right)$$
$$+ \frac{e^{-ib_1\cdot k}K_0(\Omega_1)\, u_1\cdot\tilde{e}_\theta\omega_2 - e^{-ib_2\cdot k}K_0(\Omega_2) u_2\cdot\tilde{e}_\theta\omega_1}{\sqrt{\sigma^2-1}\sqrt{\mathcal{P}}}$$
$$\times\left(\left(\sigma^2-1\right)\left(4\mathcal{P}\sigma - c_0\omega_1\omega_2\right) - c_0\mathcal{P}\sigma\right)\Big]$$

and

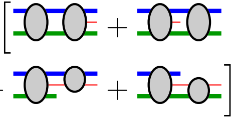
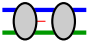
$$\frac{\tilde{\mathcal{W}}_{\mathrm{irr},+}}{\sqrt{8\pi G}} = \frac{G m_1 m_2}{\sqrt{\sigma^2-1}}\int_0^1 dx\, e^{-ib(x)\cdot k}\Big[\frac{(b\cdot e_\phi)^2}{b^2} c_0 K_1(\Omega(x))\Omega(x) - c_0 K_0(\Omega(x))$$
$$+ \frac{b^2 K_1(\Omega(x))}{\Omega(x)\mathcal{P}}\left(c_0\omega_1^2\omega_2^2 + 2\mathcal{P}^2 - 4\sigma\omega_1\omega_2\mathcal{P}\right)\Big].$$

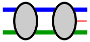
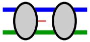
[Brandhuber et al. ’23; Herderschee, Roiban, Teng ’23] [Georgoudis, C.H., Vazquez-Holm ’23]

$$\mathcal{W}_0^{\mu\nu} = \mathcal{B}_0^{\mu\nu} = \mathcal{A}_0^{\mu\nu}\,, \qquad \mathcal{W}_1^{\mu\nu} = \mathcal{B}_1^{\mu\nu} + \frac{i}{2}$$  $$+\frac{i}{2}$$ 

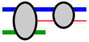
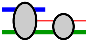
Indeed, $$-i\langle \text{out}|a_{\mu\nu}(k)|\text{out}\rangle = \langle \text{in}|a_{\mu\nu}(k)N|\text{in}\rangle$$

$$+\frac{i}{2}\langle \text{in}|a_{\mu\nu}(k)N^2|\text{in}\rangle - i\langle \text{in}|Na_{\mu\nu}(k)N|\text{in}\rangle + \cdots$$

and (only) the “iteration” cuts ,  cancel out.

IR Divergences in the Waveform Kernel

- The IR divergence appearing in $\frac{i}{2}$  $+ \frac{i}{2}$  obeys [Weinberg '65]

$$\mathcal{W}_1^{\mu\nu} = \mathcal{B}_1^{\mu\nu} - i\bar{\mu}^{2\epsilon}\frac{G}{\epsilon}(m_1\omega_1 + m_2\omega_2)\mathcal{B}_0^{\mu\nu} + i\,\mathrm{Im}\,\mathcal{A}_1^{\mu\nu[-1,0]} + \mathcal{O}(\epsilon)$$

- Therefore, it exponentiates according to

$$\mathcal{W}^{\mu\nu} = e^{-i\frac{G}{\epsilon}(m_1\omega_1+m_2\omega_2)}\left[\mathcal{B}_0^{\mu\nu} + \mathcal{B}_1^{\mu\nu} + i\mathcal{M}_1^{\mu\nu}\right] + \mathcal{O}(\epsilon) + \mathcal{O}(G^{7/2})$$

where

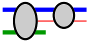
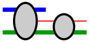
$$\mathcal{M}_1^{\mu\nu} = G(m_1\omega_1 + m_2\omega_2)\mathcal{B}_0^{\mu\nu}\log\bar{\mu}^2 + \mathrm{Im}\,\mathcal{A}_1^{\mu\nu[-1,0]}\,.$$

- The IR divergence and $\log\bar{\mu}^2$ do not enter the spectral emission rate

$$d\tilde{\rho} = |\tilde{\mathcal{W}}|^2\theta(k^0)\,2\pi\delta(k^2)\,\frac{d^4k}{(2\pi)^4}$$

The Tail Effect

[Brandhuber et al. '23; Herderschee, Roiban, Teng '23] [Georgoudis, C.H., Vazquez-Holm '23]

- IR-divergent rescattering cuts $\frac{i}{2}$  $+\ \frac{i}{2}$  enter the waveform,

$$h^{\mu\nu} \sim \frac{1}{4\pi r} \int_0^\infty \frac{d\omega}{2\pi}\, e^{-i\omega\left(u - \frac{G}{\epsilon}(m_1 u_1 + m_2 u_2)\cdot n\right)} \left[\tilde{\mathcal{B}}_0^{\mu\nu} + \tilde{\mathcal{B}}_1^{\mu\nu} + i\tilde{\mathcal{M}}_1^{\mu\nu}\right]_{k=\omega n}$$
$$+ \text{(c.c.)}.$$

- The divergence can be formally reabsorbed via a constant shift of the detector's retarded time [Goldberger, Ross '12; Porto, Ross, Rothstein '12]

$$u \mapsto u - G\left(-\frac{1}{\epsilon} + \log\bar{\mu}^2\right)(m_1 u_1 + m_2 u_2)\cdot n$$

- That is, one must fix experimentally the origin of the u-axis!
(Tail effect [see e.g. Blanchet '13])

Summary and Outlook

- An operator version of the eikonal exponentiation emerges from $2 \to 2$ and $2 \to 3$ amplitude at one loop
- The FT of the (suitably subtracted) amplitude is the waveform.
- Rescattering induces the tail effect in the subleading waveform kernel.

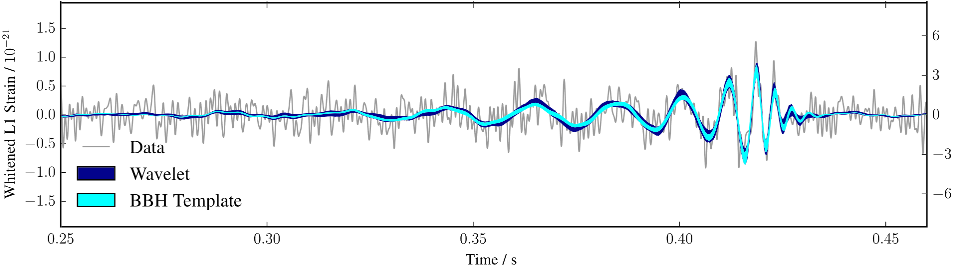
For the future:

- Analytic results for the subleading waveform? Probably simpler in time domain
- Beyond coherent-state approximation? E.g. $a^\dagger a$
- Differential spectrum? Integrated P_{rad} and J_{rad} at $\mathcal{O}(G^4)$?
- Subleading $\log \omega$ soft theorems?
- Comparison with the PN literature

ADDITIONAL MATERIAL

Waveform Templates

[LIGO Scientific Collaboration '16]



Inspiral	Merger	Ringdown
Weak gravity	Strong gravity	Small oscillations

Analytical Approximation Methods

- Post-Newtonian (PN): expansion "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$

- Self-Force: expansion in the near-probe limit $m_2 \ll m_1$ or

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \ll 1\,.$$

General Relativity from Scattering Amplitudes

Idea

Extract the PM gravitational dynamics from scattering amplitudes.

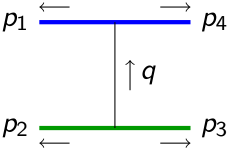
- Weak-coupling expansion $\leftrightarrow$ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM	4PM State of the art ...

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...)

Example: the 1PM Eikonal

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



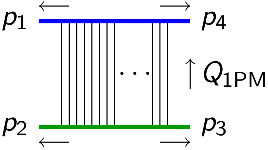
$$\mathcal{A}_0(s, q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0(s, b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al.'18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\to} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0 .$$

- From $Q = \partial_b 2\delta$, we obtain the leading-order deflection



$$Q_{1PM} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2 - 1}}$$

$$\Theta_{1PM} = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2 - 1)} .$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy

[Di Vecchia, C.H., Russo, Veneziano 2008.12743, 2101.05772, 2104.03256]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- the *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$,
- its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs\, \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

The Initial State

[Kosower, Maybee, O'Connell '18]

- We model the initial state by $|\text{in}\rangle = |1\rangle \otimes |2\rangle$, with

$$|1\rangle = \int_{-p_1} \varphi_1(-p_1)\, e^{ib_1\cdot p_1} |-p_1\rangle$$
$$|2\rangle = \int_{-p_2} \varphi_2(-p_2)\, e^{ib_2\cdot p_2} |-p_2\rangle$$

and $\int_{-p_i} = \int 2\pi\delta(p_i^2 + m_i^2)\theta(-p_i^0)\frac{d^D p_i}{(2\pi)^D}$ the LIPS measure.

- Wavepackets $\varphi_i(-p_i)$ peaked around the classical incoming momenta.
- Impact parameter $b^\mu = b_1^\mu - b_2^\mu$ lies in the transverse plane $b \cdot p_1 = 0 = p_2 \cdot b$.

- Elastic Fourier transform:

$$\mathrm{FT}\,\mathcal{A}^{(4)} = \int \frac{d^D q}{(2\pi)^D}\, 2\pi\delta(2\bar{m}_1 u_1\cdot q)\, 2\pi\delta(2\bar{m}_2 u_2\cdot q) e^{ib\cdot q}\mathcal{A}^{(4)}(q)$$
$$\simeq \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}(s,q) = \tilde{\mathcal{A}}^{(4)}\,.$$

- Inelastic Fourier transform:

$$\mathrm{FT}\,\mathcal{A}^{(5)} = \int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k)$$
$$\times\, 2\pi\delta(2\bar{m}_1 u_1\cdot q_1)2\pi\delta(2\bar{m}_2 u_2\cdot q_2)e^{ib_1\cdot q_1+ib_2\cdot q_2}\mathcal{A}^{(5)}(q_1,q_2,k)$$
$$\simeq \tilde{\mathcal{A}}^{(5)}(k)\,.$$

L.O. Gravitational Waveform

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22; Cristofoli et al. '21] [Di Vecchia, CH, Russo, Veneziano (to appear)]

- Even parity under time reversal implies

$$h^{\mu\nu}_{\text{L.O.}} \sim \frac{1}{4\pi r}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega u}\,\tilde{\mathcal{W}}^{\mu\nu}_0(k=\omega n)$$

- We isolate terms in which either $1/q_1^2$ or $1/q_2^2$ cancels, whose Fourier transform is simpler,

$$\tilde{\mathcal{W}}^{\mu\nu}_0(k) = \tilde{\mathcal{W}}^{\mu\nu}_{0,12}(k) + \tilde{\mathcal{W}}^{\mu\nu}_{0,\text{irr}}(k)$$

Eikonal Final State

[Di Vecchia, C.H., Russo, Veneziano 2210.12118] [cf. Cristofoli et al.'21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$|\text{out}\rangle \simeq \int_{p_3}\int_{p_4} e^{-ib_1\cdot p_4 - ib_2\cdot p_3} \int \frac{d^D Q_1}{(2\pi)^D}\int \frac{d^D Q_2}{(2\pi)^D}\, \Phi_1(p_4 - Q_1)\,\Phi_2(p_3 - Q_2)$$
$$\times \int d^D x_1 \int d^D x_2\, e^{i(b_1 - x_1)\cdot Q_1 + i(b_2 - x_2)\cdot Q_2}\, e^{2i\hat{\delta}(x_1,x_2)} |p_4, p_3, 0\rangle$$

with

$$e^{2i\hat{\delta}(x_1,x_2)} = e^{i\,\text{Re}\,2\delta(b)}\, e^{i\int_k \left[\tilde{\mathcal{A}}_j(x_1,x_2,k) a_j^\dagger(k) + \tilde{\mathcal{A}}_j^*(x_1,x_2,k) a_j(k)\right]}\,.$$

Here

- b is the projection of $x_1 - x_2$ orthogonal to $p_4 - Q_1/2$, $p_3 - Q_2/2$,
- $\tilde{\mathcal{A}}_j = \varepsilon_{j\mu\nu}\,\tilde{\mathcal{A}}^{\mu\nu}$ is the impact-parameter space $2 \to 3$ amplitude.

Physical Polarizations

- Transverse polarization vectors,

$$\tilde{e}_\theta \cdot \tilde{e}_\theta = 1 = e_\phi \cdot e_\phi \,, \qquad \tilde{e}_\theta \cdot e_\phi = 0 \,, \qquad \tilde{e}_\theta \cdot k = e_\phi \cdot k = 0$$

and

$$\tilde{e}_\theta \cdot b = 0 \,, \qquad e_\phi \cdot u_i = 0 \,, \qquad -\frac{u_1 \cdot \tilde{e}_\theta}{\sigma\omega_1 - \omega_2} = \frac{u_2 \cdot \tilde{e}_\theta}{\sigma\omega_2 - \omega_1} = \frac{1}{\sqrt{\mathcal{P}}} \,.$$

- Transverse-traceless polarization tensors

$$\varepsilon_\times^{\mu\nu} = \frac{1}{2}(\tilde{e}_\theta^\mu e_\phi^\nu + \tilde{e}_\theta^\nu e_\phi^\mu) \,, \qquad \varepsilon_+^{\mu\nu} = \frac{1}{2}(\tilde{e}_\theta^\mu \tilde{e}_\theta^\mu - e_\phi^\mu e_\phi^\nu)$$

so $\tilde{\mathcal{W}}_{\times,+} = \varepsilon_{\times,+\,\mu\nu} \tilde{\mathcal{W}}^{\mu\nu}$.