Review Talk:
Recent Developments in Categorical Symmetries

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Global Symmetries in 2025

Finite global symmetries form **fusion (higher) categories** =topological defects of dimensions $d-1,\cdots,0$, that can be composed and satisfy consistency conditions.

Quick recap for those who have not paid attention in the last 10 years:

- Higher form symmetries: [Gaiotto, Kapustin, Seiberg, Willett]
  Group-like symmetries acting on extended operators.

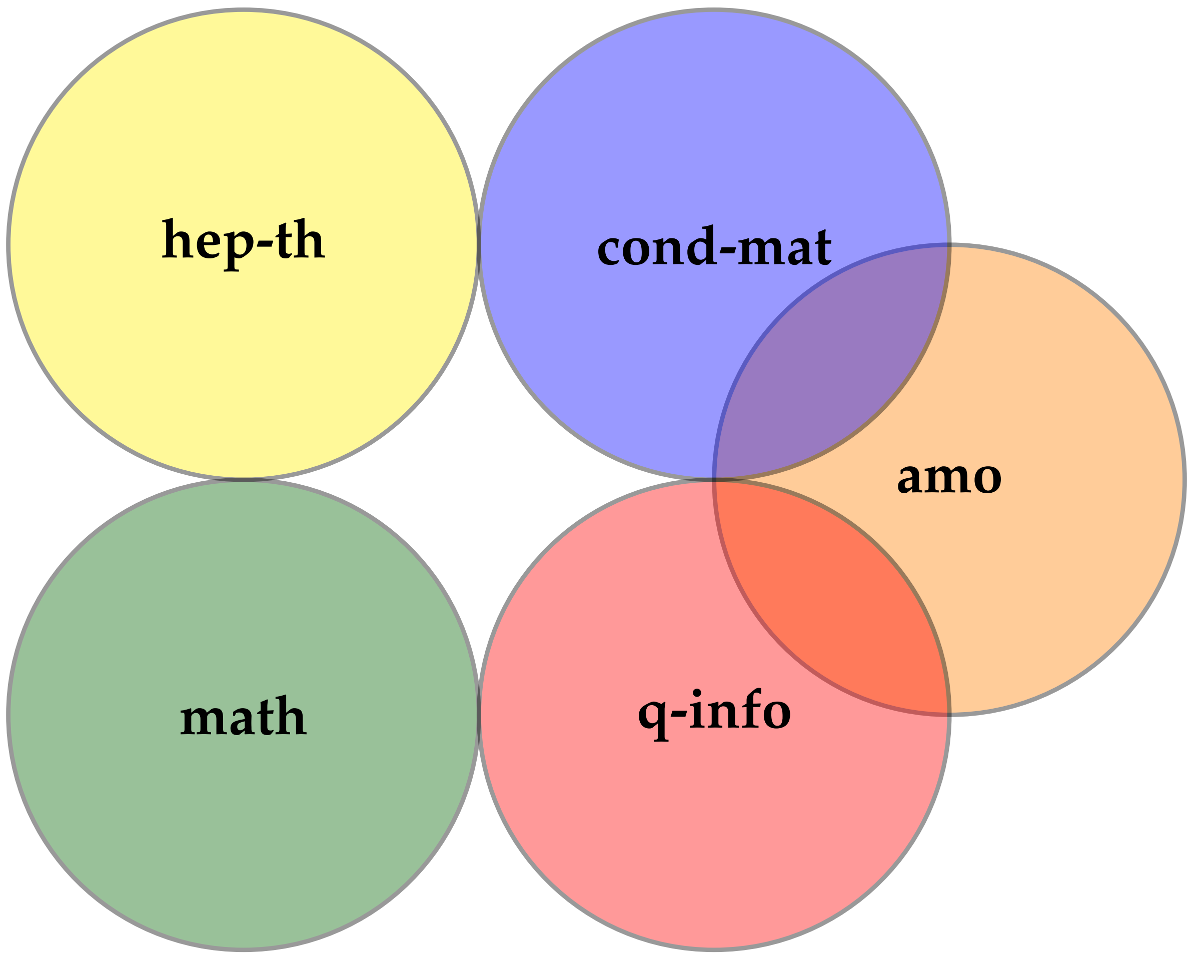
- Non-invertible symmetries: [Kaidi, Ohmori, Zheng][Choi, Cordova, Hsin, Lam, Shao][Bhardwaj, Bottini, SSN, Tiwari] $p=0,\cdots,d-1$ dimensional defects compose non-invertibly

$$D_p^a \otimes D_p^b = \bigoplus_c \mathcal{N}_{ab}^c D_p^c$$

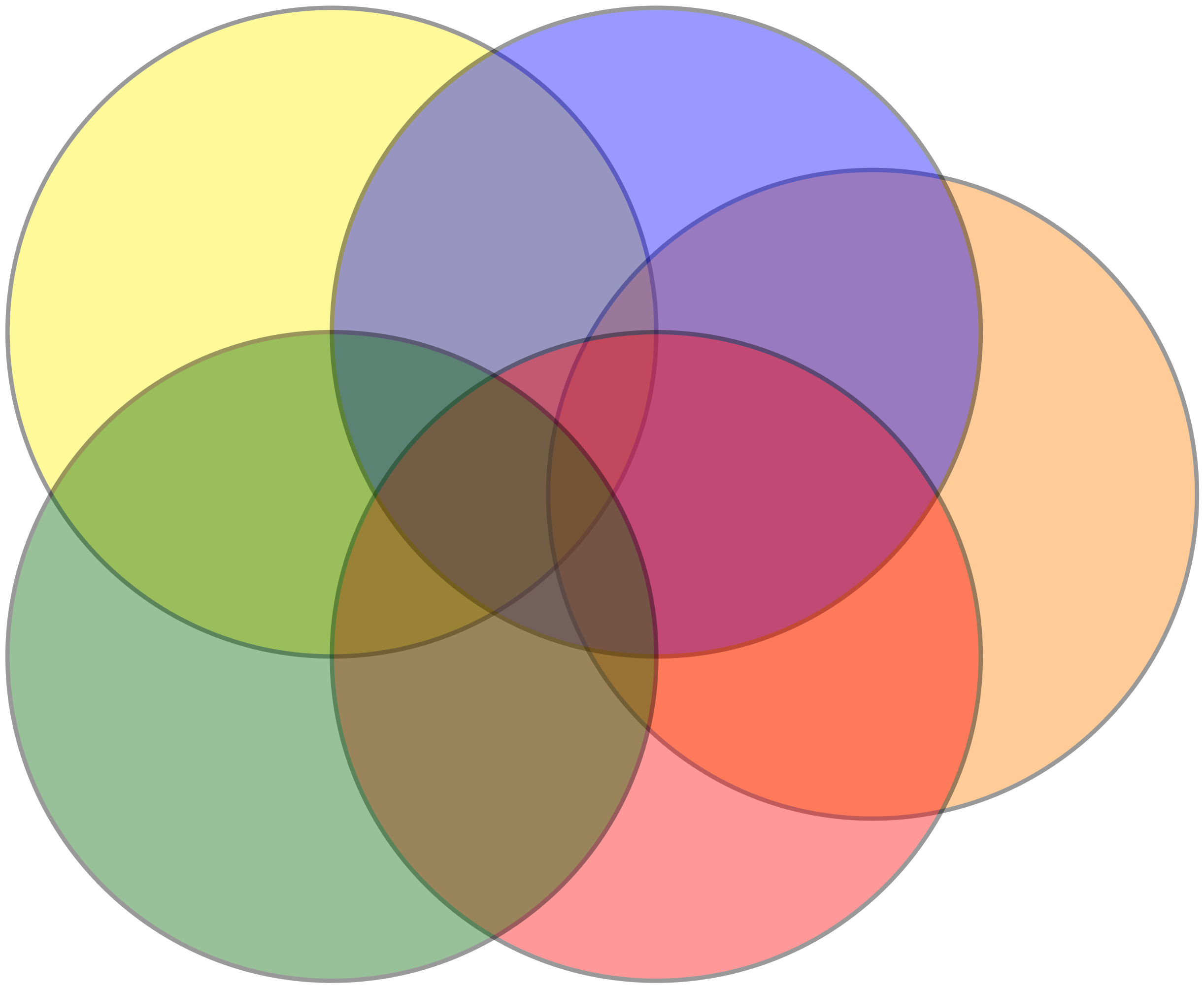
  These things form **fusion higher-categories**.

Many reviews: [SSN: 2305.18296] [Brennan,Hong: 2306.00912] [Bhardwaj et al: 2307.07547] [Luo, Wang^2: 2307.09215] [Shao: 2308.00747]
Overview talks for hep-th audiences:
Strings 2025 [McGreevy], Eurostrings 2023 [Shao], Strings 2024 [SSN].

Main development:
The synergy between hep-th, with cond-mat, math, and even quantum info and amo, has become invaluable.



”Synergy by Symmetries”



See also our KITP program from spring 2025
https://www.kitp.ucsb.edu/activities/gensym25

math Theory and classification of higher-fusion categories.

Progress: Classification of fusion 2-categories, i.e. finite symmetries of 2+1d theories. [Decoppet, et al 2024:] Any (all boson – where the transparent, i.e. braid trivially with everything else, fermion lines have +1 spin) fusion 2-category is related by finite gauging to

$$(G^{(0)}, \omega) \boxtimes \text{condensation completion of } \mathcal{B}$$

where $\mathcal{B}$ is a braided fusion category. I.e. the SymTFT of any fusion 2-category is

$$\text{Dijkgraaf Witten theory for } G \text{ with twist } \omega$$

cmt

Goal: **study phases and phase transitions, using microscopic lattice models**

- Studying beyond Landau phases and phase transitions
  $\Rightarrow$ lattice models with categorical symmetries [Aasen, Fendley, Mong][Inamura, Ohmori]

- Development of tensor network methods to numerically study phases [Verstrate group]

- Anomalies: can one realize a symmetry in a lattice model with tensor product Hilbert space, and on-site action? E.g. [Else, Nayak][Kapustin]

qi/amo

Fault-tolerant, error-correcting universal quantum codes, implementable in near future quantum devices/amo setups

- Overcome no-go theorem for fault tolerant transversal quantum codes (Eastin-Knill) using lattice models with generalized symmetries
- Real-wold setups: characterize open quantum systems (mixed states), symmetries in mixed states. [Ma, Turzillo, Meng Cheng, Ellison, etc]
- Magic state (states with actual quantum advantage) preparation from non-abelian topological order [Davydova et al][Huang, Chen]

If you still find yourself (secretly?) thinking:

"... categorical symmetries are exotic, physically useless, mathematical constructs for a niche subcategory of theorists in hep-th..."

hopefully this talk will debunk this misconception.

Plan:

1. Non-Invertible Symmetries in hep-th
2. SymTFT Diplomacy: Synergies with other fields: HEP, CM, AMO/QI
   - Pure and Mixed State Phases from SymTFTs **hep-th** **cmt** **qi/amo**
   - Categorical Anomalies from SymTFT **hep-th** **cmt**
   - Quantum Simulators and Computing using SymTFTs **qi/amo**

1. Non-Invertible Symmetries in hep-th

Duality Defects

Self-dualities give rise to non-invertible symmetries.

- 1+1d Kramers-Wannier (KW) duality symmetries:
  Critical Ising CFT has a $\mathbb{Z}_2$ spin flip symmetry η and

  $$N \otimes N = 1 \oplus \eta$$

  This originates from the KW duality $g \to 1/g$ of the transverse field Ising chain

  $$H = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x \,.$$

  At $g = 1$, this becomes a non-invertible symmetry.

- 3+1d theories: [Kaidi, Ohmori, Zheng][Choi, Cordova, Hsin, Lam, Shao] using self-duality symmetries D

  $$\text{QFT} \cong \text{QFT}/D \qquad \Rightarrow \qquad \mathcal{N}_3 \otimes \mathcal{N}_3^\dagger = \mathcal{C} = \text{condensation defect}$$

Exist also in any even spacetime dimensions, $d = 2n$, where gauging an $(n-1)$-form symmetry gives back an $(n-1)$-form symmetry.

- 6d (2,0) theories:

  [Lawrie, Yu, Zhang][Apruzzi, SSN, Warman][Bonetti, del Zotto, Minasian]

  Self-duality from **Green-Schwarz (GS) automorphisms**, i.e. automorphisms of lattice of BPS string charges

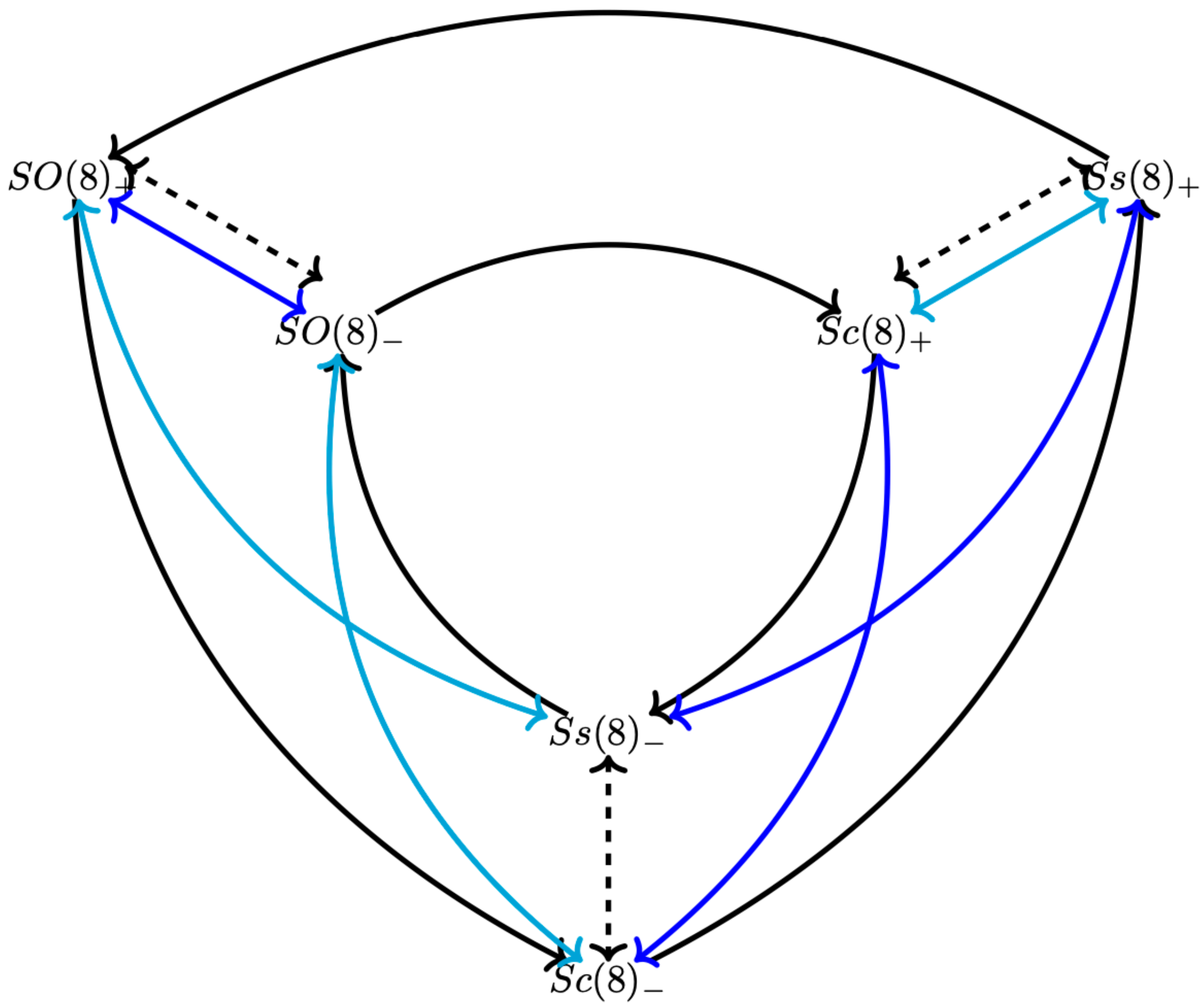
  $$\text{GS}: \qquad \Lambda_{\text{BPS}} \to \Lambda_{\text{BPS}}$$

  combined with

  - stacking a 2-form symmetry SPT $\exp(i\pi \int_{M_6} C_3 \cup C_3)$
  - gauging the 2-form symmetry, i.e. summing over background fields C_3

  result in non-invertible G-ality defects, where G= group formed by the GS-automorphisms.

For example $\mathfrak{so}(8)$ theory has a non-invertible, duality, and S_3-ality symmetry: The order 2 and 3 GS-automorphisms are shown in black, and blue arrows are gauging/stacking TQFTs.



Gauging Outer Automorphisms

Any outer automorphism can be gauged to give rise to a non-invertible symmetry [Bhardwaj, Bottini, SSN, Tiwari][Hsin, Kobayashi, Zhang]

Example: $O(2)$ gauge theory as $U(1)/\mathbb{Z}_2^{\text{cc}}$, gauging charge conjugation.

There is a 1-form symmetry generated by $D_\alpha := e^{i\alpha \int *F}$.

Charge conjugation maps $*F \to - * F$ and so

$$\mathbb{Z}_2^{\text{cc}} : \; D_\alpha \to D_{-\alpha}$$

The invariant combination is

$$D_\alpha^{\text{inv}} = D_\alpha \oplus D_{-\alpha}$$

which has non-invertible fusion* is

$$D_\alpha^{\text{inv}} \otimes D_\alpha^{\text{inv}} = 1 \oplus D_{2\alpha}^{\text{inv}}$$

* this depends on α and also should include condensation defects on the RHS.

ABJ Anomalies

Any ABJ anomaly – usually viewed as a non-symmetry – can be reinterpreted as a non-invertible symmetry. [Choi, Lam, Shao][Cordova, Ohmori]

Example: 4d QED with massless charge 1 Dirac fermion

$$\mathcal{L}_{\mathrm{QED}+\Psi} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}\left(\partial_\mu - iA_\mu\right)\gamma^\mu \Psi$$

the axial current $j_\mu = \frac{1}{2}\bar{\Psi}\gamma_5\gamma_\mu\Psi$ is not conserved due to the ABJ anomaly

$$d \star j = \frac{1}{8\pi^2} F \wedge F$$

Define an operator dressed by 3d Topological QFT that has opposite anomaly

$$\mathcal{N}_{\frac{1}{N}}(M_3) = \int [Da] \exp\left(\int_{M_3} \frac{2\pi i}{N} \star j + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right).$$

It is topological, but satisfies non-invertible fusion

$$\mathcal{N}_{\frac{1}{N}} \times \mathcal{N}^{\dagger}_{\frac{1}{N}} = \mathcal{C} = \text{condensation defect for 1-form symmetry}$$

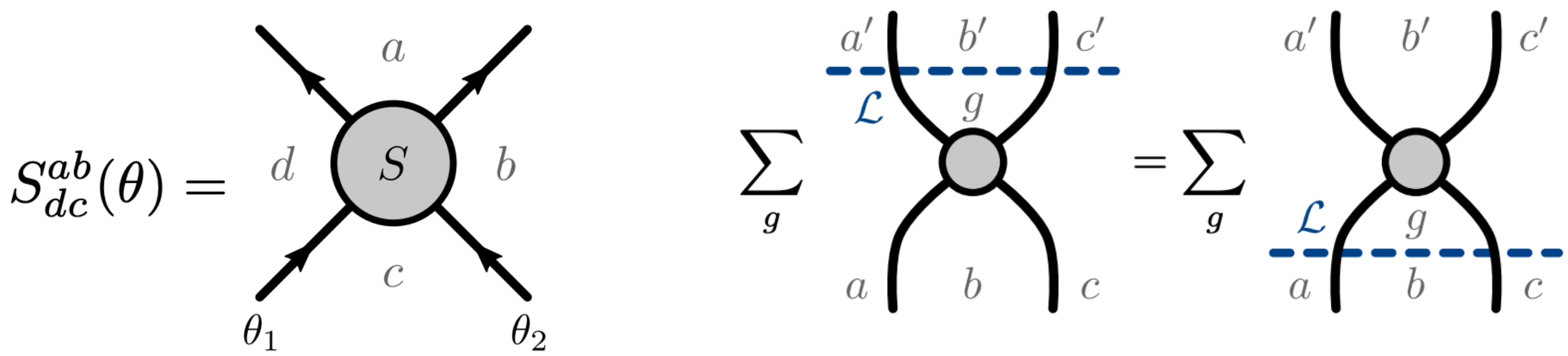
Physical Implications I: Modified Crossing Relations

Non-invertible symmetries lead to modified crossing relations for S-matrices!

Example: (1+1)d CFTs have non-invertible symmetries, generated by lines $\mathcal{L}$

Relevant, integrable deformations can preserve some of $\mathcal{L}$.

$\Rightarrow$ IR are gapped vacua. $\mathcal{L}$ constrains S-matrix of kinks through Ward ids:



[Copetti, Lucia Cordova, Komatsu] showed: crossing incompatible with symmetry/integrability/unitarity. Consistency implies modified crossing

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}}\, S^{bc}_{ad}(i\pi - \theta)\,, \qquad d_a = \langle \mathcal{L}_a \rangle \tag{1}$$

Modified crossing 2+1d see: [Mehta, Minwalla, Patel, Prakash, Sharma]

Status: Modified crossing direct implication of non-invertible symmetries. Even more compelling if extendable to higher dims.

Physical Implications II: Classifying Symmetric Phases

Non-invertible Symmetries lead to new IR phases, and new second order Phase Transitions!

Landau paradigm:

A 2nd order phase transition is a symmetry breaking transition for a group G.

- Gapped Phases: G spontaneously broken (SSB) to subgroup H. Phase has $|G/H|$ vacua, which are acted upon by the broken symmetry.
- Phase transitions:
  Unbroken symmetry group $H_i \subset G$ in each gapped phase, then there is a transition if $H_1 \subset H_2$
- Order Parameters:
  field transforming trivially in H_1, non-trivially in H_2.

Categorical Landau paradigm: [Bhardwaj, Bottini, Pajer, SSN][Bhardwaj, Pajer, SSN, Warman]:

$\mathcal{S}$ be a non-invertible symmetry, in particular very sharp for 1+1d fusion category symmetries.

- Gapped Phases:
  1-1 with gapped BCs of the SymTFT for $\mathcal{S}$, which are 1-1 with Lagrangian algebras $\mathcal{L}$ of the Drinfeld Center of $\mathcal{S}$.

- Phase Transitions:
  Two gapped phases are connected by a phase transition if the associated algebras $\mathcal{L}_i$ share a common, subalgebra $\mathcal{A} \subset \mathcal{L}_1 \cap \mathcal{L}_2$.

$$\boxed{\mathcal{S}\text{ Gapped Phase}} \quad \longleftarrow \quad \boxed{\text{CFT}} \quad \longrightarrow \quad \boxed{\mathcal{S}\text{ Gapped Phase'}}$$

- Order Parameters:
  Are topological defects that are shared between the symmetry Lagrangian algebra and the condensable algebra

$\Rightarrow$ **Categorical Landau Paradigm** [Bhardwaj, Bottini, Pajer, SSN]

Status: systematic approach to classification, and new phases

2. Synergies with other Fields

The Synergy through SymTFT Sandwiches

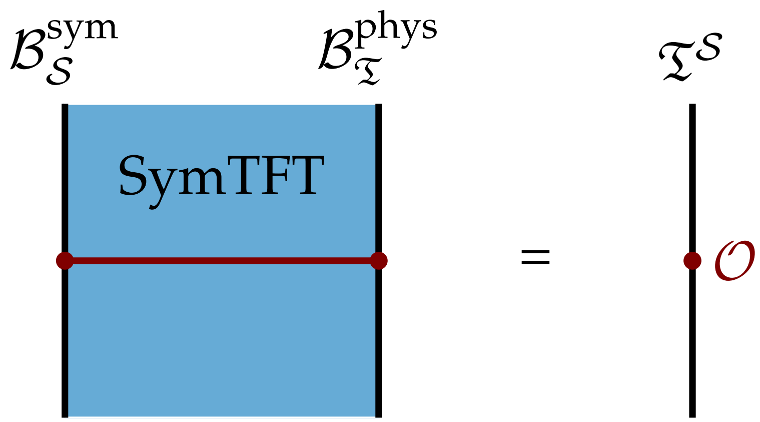
- Pure and Mixed State Phases from SymTFTs **hep-th** **cmt** **qi/amo**
- Categorical Anomalies from SymTFT **hep-th** **cmt**
- Quantum Simulators and Computing using SymTFTs **qi/amo**

All three cross-connections are done through the so-called SymTFT sandwich construction. Lets recap this.

Symmetry TFT (SymTFT) Sandwich

[Ji, Wen][Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-E, Hosseini, SSN][Freed, Moore, Teleman]

Let $\mathfrak{T}$ be a QFT with finite symmetry $\mathcal{S}$ in d dimensions. The SymTFT is a $d+1$ dimensional TQFT obtained by gauging $\mathcal{S}$ in $(d+1)$ dims:



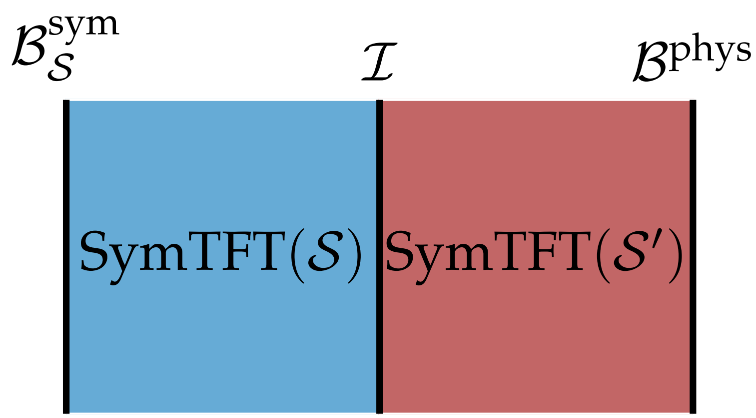
For G a group, this is the Dijkgraaf Witten theory for G.

- Topological defects of the SymTFT: "Drinfeld center" of the symmetry category. In $d+1=3$: anyons, i.e. topological lines
- $\mathcal{B}^{\text{sym}}_{\mathcal{S}}$ = Symmetry boundary, gapped, realizes symmetry. Dirichlet gives $\mathcal{S}$.
- $\mathcal{B}^{\text{phys}}_{\mathfrak{T}}$ = Physical boundary, encodes dynamics
- Generalized charges: local operators end on both BCs

SymTFT has natural origin in holography, e.g. topological couplings in supergravity, and in fact in CM is referred to as "topological holography"

Interfaces and SymTFT Club Sandwiches

We can study topological interfaces from the SymTFT of $\mathcal{S}$ to other TQFTs (or topological orders), by condensing topological defects:

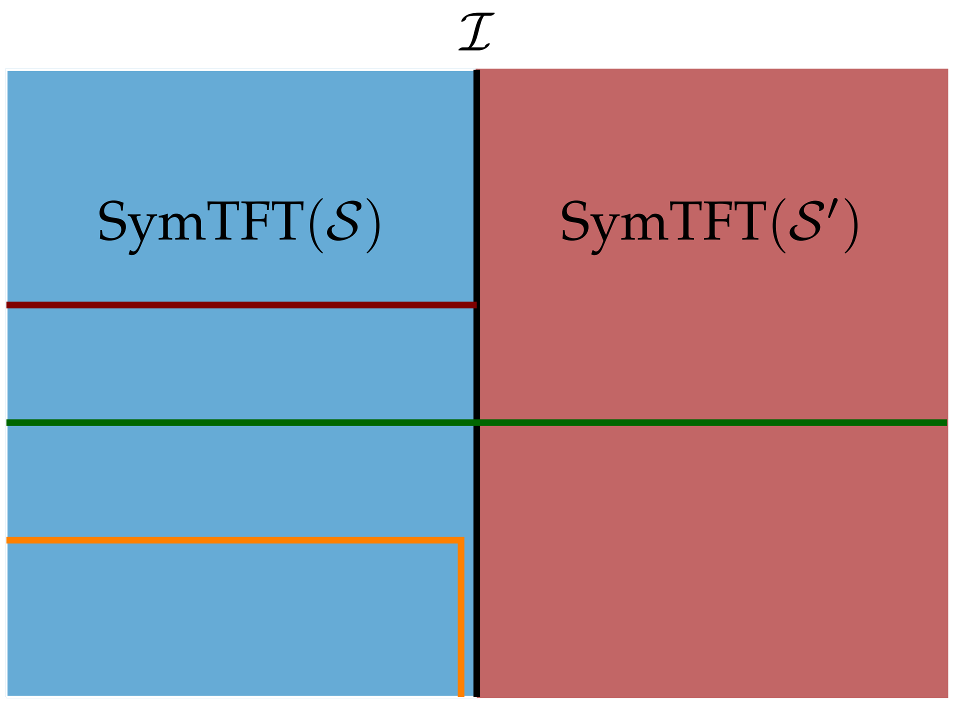


Studying this configurations, allows us to

- categorical Landau phases, phase transitions for pure and mixed states
- anomalies of categorical symmetries
- derive new magic state preparations.

Club Quiche

For concreteness consider 1+1d, with 0-form non-invertible, fusion category symmetry $\mathcal{S}$. The interface (club quiche) is characterized as follows:



- Anyons that end (condense) on $\mathcal{I}$ are given by a condensable algebra $\mathcal{A} = \oplus_a n_a a$
- Anyons that are mutually non-local with those condensed confine
- Uncondensed anyons that braid trivially with $\mathcal{A}$ are de-confined $\Rightarrow$ give rise to map of anyons between the TOs

SymTFT($\mathcal{S}'$)= Trivial: $\mathcal{I}$= gapped BC, $\mathcal{A}$= Lagrangian algebra

Example: SymTFT for G

The anyons in the SymTFT for a (non-abelian) group G, are labeled by

$$([g], R)$$

- conjugacy classes $[g]$
- representations of the centralizer group H_g of $g \in [g]$.

Condensable algebras for G are classified [Davidov, Simmons][Gai, SSN, Warman to appear for anomalous G]: they are given by (H, N, γ, ϵ), where $N \triangleleft H < G$.

Examples:

$$\mathcal{L}_G = \bigoplus_{\mathbf{R} \in \text{irrep of } G} ([\text{id}], \mathbf{R}). \qquad \text{As } \mathcal{B}^{\text{sym}}\text{: } G$$

$$\mathcal{L}_{\text{Rep}(G)} = \bigoplus_{[g]} ([g], 1). \qquad \text{As } \mathcal{B}^{\text{sym}}\text{: non-invertible symmetry } \text{Rep}(G).$$

Note $\text{Rep}(G)$ is non-invertible for all non-abelian groups.

Defusing SymTFTs

$G = \mathbb{Z}_N$ 0-form symmetry in 1+1d. The SymTFT is the DW theory for $\mathbb{Z}_N$ with action

$$N \int b_1 \wedge da_1$$

Topological defects are

$$\mathbf{e}(\gamma) = e^{2\pi i/N \int_\gamma b_1} \qquad \mathbf{m}(\gamma) = e^{2\pi i/N \int \gamma a_1}$$

with $\mathbf{e}^N = \mathbf{m}^N = 1$. They braid non-trivially

$$\mathbf{e}(\gamma_1)\mathbf{m}(\gamma_2) = e^{2\pi i/N \mathrm{link}(\gamma_1,\gamma_2)}\mathbf{me}$$

Gapped Boundary Conditions: vanilla Dirichlet and Neumann

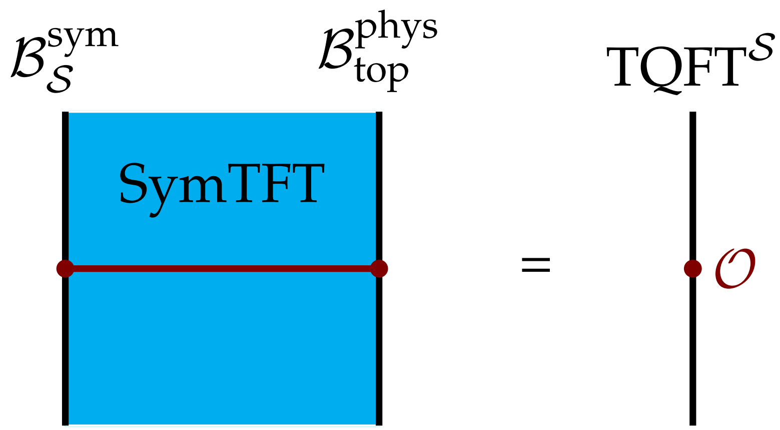
$$\mathcal{L}_e = \bigoplus_i \mathbf{e}^i\,, \qquad \mathcal{L}_m \bigoplus_i \mathbf{m}^i$$

But there are more condensable algebras, labeled by subgroups and cocycles.

Application I: Pure State Phases from SymTFTs

hep-th cmt

Pure state gapped phases: [Bhardwaj, Bottini, Pajer, SSN] Physical boundary is gapped, i.e. given by a Lagrangian algebra.



- **Classifying gapped phases $\Longleftrightarrow$ classifying gapped BCs of the SymTFT.**
  - SPT (symmetry protected topological phase): no non-trivial local OP $\mathcal{O}$
  - SSB (spontaneous symmetry breaking): $\mathcal{O}_i$, $i = 1, \cdots, n$ local OPs
- # of vacua = # of defects that end on both boundaries (order parameters)
- Symmetry action from braiding in the SymTFT

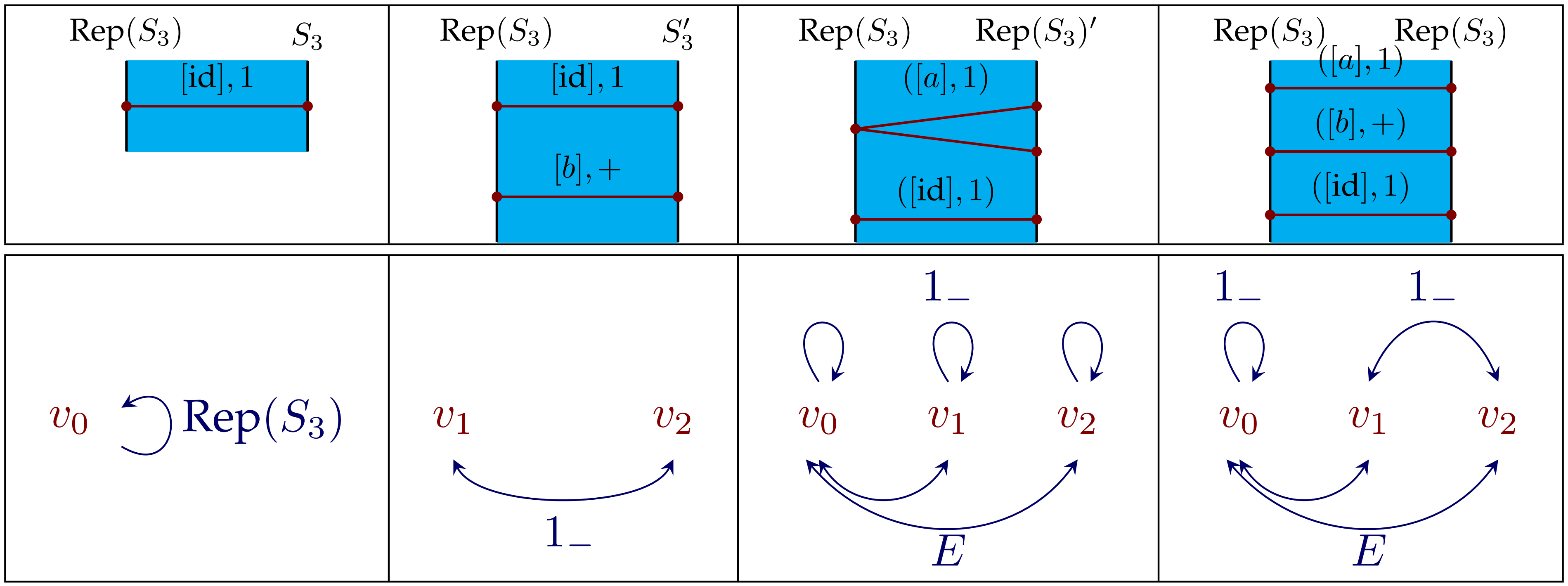
Example: Rep(S_3)-symmetric Gapped Phases

Rep(S_3) ($1, 1_-, E$ irreps) $E \otimes E = 1 \oplus 1_- \oplus E$. SymTFT has 4 gapped BCs:

$$\mathcal{L}_{S_3} = ([\mathrm{id}], 1) \oplus ([\mathrm{id}], 1_-) \oplus 2([\mathrm{id}], E) \qquad \mathcal{L}_{\mathrm{Rep}(S_3)'} = ([\mathrm{id}], 1) \oplus ([\mathrm{id}], 1_-) \oplus 2([a], 1)$$

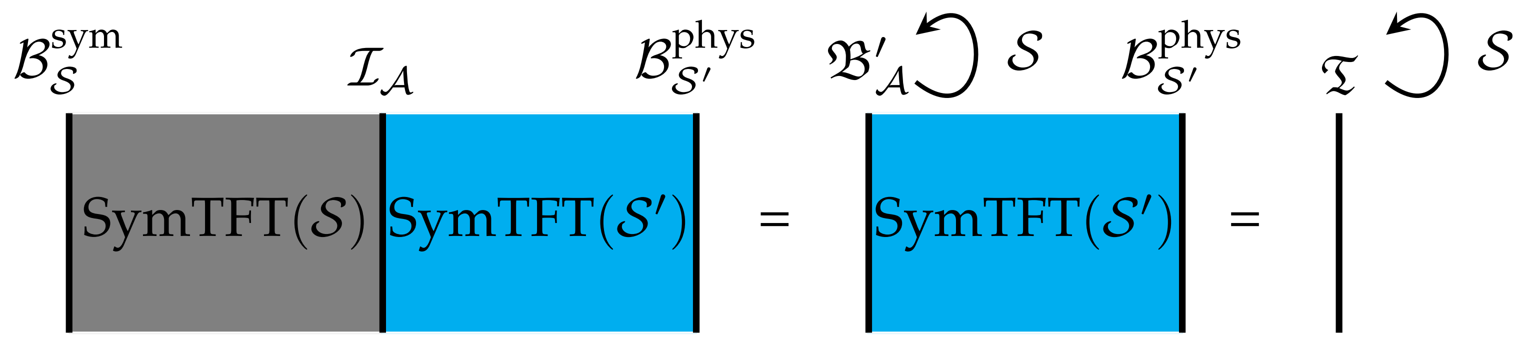
$$\mathcal{L}_{S_3'} = ([\mathrm{id}], 1) \oplus ([\mathrm{id}], E) \oplus ([b], 1) \qquad \mathcal{L}_{\mathrm{Rep}(S_3)} = ([\mathrm{id}], 1) \oplus ([a], 1) \oplus ([b], 1)$$

Trivial	$\mathbb{Z}_2$ SSB	Rep(S_3)/$\mathbb{Z}_2$ SSB	Rep(S_3) SSB



Gapless Phases: Club Sandwich

[Chatterjee, Wen][Bhardwaj, Bottini, Pajer, SSN][Bhardwaj, Pajer, SSN, Warman] Use an interface to construct a phase transition by inputting a 2nd order phase transition for the smaller symmetry $\mathcal{S}'$:



The Interface provides a map between the two TQFTs, and thus also between the symmetry actions. It allows mapping an $\mathcal{S}'$-symmetric phase transition, e.g. for $\mathbb{Z}_2$ the Ising transition, into an $\mathcal{S}$-symmetric one.

This constructs an $\mathcal{S}$-symmetric gapless phase, in which only the symmetry $\mathcal{S}'$ acts faithfully on the gapless degrees of freedom.

Examples:

Rep$G$ symmetric CFTs, Haagerup-symmetric gapless phase [Bottini, SSN]

Physical Application II: Mixed State Phases from SymTFT

hep-th **cmt** **qi/amo**

Many motivations, from holography, to real-world systems to quantum computation setups, that are not isolated pure states, but usually system in mixed states. What can we say about the possible symmetric phases?

Fix a Hilbert space $(\mathcal{H}, \langle , \rangle)$.
Density matrix is a hermitian operator $\rho = \rho^\dagger$

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n| \,, \quad 0 \leq p_n \leq 1 \,, \ \sum_n p_n = 1 \,.$$

Pure: $\rho^2 = \rho$, i.e. $\rho = |\psi\rangle\langle\psi|$.

Mixed States and Symmetries

Consider finite group symmetries G at first: U_g unitary representation on $\mathcal{H}$

- Strong Symmetry: $U_g \rho = e^{i\theta_g} \rho$ for all $g \in G$
- Weak Symmetry: $U_g \rho U_g^\dagger = \rho$ for all $g \in G$

For non-invertible symmetries $s \in \mathcal{S}$:
We require that for all $s \in \mathcal{S}$ there is $s^\dagger \in \mathcal{S}$ such that

$$s \otimes s^\dagger = 1 \oplus \bigoplus_i m_i a_i \equiv 1 \oplus a$$

Let D_s be a representation on $\mathcal{H}$, with $D_s^\dagger = D_{s^\dagger}$:

- Strong Symmetry: $D_s \rho = c_s \rho$ for all $s \in \mathcal{S}$
- Weak Symmetry: $D_s \rho D_s^\dagger = |c_s|^2 \rho$ for all $s \in \mathcal{S}$

Purification

Define a pure state on $\mathcal{H}_L \otimes \mathcal{H}_R$, where $\mathcal{H}_L \cong \mathcal{H}_R \cong \mathcal{H}$:

$$|\rho\rangle\rangle \equiv \frac{1}{\mathrm{Tr}(\rho^2)} \sum_n p_n |\psi_n\rangle |\overline{\psi}_n\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R \,.$$

The strong symmetry action becomes:

$$\begin{aligned} D_s \rho = c_s \rho \quad &\rightarrow \quad (D_{s,L} \otimes \mathbf{1}_R)|\rho\rangle\rangle = c_s |\rho\rangle\rangle \\ \rho D_s^\dagger = \overline{c}_s \rho \quad &\rightarrow \quad (\mathbf{1}_L \otimes \overline{D}_{s,R})|\rho\rangle\rangle = \overline{c}_s |\rho\rangle\rangle \,. \end{aligned}$$

The weak symmetry action becomes:

$$D_s \rho D_s^\dagger = |c_s|^2 \rho \quad \rightarrow \quad D_{s,L} \otimes \overline{D}_{s,R} |\rho\rangle\rangle = |c_s|^2 |\rho\rangle\rangle$$

SymTFT for Mixed States

Consider the SymTFT for the Choi state: The symmetry is $\mathcal{S}_L \boxtimes \mathcal{S}_R$, satisfying the SWAP* invariance:

$$\mathcal{S}_R = \overline{\mathcal{S}_L}\,.$$

So the SymTFT we consider is

$$\mathrm{SymTFT}(\mathcal{S} \boxtimes \overline{\mathcal{S}})$$

Questions:

Which BCs give rise to strong symmetries?
Which BCs correspond to gapped phases, that are mixed phases, i.e. after tracing out one Hilbert space, give rise to a consistent density matrix?
Similar for weak symmetries.

SymTFT solution for categorical symmetries [SSN, Tiwari, Warman, Zhang –quant-ph]
SymTFT solution for groups: [Luo, Y.-N. Wang, Bi][Qi, Sohal, Chen, Stephen, Prem – cond-mat]

Mixed State (Lagrangian) Algebras

Not all Lagrangians in SymTFT($\mathcal{S} \boxtimes \overline{\mathcal{S}}$) are admissible. Let

$$\mathcal{L} = \bigoplus_{a_L, b_R} n_{a_L, b_R} a_L \overline{b}_R \,,$$

What is the strong symmetry BCs?

$$\mathcal{L}_{\mathcal{S}}^{\text{strong}} = (\mathcal{L}^{\mathcal{S}})_L \otimes (\mathcal{L}^{\overline{\mathcal{S}}})_R$$

Conditions on gapped boundary conditions $\mathcal{L}$:

1. T**=SWAP* Invariance:**

$$\boxed{n_{a_L, b_R} = n_{\overline{b}_L, \overline{a}_R}}$$

2. **Positivity:**

$$\boxed{n_{a_L, b_R} \leq n_{a_L, \overline{a}_R} n_{\overline{b}_L, b_R}}$$

We call such algebras mixed state (Lagrangian) algebras.

Mixed Phases

Depending on the properties of $\mathcal{L}$ we get different type of gapped phases:

- Pure state phases: from factorized Lagrangians
- Non-factorized $\mathcal{L}$: Mixed state SSB or SPT.
  A particularly interesting type of algebra is condensing diagonal charges:

  $$n_{a_L,b_R} \neq 0\,, \qquad n_{a_L,1} = n_{1,b_R} = 0\,,$$

  i.e. $\mathcal{L}$ contains only the diagonal gauge charge, without the off-diagonal terms, then the resulting gapped phase will have a weak symmetry.

  $\Rightarrow$ **Strong-Weak SSB (SWSSB)**

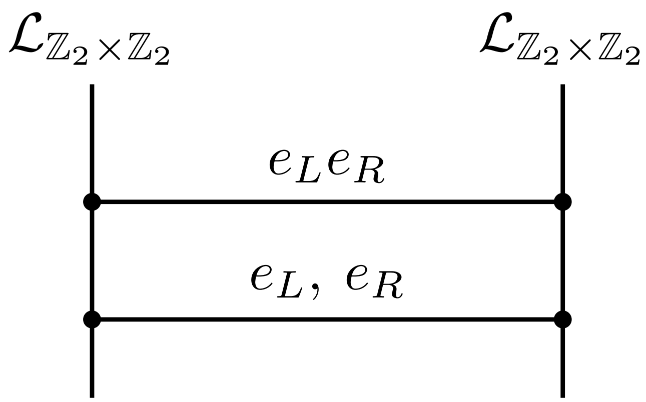
Examples: Strong Symmetric Phases

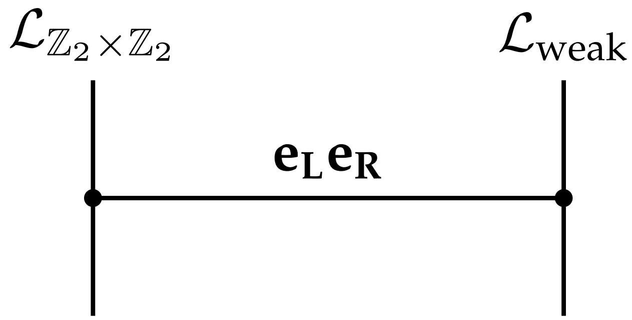
Assume that there is no explicit breaking of the strong symmetry, i.e. $\mathrm{Tr}(\rho\,\mathcal{O}_i) = 0 = \mathrm{Tr}(\rho\,\mathcal{O}_i\rho\mathcal{O}_i^\dagger)$.

Warmup: Phases with Strong $\mathbb{Z}_2$ Symmetry The mixed state Lagrangians are

$$
\begin{aligned}
\mathcal{L}_{\mathrm{sym}} = \mathcal{L}_{\mathbb{Z}_2\times\mathbb{Z}_2} &= (1\oplus e_L)\otimes(1\oplus e_R) \\
\mathcal{L}_{\mathrm{Rep}(\mathbb{Z}_2\times\mathbb{Z}_2)} &= (1\oplus m_L)\otimes(1\oplus m_R)\,. \\
\mathcal{L}_{\mathrm{weak}} &= 1\oplus e_Le_R\oplus m_Lm_R\oplus e_Lm_Le_Rm_R
\end{aligned}
$$

The first two factorize, the last is a proper mixed phase:





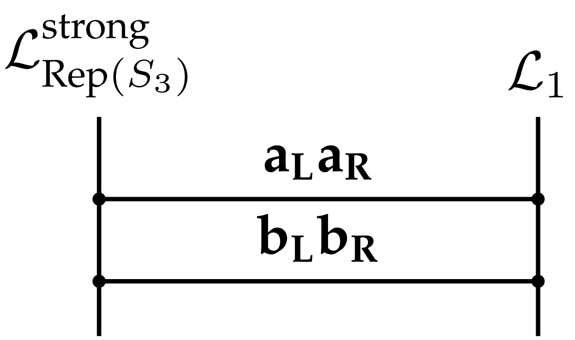
$\implies \mathbb{Z}_2$ SSB $\qquad\qquad \implies \mathbb{Z}_2$ SWSSB

In the SWSSB the combination m_Lm_R braids trivially with the OP and thus remains a, now weak, symmetry.

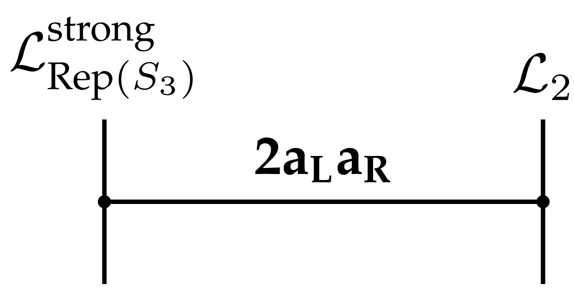
Non-Invertible Symmetric Mixed Phases

The main advantage of the SymTFT approach is to tackle non-invertible symmetries, and higher dims.

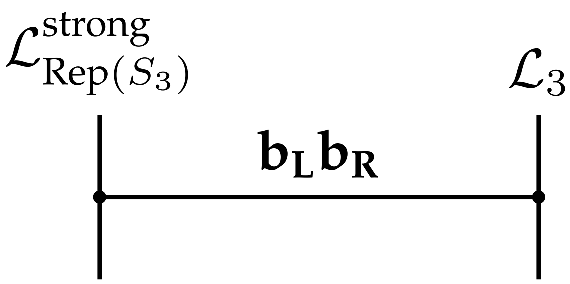
Example: Strong Rep(S_3) **Symmetry.**



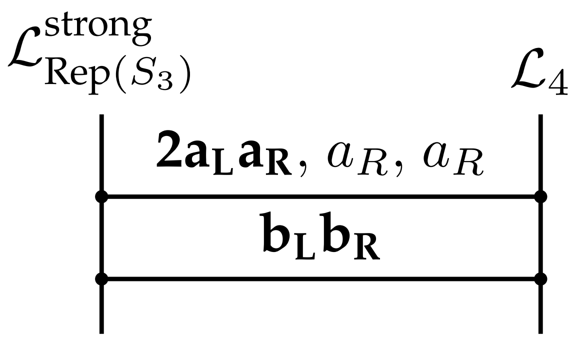
$\Longrightarrow$ Rep(S_3) SWSSB



$\Longrightarrow$ (Rep$(S_3)/\mathbb{Z}_2$) SWSSB



$\Longrightarrow$ $\mathbb{Z}_2$ SWSSB



$\Longrightarrow$ (Rep$(S_3)/\mathbb{Z}_2$) SSB, $\mathbb{Z}_2$ SWSSB

Corroborated with lattice models.

Weak and Strong Symmetric Phases

So far we started with a strong symmetry. How to get a weak symmetry $\mathcal{W}$ as a starting point?

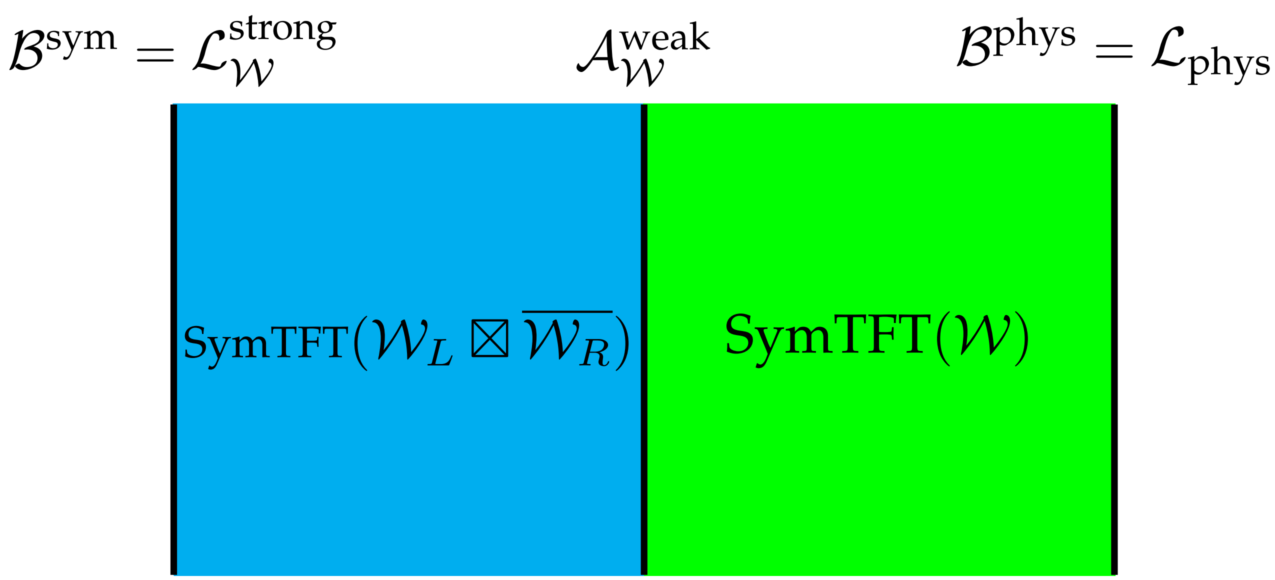
Start with $\mathcal{W}_L \boxtimes \mathcal{W}_R$ symmetry, and condense a diagonal $\mathcal{W}$ symmetry:

$$\text{SymTFT}(\mathcal{W}_L \boxtimes \mathcal{W}_R) \to \text{SymTFT}(\mathcal{W})\,,$$

Input the strong symmetry $\mathcal{L}_{\mathcal{W}}^{\text{strong}} = \mathcal{L}_{\mathcal{W},L} \otimes \mathcal{L}_{\mathcal{W},R}$ but condense

$$\mathcal{A}_{\mathcal{W}}^{\text{weak}} = \bigoplus_w n_w w_L \overline{w_R}\,.$$

Club-sandwich setup:



Not all Symmetries can be Weak

Consider the general fusion

$$W_a \otimes W_b = \oplus_c N_{ab}^c W_c$$

these are not always bompatible with the weak symmetry:

$$D_w \rho D_w^\dagger = |c_w|^2 \rho$$

Consistency of this with the fusion means

$$|c_a|^2 |c_b|^2 \rho = W_a (W_b \rho W_b^\dagger) W_a^\dagger = \left(\bigoplus_c n_{ab}^c W_c \right) \rho \left(\bigoplus_d n_{ab}^d W_d^\dagger \right)$$

If all W_a are weak:

$$\text{RHS} = \bigoplus_f m_f W_f \rho W_f^\dagger = \sum_f |c_f|^2 \rho$$

i.e. the fusion is invertible and no sum over f.

Example of Weak Non-Invertible Phases

Consider Ising with the initial doubled TO

$$\mathrm{SymTFT}(\mathsf{Ising}_L \boxtimes \mathsf{Ising}_R) = \mathrm{SymTFT}(\mathsf{Ising}_L) \boxtimes \mathrm{SymTFT}(\mathsf{Ising}_R)$$

Each Ising center is a double, and we have simple lines 1, N, η for L and R:

$$N^2 = 1 \oplus \eta\,, \qquad \eta^2 = 1$$

This is also the Tambara-Yamagami category for $\mathbb{Z}_2$. We make the duality N weak and keep the $\mathbb{Z}_2$ strong: The resulting SymTFT is that of

$\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2):$ diagonal bi-character and trivial Frobenius-Schur indicator, i.e. $\mathrm{Rep}(H_8)$

Full classification of phases possible via club sandwich setup.

Future Directions

- Phases in higher dimensions – IR phases of gauge theories.
- Key input: SymTFT. For 2+1d: fully classified. [Decopet, et al] results in full classification of gapped phases (assuming input braided fusion 1-categories) [Rui Wen][Bhardwaj, et al]
  Example of new types of phases:
  Spontaneously Non-uniform Entangled Phase (SNEP):
  multiple vacua, with different TOs, e.g. from a $2\text{Rep}(\mathbb{G}^{(2)})$ symmetry

$$\begin{array}{ccc} \mathrm{DW}(\mathbb{Z}_2) & \boxplus & \mathrm{Triv} \\ (\mathbb{Z}_2^{(1)}\text{-SSB}) & & (\mathbb{Z}_2^{(1)}\text{-Triv}) \end{array}$$

- Lattice models: again for 2+1d possible thanks to classification result [Inamura, Huang, Tiwari, SSN]
- Applications to mixed phases in holographic/string theory settings, see also [Heckman, Hubner, Murdia]

Physical Application III: Categorical Anomaly Matching

’t Hooft anomalies for global symmetries provide stringent constraints on RG-flows. Given a UV and an IR theory, connected by RG flow

$$\mathcal{T}_{\text{UV}} \to \mathcal{T}_{\text{IR}}$$

the question is how symmetries of these theories constrain the possible IR outcomes.

Coarse criterion: an anomalous symmetry cannot be trivialized in the IR. ”There is no fiber functor”.

Many interesting effects already for higher-form symmetries: symmetries in the UV and IR might not be the same!

- Part of the symmetry might trivialize
- There can be emergent symmetries
- Fractionalization/Transmutation: ”Symmetries don’t stick to their symmetry degree”, e.g. 0-form symmetries can act on extended operators. [Barkeshli et al][Seifnashri, Seiberg]

Anomalies for Non-Invertible Symmetries

This problem is even more severe for non-invertible symmetries: no quantification of anomalies until now.

We have a way of specifying whether a categorical symmetry is anomalous or not (”fiber functor”), e.g. [Thorngren, Wang]

$$F: \qquad \mathcal{C} \to \mathrm{Vec}$$

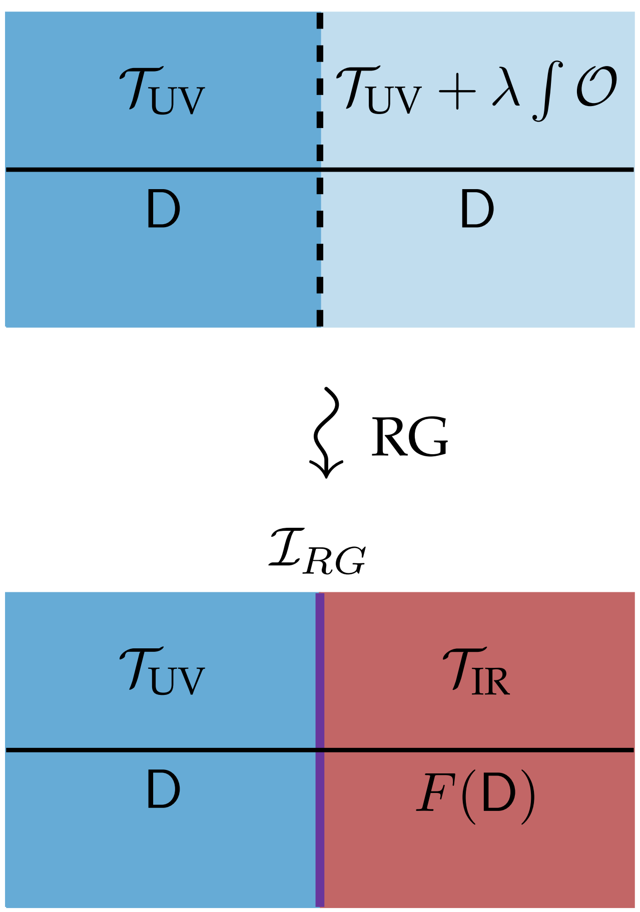
but not how to do anomaly matching.

$\Rightarrow$ New Categorical Anomaly Matching framework [Antinucci, Copetti, Gai, SSN]

... Using SymTFT.

Motivation

- Global symmetries are topological defects: should have some robustness along RG-flows.
- RG-interfaces [Gaiotto]



Adding a relevant deformation along half-space in the UV: RG-interface

The map on the symmetry generators D that form groups is

$$\varphi : \quad G_{\mathrm{UV}} \to G_{\mathrm{IR}}\,, \qquad \varphi^* \omega_{\mathrm{IR}} = \omega_{\mathrm{UV}}$$

What replaces this for categories? Tensor functor

$$F : \mathcal{C}_{\mathrm{UV}} \to \mathcal{C}_{\mathrm{IR}}$$

Satisfies consistency conditions with the fusion and associators:
$F(\mathsf{D}_1) \otimes F(\mathsf{D}_2) \cong F(\mathsf{D}_1 \otimes \mathsf{D}_2)$

How to quantify anomalies of categorical symmetries?

[Antinucci, Copetti, Gai, SSN]

Consider short exact sequences of tensor functors:

$$\mathcal{N} \xrightarrow{I} \mathcal{C} \xrightarrow{P} \mathcal{S}$$

here I is injective, and P surjective.

$$\mathrm{im}(I) = \ker(P)$$

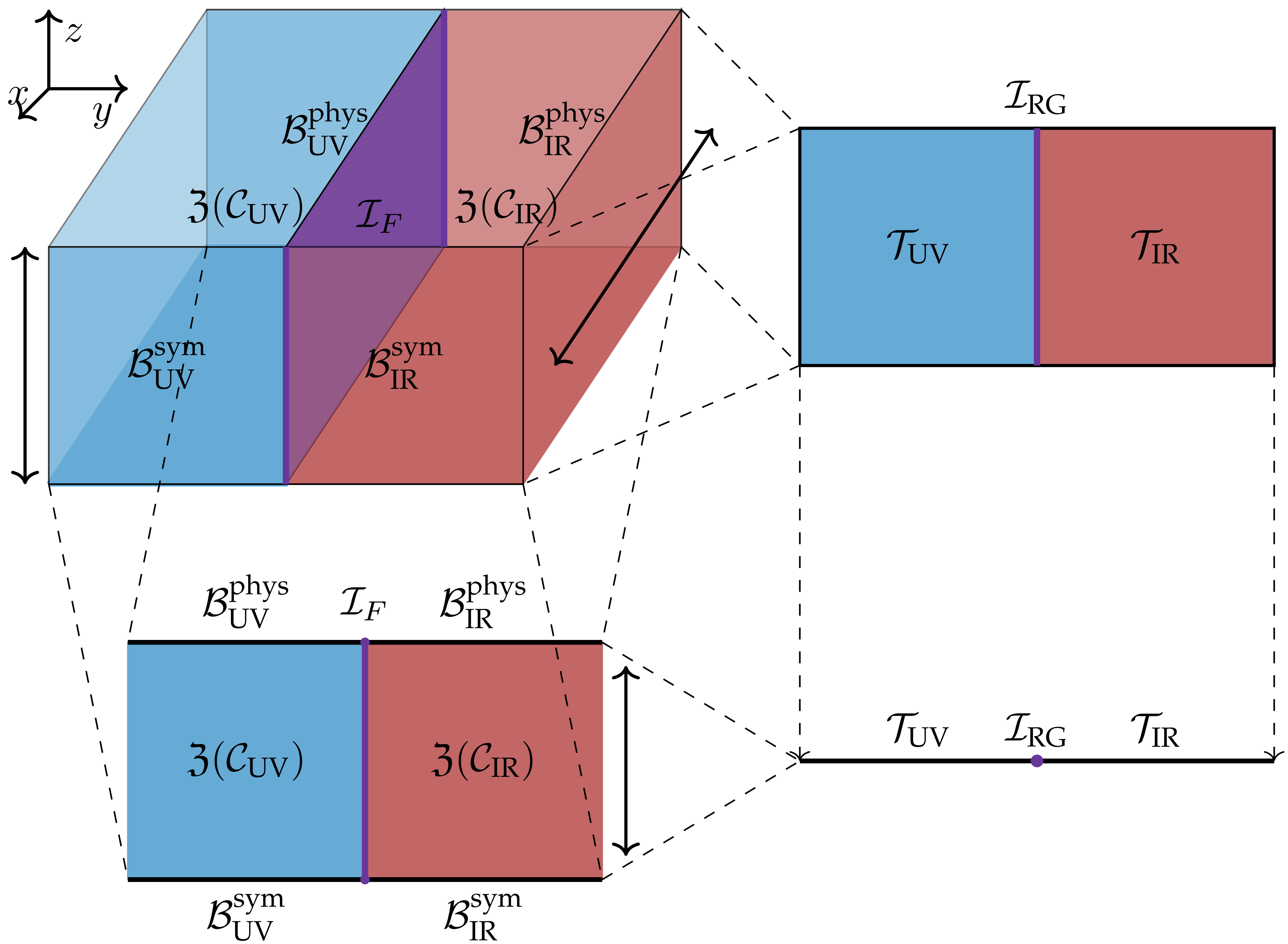
Exactness implies that $\mathcal{N}$ is anomaly free – it has a map to the trivial symmetry, Vec.

$\mathcal{N}$ is a normal subcategory, i.e. the kernel of surjective functor. Similar to normal subgroups.

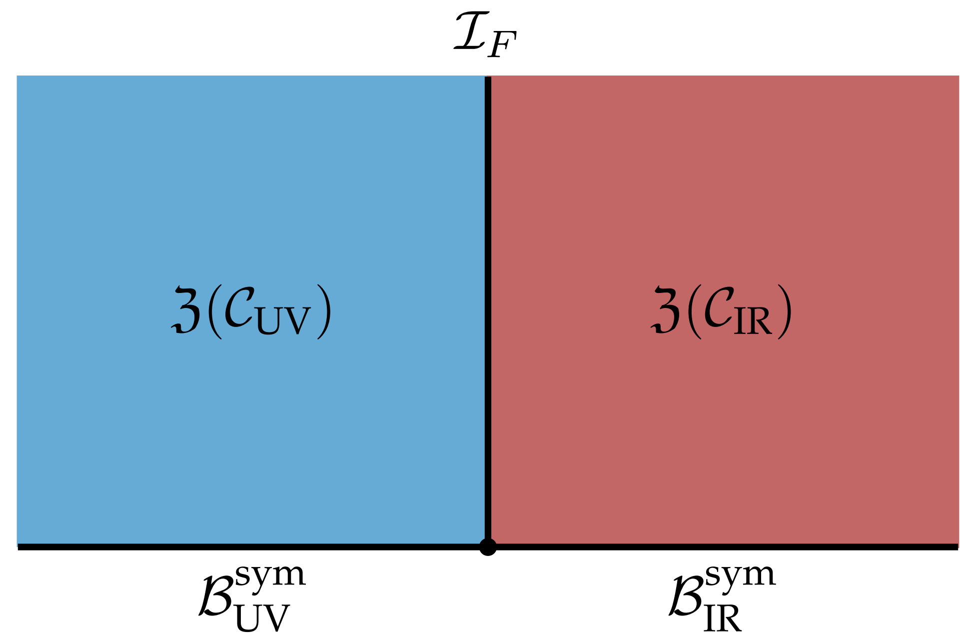
We define a category that does not have a non-trivial normal subcategory to be an **anomalous simple category (ASCy)**. Any category $\mathcal{S}$ that fits into an exact sequence with $\mathcal{N}$ maximal is an ASCy, and captures aspects of the anomaly of $\mathcal{C}$.

Limitations: only developed for fusion categories, and very tedious. Recast in terms of SymTFT, and extend to higher dim.

RG-Interfaces in the SymTFT



SymTFT Realization of Tensor Functors: RG-Quiche



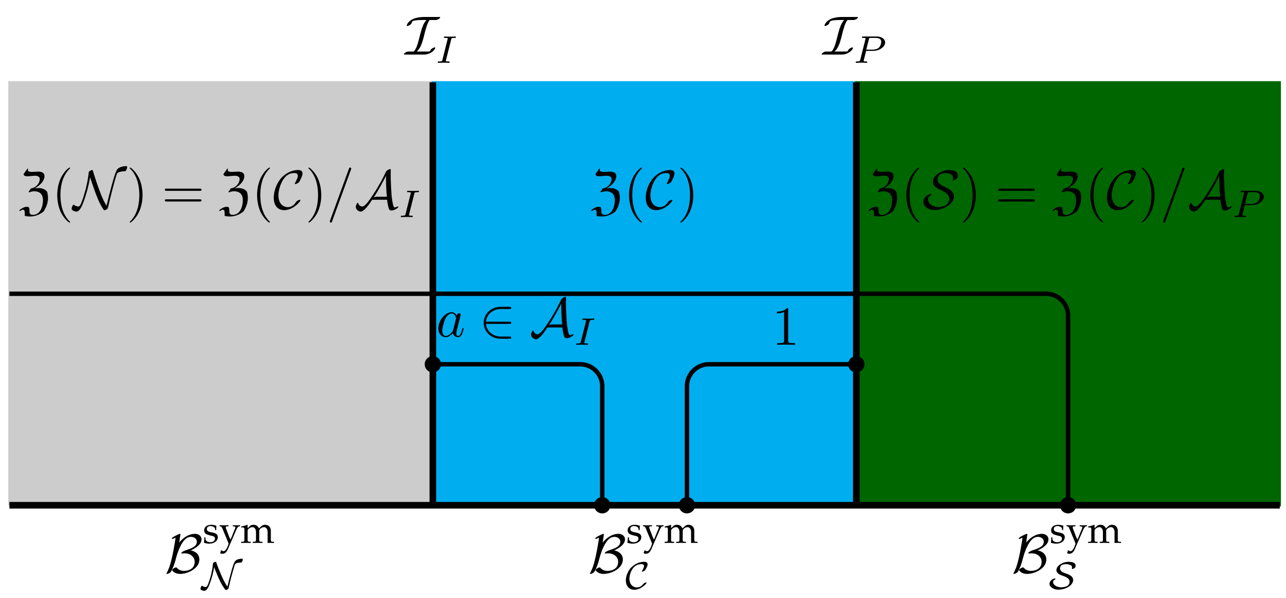
SymTFT Matching Equation (ME):
The interface corresponds to a **tensor functor** iff

$$\text{(ME)}: \quad \mathcal{B}^{\text{sym}}_{\text{UV}} \times \mathcal{I}_F = \mathcal{B}^{\text{sym}}_{\text{IR}} \,.$$

Injective and Surjective Functors

- $F : \mathcal{C}_{\text{UV}} \to \mathcal{C}_{\text{IR}}$ injective: $\mathcal{C}_{\text{UV}}$ acts faithfullyin the IR
  $\mathcal{I}_F$ is given by condensing an **electric algebra** $\mathcal{A}_F \subset \mathcal{L}_{\mathbf{IR}}$
- $F : \mathcal{C}_{\text{UV}} \to \mathcal{C}_{\text{IR}}$ surjective: no emergent symmetries
  $\mathcal{A}_F$ is given by an **magnetic algebra** $\mathcal{A}_F \cap \mathcal{L}_{\mathbf{UV}} = 1$

The short exact sequence $\mathcal{N} \to \mathcal{C} \to \mathcal{S}$ becomes a triple club sandwich, and $\mathcal{N}$ is a normal subcategory:



SymTFT Realization of Ascies

Anomalous Simple Categories (ASCies) $\mathcal{S}_i$ are categories without any non-trivial normal subcategory.

For given $\mathcal{C}$, there can be several choices of maps P_i with maximal kernel $\mathcal{N}_{\text{max},i}$ and $\mathcal{S}_i$:

$$\mathcal{N}_{\text{max},i} \xrightarrow{I_i} \mathcal{C} \xrightarrow{P_i} \mathcal{S}_i \,.$$

In the SymTFT ASCies are categories $\mathcal{S}$ **such that SymTFT$(\mathcal{S})$ does not have non-trivial magnetic algebras** $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = \{1\}$.

This criterion makes it straightforward to determine if a symmetry category also is an ASCy.

Example: Warmup Anomalous Groups

A short exact sequence of finite groups, for N is a normal subgroup of G

$$1 \to N \to G \xrightarrow{p} G/N \to 1\,,$$

gives rise to a short exact sequence of symmetry tensor categories

$$\mathrm{Vec}_N \to \mathrm{Vec}_G^{\omega} \xrightarrow{p} \mathrm{Vec}_{G/N}^{\omega'}$$

if $\omega \in H^3(G, U(1))$ is

$$\omega = p^*(\omega')$$

Consider $\mathbb{Z}_8^{\omega=4}$. The largest normal subcategory is $\mathcal{N} = \mathrm{Vec}_{\mathbb{Z}_2}$ which fits into two ses:

$$\mathrm{Vec}_{\mathbb{Z}_2} \xrightarrow{I} \mathrm{Vec}_{\mathbb{Z}_8}^{\omega=4} \xrightarrow{P_1} \mathrm{Vec}_{\mathbb{Z}_4}^{\omega=1}$$

$$\mathrm{Vec}_{\mathbb{Z}_2} \xrightarrow{I} \mathrm{Vec}_{\mathbb{Z}_8}^{\omega=4} \xrightarrow{P_{-1}} \mathrm{Vec}_{\mathbb{Z}_4}^{\omega=-1}$$

In the SymTFT the embedding of $\mathbb{Z}_2$ is realized in terms of the electric condensable algebra

$$\mathcal{A}_I = 1 \oplus e^2 \oplus e^4 \oplus e^6$$

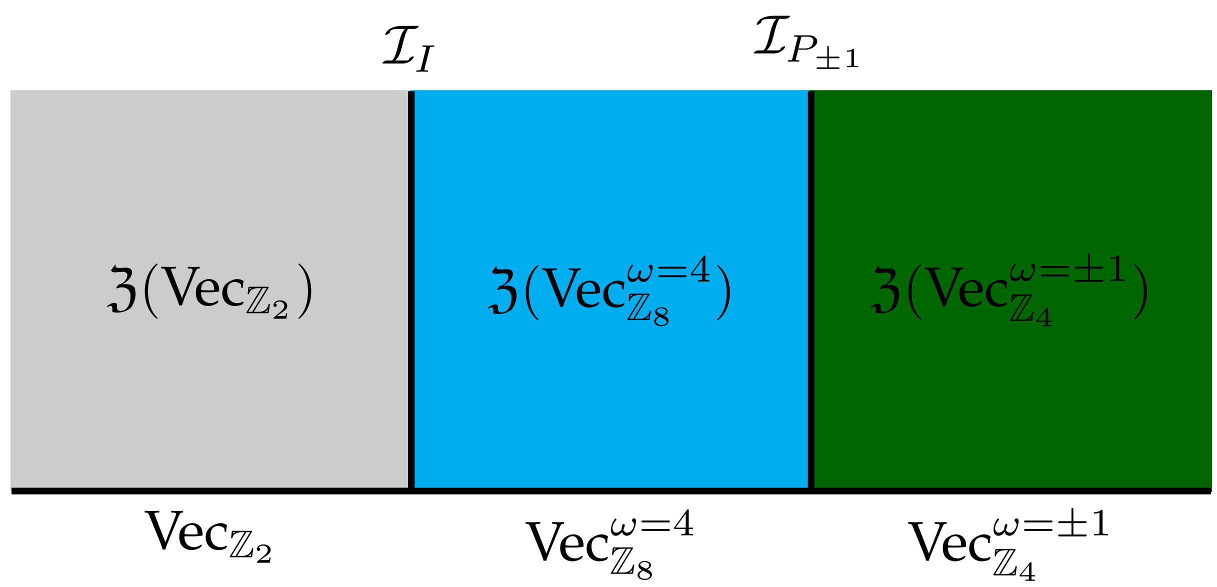
The surjection is obtained from the magnetic algebras

$$\mathcal{A}_{P_1} = 1 \oplus m^4 \,, \qquad \mathcal{A}_{P_{-1}} = 1 \oplus e^4 m^4 \,,$$

and

$$\mathrm{SymTFT}(\mathbb{Z}_8^{\omega=4})/\mathcal{A}_{P_1} = \mathrm{SymTFT}(\mathbb{Z}_4^{\omega=1}) \,,$$
$$\mathrm{SymTFT}(\mathbb{Z}_8^{\omega=4})/\mathcal{A}_{P_{-1}} = \mathrm{SymTFT}(\mathbb{Z}_4^{\omega=-1}) \,.$$

The situation can be summarized succinctly via the RG-quiche:

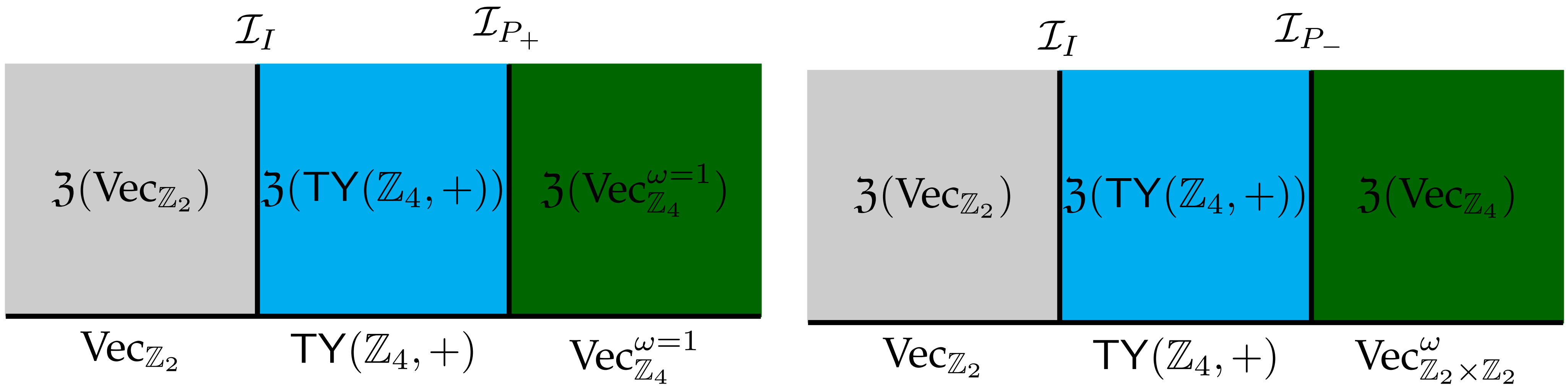


ASCies for Non-Invertible Symmetries

$\mathsf{Ising} = \mathsf{TY}(\mathbb{Z}_2)$ has a $\mathbb{Z}_2$ subsymmetry, which is not normal: Ising is an ASCy itself.

$$\mathsf{TY}(\mathbb{Z}_4,+): \qquad N\otimes N = 1 + a + a^2 + a^3\,, \qquad a^4 = 1$$

From the analysis of algebras in the SymTFT$(\mathsf{TY}(\mathbb{Z}_4,+))$ we find



There are two inequivalent $\mathbb{Z}_2$ ASCies:

$$\begin{tikzcd}
 & & \mathsf{Vec}_{\mathbb{Z}_4}^{\omega=1} \\
\mathsf{Vec}_{\mathbb{Z}_2} \arrow[r, "I"] & \mathsf{TY}(\mathbb{Z}_4,+) \arrow[ur, "P_+"] \arrow[dr, "P_-"] & \\
 & & \mathsf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}^{\omega}
\end{tikzcd} .$$

I.e. In an RG-flow, starting with a UV symmetry $\mathsf{TY}(\mathbb{Z}_4,+)$ one can trivialize $\mathbb{Z}_2$ in two ways, reaching either $\mathrm{Vec}_{\mathbb{Z}_4}^{\omega=1}$ or $\mathrm{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}^{\omega}$.

Future Directions

SymTFT approach to ASCies and Categorical Anomaly Matching extends to higher-dimensions, e.g. connection to Wang-Wen-Witten, symmetry fractionalization/transmutation and Lieb-Schulz-Matthis anomalies (including discrete spacetime symmetries). See [Antinucci, Copetti, Gai, SSN]

- Classification of ASCies
- Tensor Functors for higher fusion categories
- Continuous Symmetries – see SymTFTs in [Brennan, Sun][Antinucci, Benini][Bonetti, del Zotto, Minasian][Apruzzi, Bedogna, Dondi]

Clearly lots more to explore here!

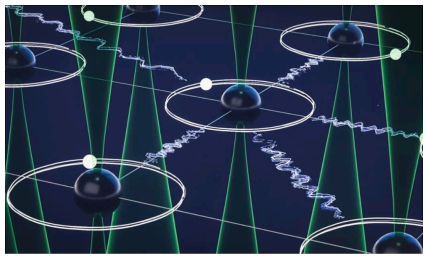
IV. Quantum Simulators and Quantum Codes

qi/amo

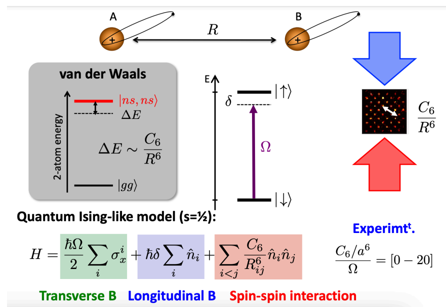
Two totally different questions:

1. **Can we test the phase diagram, including second order phase transitions in the presence of categorical symmetries?**
   Yes, using simple lattice model realizing the gapped and gapless phases of $\mathrm{Rep}(D_4)$ on qubits [Warman, Yang, Tiwari, Pichler, SSN]

   Atoms as Qubits: Using quantum simulators realized in Rydberg atoms [Jaksch et al PRL 2010],[Weimer et al], many groups: Lukin, Saffman, Molmer, Pollman, ...]



Optical tweezer array



Rydberg blockade: Rydberg states are **highly excited states** in atoms with very large principal quantum numbers. e^- far from the nucleus have high polarizability and lifetimes. Dipole interaction ensures only one atom within a certain distance (blockade radius) can be excited. Introduces spatial correlations, mimicking interaction terms in a Hamiltonian.

2. **Can one use categorical symmetries to improve quantum codes?**
   (a) Using higher-form symmetries, i.e. DW theories for higher-form and higher-group symmetries [Barkeshli, Hsin, Kobayashi, Zhu, ...]
   (b) **Magic state preparation using SymTFT**(D_4) [Davydova, Bauer, de la Fuente, Webster, Williamson, Brown][Huang, Chen]

Magic state preparation using SymTFT Sandwiches

[Davydova, Bauer, de la Fuente, Webster, Williamson, Brown][Huang, Chen]

Idea: use lattice models that realize the SymTFT$(G) = D(G)$ – Kitaev surface codes – to do fault tolerant universal quantum computing. These can do so-called Clifford operators (Pauli operators, and things that leave these invariant), but require one non-Clifford gate to realize a universal gate set.

Example of a non-Clifford magic state:
logical T :

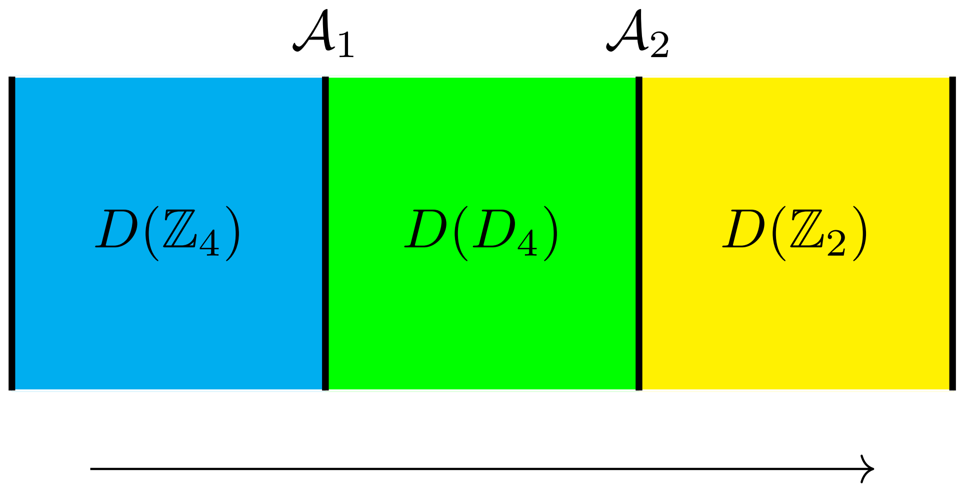
$$|T\rangle = |1\rangle + \zeta_8 |e\rangle$$

in $\mathbb{Z}_2$, where $\zeta_8 = e^{2\pi i/8}$.

Idea in [Huang, Chen]: using a stabilizer (Clifford) S-gate in $D(\mathbb{Z}_4)$

$$|S\rangle = |1\rangle + \zeta_8 |e\rangle - |e^2\rangle + \zeta_8 |e^3\rangle$$

as input and apply the following triple-sandwich:



$\mathcal{A}_1$ corresponds to gauging $\mathbb{Z}_2$. Both algebras give rise to maps between these topological orders, via a club quiche [Bhardwaj, Pajer, SSN, Warman].

$$\varphi_1 : \quad \begin{cases} 1 & \to \quad 1\,, \\ e^2 & \to \quad e_r\,, \\ e \oplus e^3 & \to \quad E\,. \end{cases} \qquad \varphi_2 : \quad \begin{cases} 1 & \to \quad 1\,, \\ e_r e_x & \to \quad 1\,, \\ E & \to \quad e\,, \end{cases}$$

Such that

$$\varphi_2 \circ \varphi_1(|S\rangle) = |T\rangle$$

[Davydova et al] use the same TQFT, with a slightly different protocol, to prepare a magic state for CCP (CC phase gate).

Many extensions to qudit magic states, and lattice surgery [Huang, Chen, Warman, SSN – in progress].

Conclusions and Outlook

- Categorical symmetries have imprinted many areas of theoretical physics.
- Clear implications on QFT, quantum many-body system phases.
- SymTFT ”sandwich spread” key to many of these studies.

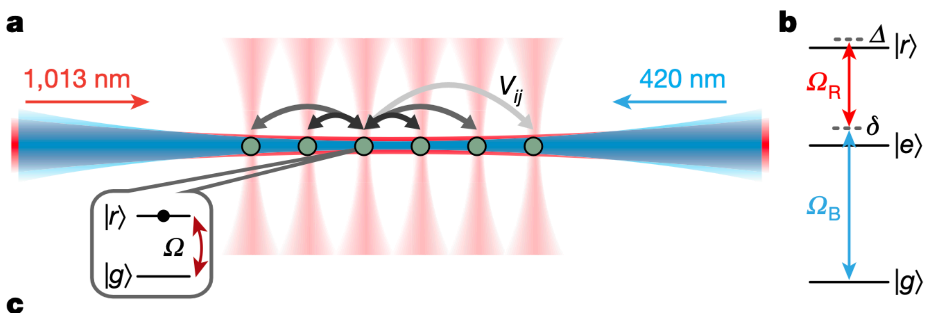
The Future:

- A lot of progresson finite symmetries. What about continuous? Using the SymTFTs for continuous symmetries in [Brennan, Sun][Antinucci, Benini][Bonetti, del Zotto, Minasian][Apruzzi, Bedogna, Dondi]
- Spacetime symmetries – [Apruzzi, Dondi, Garcia-Extebarria, Lam, SSN to appear]
- Explore the full scope of categorical anomalies, in various dimensions.

Backup Slides

Quantum Simulation of Spin-Models with Rydberg Atoms

The Rydberg blockade can simulate 1+1d quantum Hamiltonians in this way, e.g. using ^{87}Rb atoms: [Bernien, ..., Lukin (Nature)]



Example: Ising-like Model

$$H = \sum_i \frac{\Omega}{2}\sigma_i^x - \Delta \sum_i n_i + \sum_{i<j} V_{ij} n_i n_j$$

Terms and Mapping:

- $\Omega\sigma_i^x$: Rabi frequency Ω driving transitions between $|g\rangle$ and $|r\rangle$=Rydberg state.
- $-\Delta n_i$: Detuning controls on-site potential, where $n_i = |1\rangle\langle 1|$ acts as $(1-\sigma_z)/2$
- $V_{ij}n_in_j$: Repulsive van der Waals or dipole-dipole interactions.

Spin Models for Rep(D_8)-Gapped Phases

[Warman, Yang, Tiwari, Pichler, SSN]

Rep(D_8) phases can be realized on a simple tensor product Hilbert space based on qubits only:

$$\mathcal{H} = \mathbb{C}^{3L} = (\mathbb{C}^{\mathrm{I}}, \mathbb{C}^{\mathrm{II}}, \mathbb{C}^{\mathrm{III}})^L \,.$$

with Hamiltonian

$$H = \sum_i V_{i,i+1} + \sum_i P_i \,,$$

where V is nearest neighbor and P is a projection operator.

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB with non-invertible symmetry action

$$H_{(\mathbb{Z}_2^a \times \mathbb{Z}_2^b)^+} = -\frac{1}{4} \sum_i (2\, \mathbb{I}_i \mathbb{I}_{i+1} + X_i^{\mathrm{II}} X_{i+1}^{\mathrm{II}} + X_i^{\mathrm{III}} X_{i+1}^{\mathrm{III}}) - \frac{1}{2} \sum_i \left[\mathbb{I} + Z^{\mathrm{I}}\right]_i$$

X and Z are acting only on each qubit as Pauli operators.

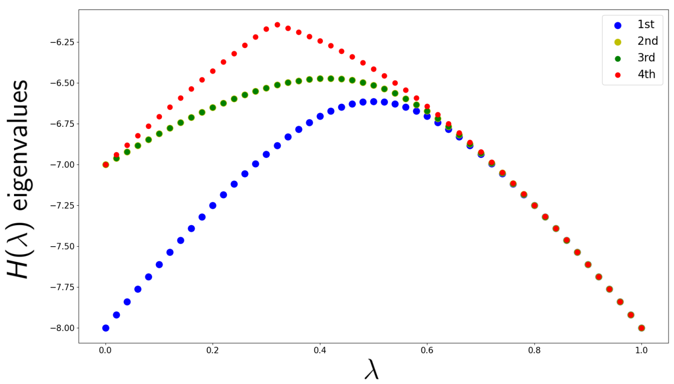
Trivial	$H_1 = -\sum_i \left[\mathbb{I} + \frac{1}{8}(\mathbb{I}+Z^{\mathrm{I}})(\mathbb{I}+Z^{\mathrm{II}})(\mathbb{I}+Z^{\mathrm{III}})\right]_i\,, \quad \lvert\mathrm{GS}\rangle$
$\mathbb{Z}_2$ SSB	$H_{\mathbb{Z}_2^a} = -\frac{1}{2}\sum_i \left(\mathbb{I}_i\mathbb{I}_{i+1} + X_i^{\mathrm{II}}X_{i+1}^{\mathrm{II}}\right) - \frac{1}{4}\sum_i \left[(\mathbb{I}+Z^{\mathrm{I}})(\mathbb{I}+Z^{\mathrm{III}})\right]_i$ $\lvert\mathrm{GS},+\rangle \oplus \lvert\mathrm{GS},-\rangle$
$\frac{\mathrm{Rep}(D_8)}{(\mathbb{Z}_2\times\mathbb{Z}_2)}$ SSB	$H_{\mathbb{Z}_2^{ab}} = -\frac{1}{2}\sum_i \left[\mathbb{I}_i\mathbb{I}_{i+1} + (X^{\mathrm{II}}X^{\mathrm{III}})_i(X^{\mathrm{II}}X^{\mathrm{III}})_{i+1}\right] - \frac{1}{4}\sum_i \left[(\mathbb{I}+Z^{\mathrm{I}})(\mathbb{I}+Z^{\mathrm{II}}Z^{\mathrm{III}})\right]_i$ $\lvert\mathrm{GS},+\rangle \oplus \lvert\mathrm{GS},-\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_2$ SSB	$H_{(\mathbb{Z}_2^a\times\mathbb{Z}_2^b)+} = -\frac{1}{2}\sum_i (X_i^{\mathrm{II}}X_{i+1}^{\mathrm{II}} + X_i^{\mathrm{III}}X_{i+1}^{\mathrm{III}}) - \frac{1}{2}\sum_i \left[\mathbb{I}+Z^{\mathrm{I}}\right]_i$ $\lvert\mathrm{GS},++\rangle \oplus \lvert\mathrm{GS},--\rangle \oplus \lvert\mathrm{GS},+-\rangle \oplus \lvert\mathrm{GS},-+\rangle$
SPT	$H_{(\mathbb{Z}_2^a\times\mathbb{Z}_2^b)-} = -\frac{1}{2}\sum_i \left[X_i^{\mathrm{II}}(X^{\mathrm{II}}Z^{\mathrm{III}})_{i+1}\right] - \frac{1}{2}\sum_i \left[(Z^{\mathrm{II}}X^{\mathrm{III}})_i X_{i+1}^{\mathrm{III}}\right] - \frac{1}{2}\sum_i \left[\mathbb{I}+Z^{\mathrm{I}}\right]_i\,, \quad \lvert\mathrm{GS}\rangle$
$\mathbb{Z}_2$ SSB	$H_{\mathbb{Z}_2^c} = -\frac{1}{2}\sum_i \left(\mathbb{I}_i\mathbb{I}_{i+1} + X_i^{\mathrm{I}}X_{i+1}^{\mathrm{I}}\right) - \frac{1}{4}\sum_i \left[(\mathbb{I}+Z^{\mathrm{II}})(\mathbb{I}+Z^{\mathrm{III}})\right]_i$ $\lvert\mathrm{GS},+\rangle \oplus \lvert\mathrm{GS},-\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_2$ SSB	$H_{(\mathbb{Z}_2^c\times\mathbb{Z}_2^{ab})+} = -\frac{1}{2}\sum_i \left[X_i^{\mathrm{I}}X_{i+1}^{\mathrm{I}} + (X^{\mathrm{II}}X^{\mathrm{III}})_i(X^{\mathrm{II}}X^{\mathrm{III}})_{i+1}\right] - \frac{1}{2}\sum_i \left[\mathbb{I}+Z^{\mathrm{II}}Z^{\mathrm{III}}\right]_i$ $\lvert\mathrm{GS},++\rangle \oplus \lvert\mathrm{GS},--\rangle \oplus \lvert\mathrm{GS},+-\rangle \oplus \lvert\mathrm{GS},-+\rangle$
SPT	$H_{(\mathbb{Z}_2^c\times\mathbb{Z}_2^{ab})-} = -\frac{1}{2}\sum_i \left[X_i^{\mathrm{I}}(X^{\mathrm{I}}Z^{\mathrm{II}})_{i+1}\right] - \frac{1}{2}\sum_i \left[(Z^{\mathrm{I}}X^{\mathrm{II}}X^{\mathrm{III}})_i(X^{\mathrm{II}}X^{\mathrm{III}})_{i+1}\right] - \frac{1}{2}\sum_i \left[\mathbb{I}+Z^{\mathrm{II}}Z^{\mathrm{III}}\right]_i$ $\lvert\mathrm{GS}\rangle$
$\mathbb{Z}_2$ SSB	$H_{(D_8)-} =$ (A110) $\lvert\mathrm{GS},+\rangle \oplus \lvert\mathrm{GS},-\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_2$ SSB	$H_{\mathbb{Z}_4^{ca}} =$ (A96) $\lvert\mathrm{GS},++\rangle \oplus \lvert\mathrm{GS},--\rangle \oplus \lvert\mathrm{GS},+-\rangle \oplus \lvert\mathrm{GS},-+\rangle$
$\mathrm{Rep}(D_8)$ SSB	$H_{(D_8)+} =$ (A103) $\lvert\mathrm{GS},1\rangle \oplus \lvert\mathrm{GS},2\rangle \oplus \lvert\mathrm{GS},5\rangle \oplus \lvert\mathrm{GS},3\rangle \oplus \lvert\mathrm{GS},4\rangle$

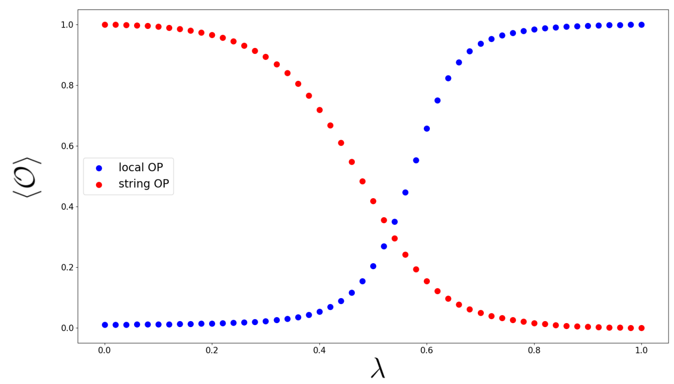
red/green: $\mathbb{Z}_2$ given by $1_{a/c}$, blue: E non-invertible symmetry $E^2 = 1 + 1_a + 1_c + 1_{ac}$

Numerical Analysis of Ground-states and Phase Transitions

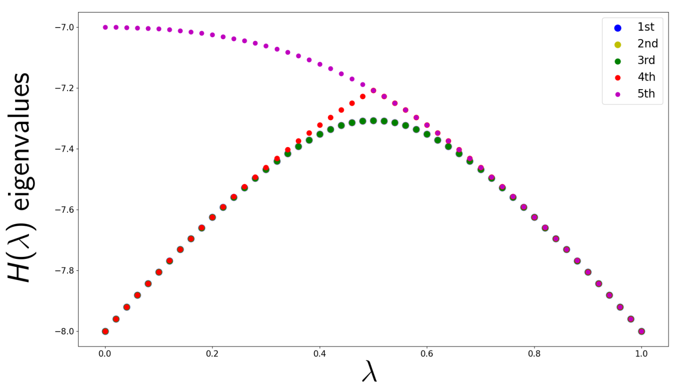
Phase transitions: $H_i + (1-\lambda)H_j$ by tuning λ.

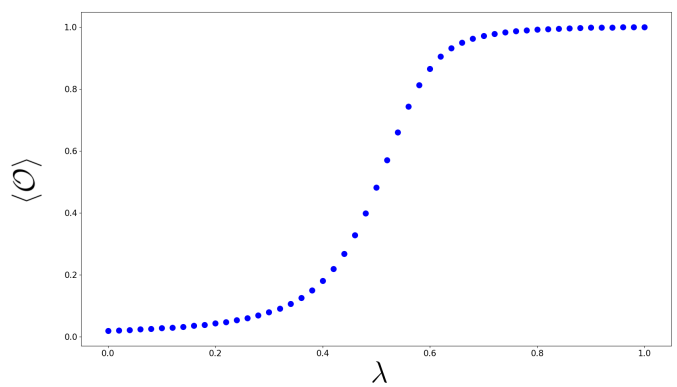
Phase transition: from 1 groundstate (GS) SPT to 4 GS $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB:



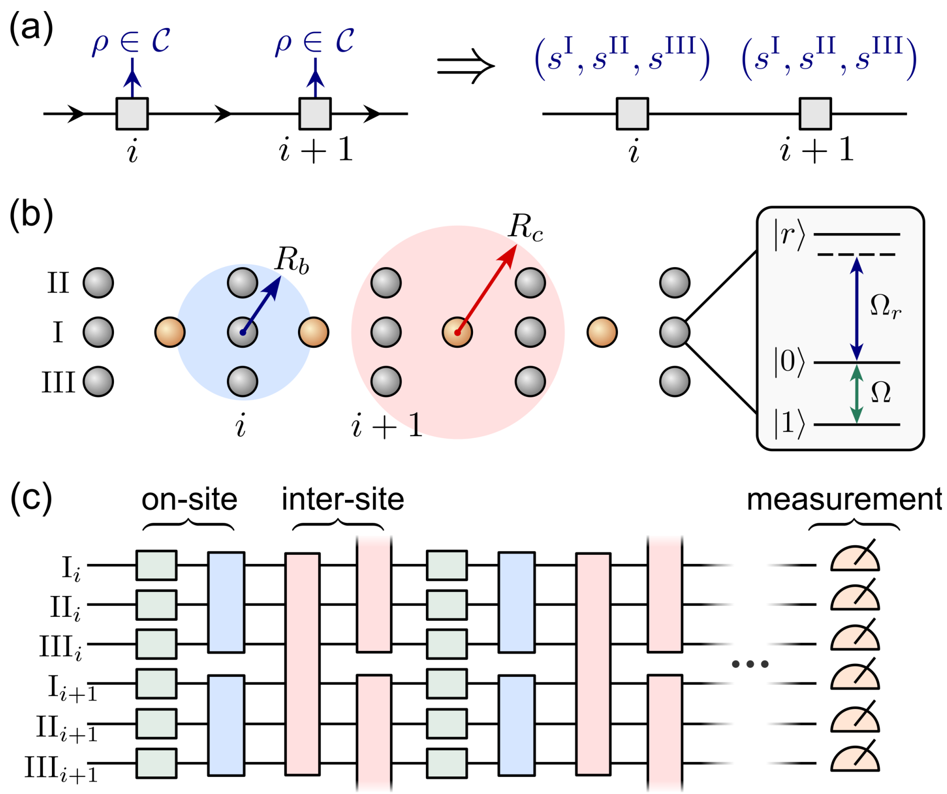


Phase transition from the 4 GS $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB to the 5 GS Rep(D_8) SSB: The order parameter is $\mathcal{O} = 1/L \sum_i X_i^{\mathrm{I}}$.



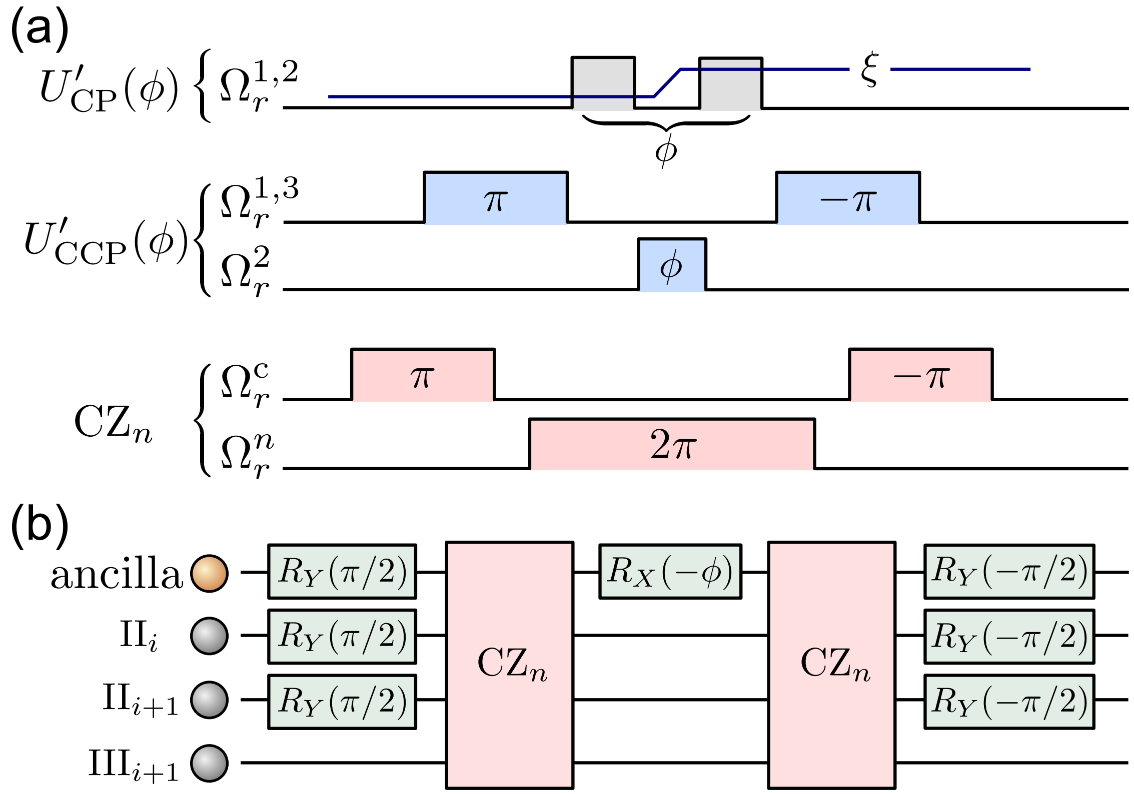


Realization in a Rydberg Quantum Simulator



Each site has three grey Rydberg atoms realizing qubits. Yellow Rydberg atoms are ancillary qubits. This can be used to realize the above Hamiltonians, and phase-transitions.

Pulse sequence to realize e.g. the interactions:



(a) elementary quantum gates: phase gate diag $(1, e^{i\phi})$, controlled phase gate (4x4 incarnation of phase gate), CZ diag$(1^3, -1)$.

(b) Gate sequence for simulating the three-body plaquette evolution $U_\square(\phi) = \exp(-\mathrm{i}\phi X_i^{\mathrm{II}} X_{i+1}^{\mathrm{II}} Z_{i+1}^{\mathrm{III}})$.