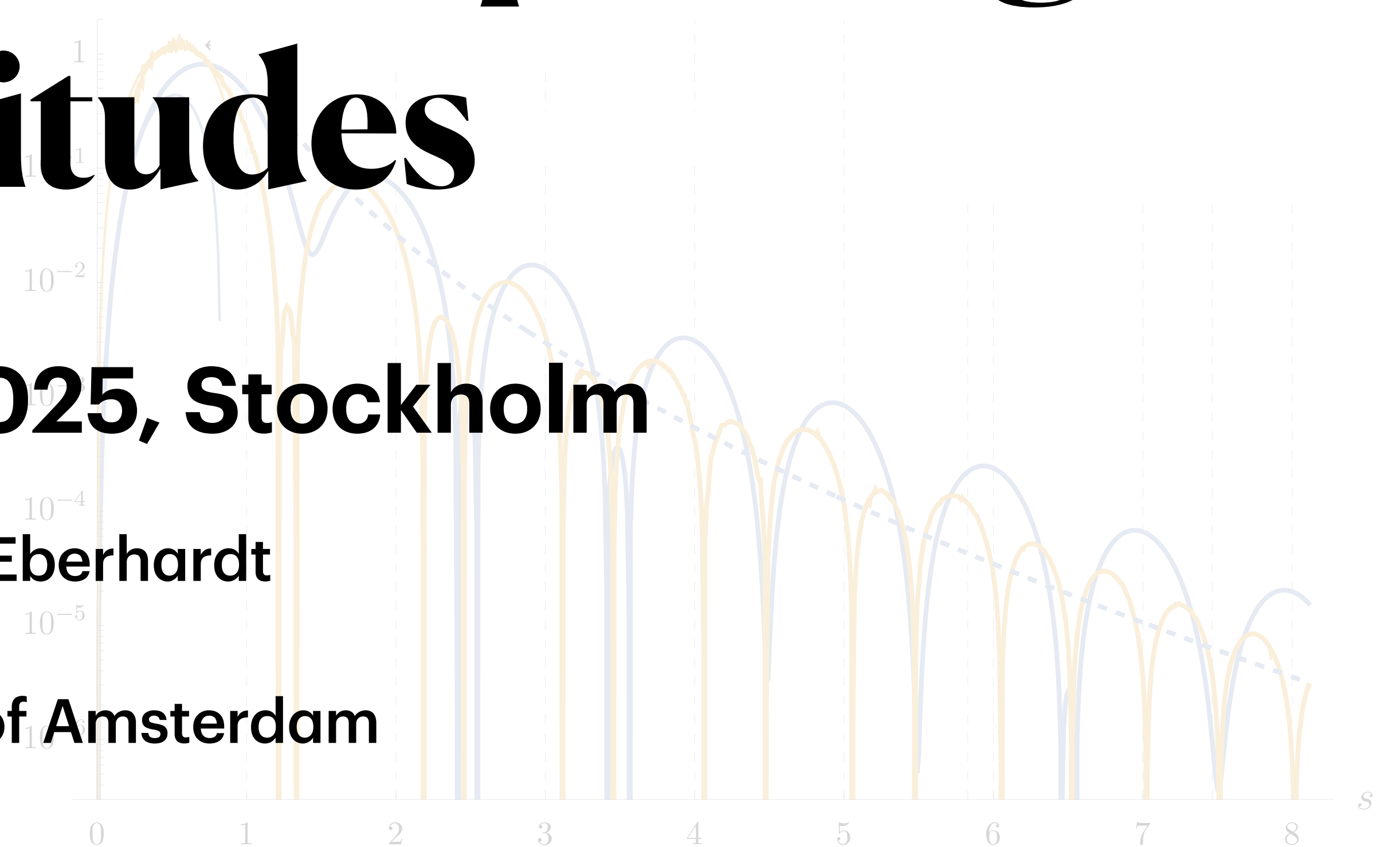


Evaluating one-loop string amplitudes

Eurostrings 2025, Stockholm

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Based on work in collaboration with Marco M. Baccianti, Jeevan Chandra, Tom Hartman, Sebastian Mizera
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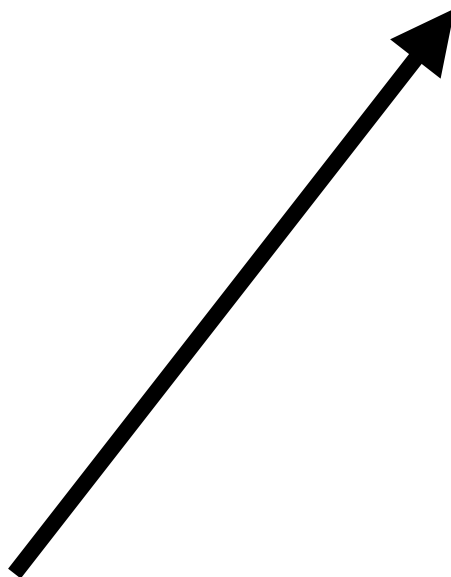
**Surprisingly little is known
about higher loop
string amplitudes**

Virasoro-Shapiro amplitude

$$\mathcal{A}(s, t) = t_8 \tilde{t}_8 \times \frac{\Gamma(-\frac{\alpha' s}{4}) \Gamma(-\frac{\alpha' t}{4}) \Gamma(-\frac{\alpha' u}{4})}{\Gamma(1 + \frac{\alpha' s}{4}) \Gamma(1 + \frac{\alpha' t}{4}) \Gamma(1 + \frac{\alpha' u}{4})}$$

Veneziano '68,
Virasoro '69,
Shapiro '69

Polarization structure



No explicit formulas for higher loops...

Virasoro-Shapiro amplitude

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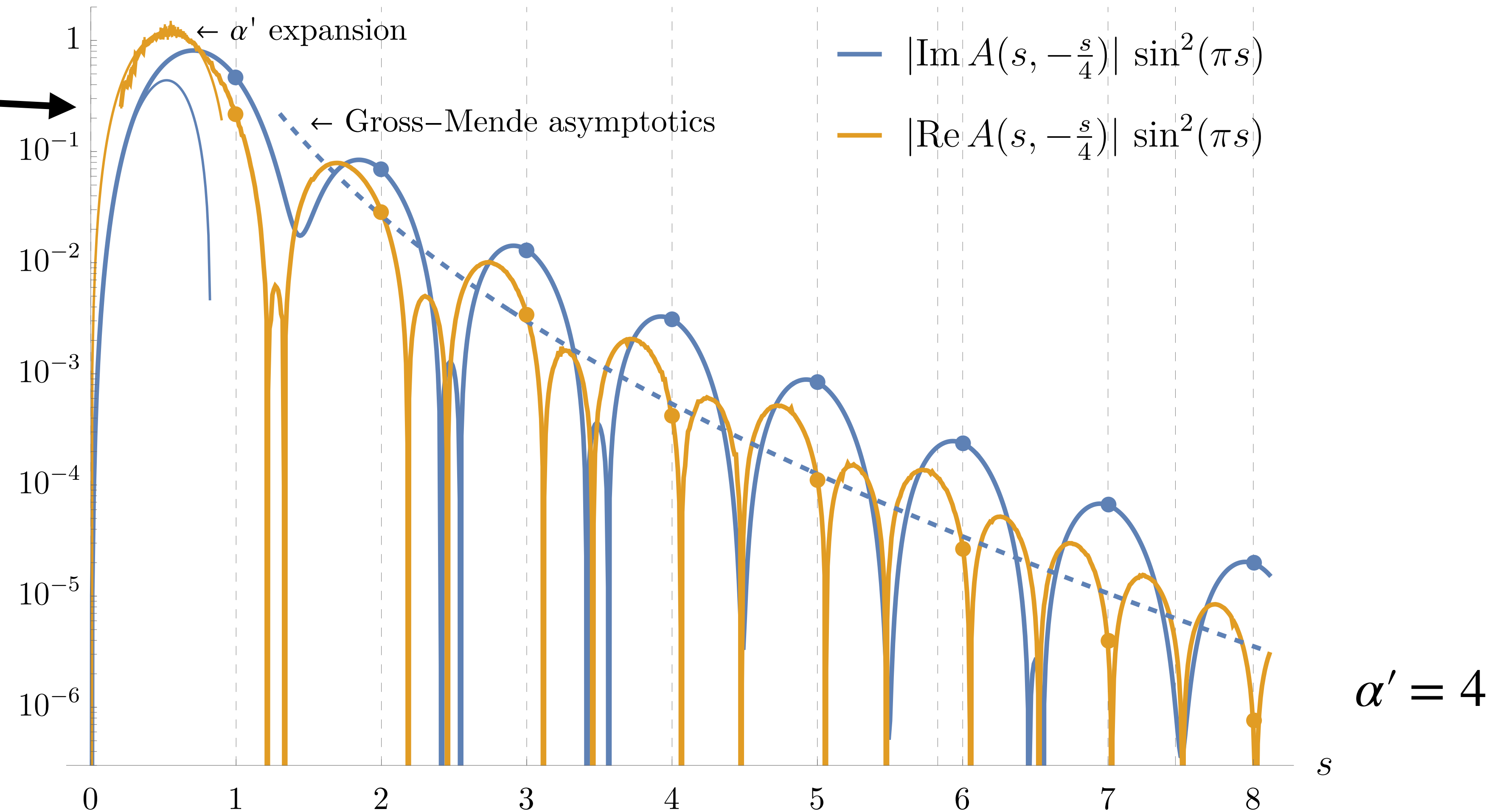
Amazing properties:

- Manifestly crossing symmetric (dual resonance)
- Regge bounded
- UV soft
- Unitary

**Today I'll explain a general method to
evaluate one-loop string amplitudes**

Type II one-loop amplitude in 10d @ 60 degrees

Field theory
+ α' corrections
[enormous literature: Green,
Schwarz, Gross, Veneziano,
Amati, Ciafaloni, Di Vecchia,
Koba, Nielsen, D'Hoker,
Phong, Martinec, Bern,
Dixon, Polyakov, Kosower,
Vanhove,
Schlotterer, Mafra, Stieberger,
Brown, Broedel, Hohenegger,
Kleinschmidt, Gerken,
Roiban, Lipstein, Mason,
Monteiro, Claasen,
Doroudiani...]



Outline

I. Defining the amplitude

II. Rademacher evaluation

III. High-energy limit

I. Defining the amplitude

Higher loop amplitudes

Textbook definition:

$$A_{g,n}(p_1, \dots, p_n) = \int_{\mathcal{M}_{g,n}} (\text{CFT correlation function})$$

Moduli space of Riemann surfaces

Integral does not converge due to expected physical singularities in the amplitude:
massive poles & branch cuts

A definition consistent with causality

We treat the worldsheet as Euclidean, but spacetime is Lorentzian!

$$A_{g,n}(p_1, \dots, p_n) = \int_{\Gamma \subset \mathcal{M}_{g,n}^{\mathbb{C}}} \quad (\text{CFT correlation function})$$

Contour consistent
with the Lorentzian spacetime

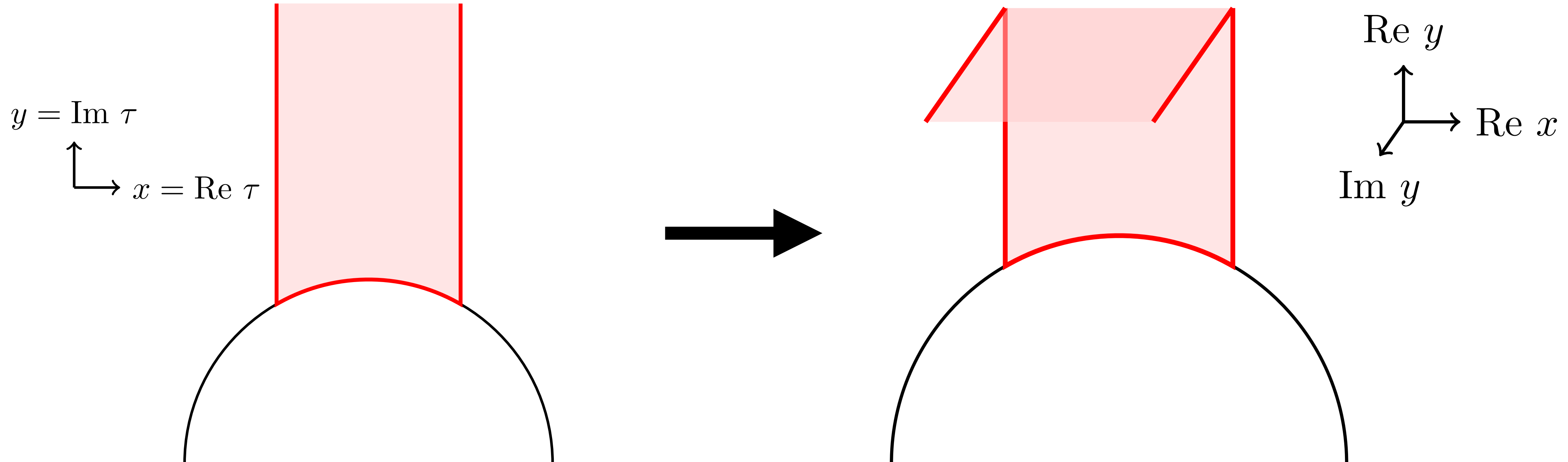
Describes complex metrics
on the worldsheet

Modify contour of integration near a degeneration of the worldsheet: stringy $i\epsilon$ prescription

The one-loop amplitude

Green, Schwarz '82

$$A = \int_{\Gamma} \frac{d^2 \tau}{(\text{Im } \tau)^5} \int_{\mathbb{T}^2} \prod_{j=1}^3 d^2 z_j \prod_{1 \leq j < i \leq 4} |\vartheta_1(z_{ij} | \tau)|^{-2s_{ij}} e^{\frac{2\pi s_{ij} (\text{Im } z_{ij})^2}{\text{Im } \tau}}$$



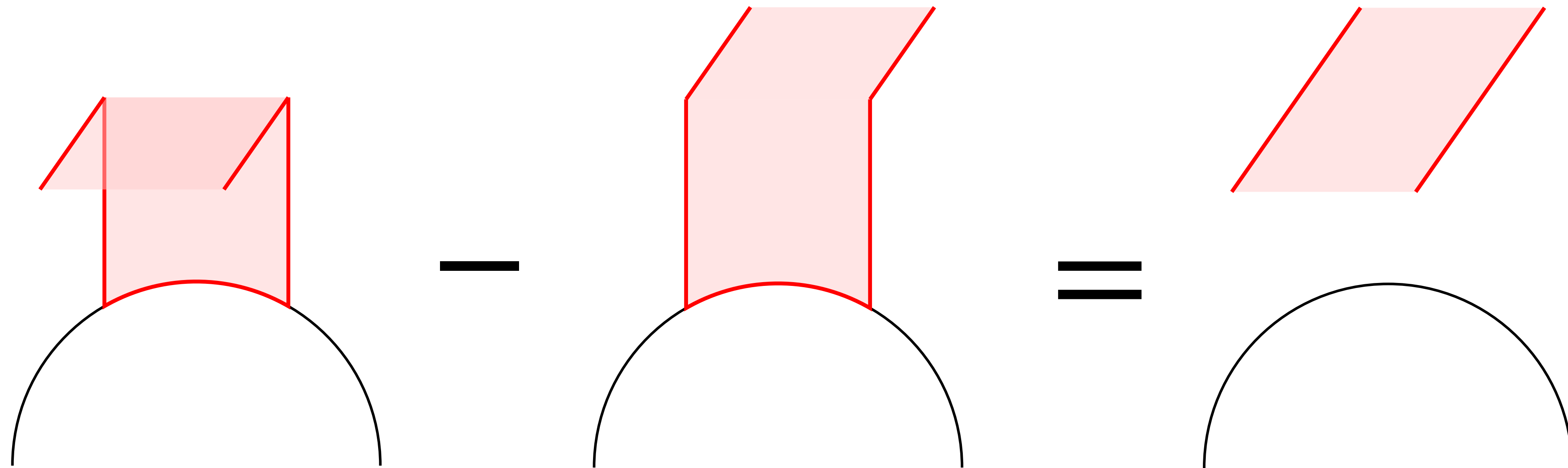
II. Evaluating the amplitude

Contour still not very useful

- Convergent representation of the amplitude, but still very bad for e.g. numerical integration
- Integrand highly oscillatory
- Use contour deformations to proceed!

Warm up: imaginary part

- Difference of contours:



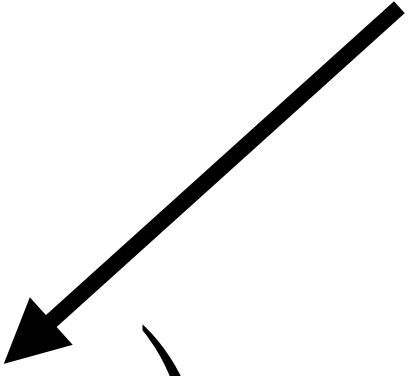
- Only need to know the local behavior of the integrand near the cusp $\text{Im } \tau \rightarrow \infty$

Toy example

Bosonic string partition function: $Z = \int_{\Gamma} \frac{d^2\tau}{(\text{Im } \tau)^{14} |\eta(\tau)^{24}|^2}$

$$\begin{aligned}
 2i\text{Im } Z &= \int_{\Gamma_{\text{Im}}} \frac{d^2\tau}{(\text{Im } \tau)^{14} |\eta(\tau)^{24}|^2} \\
 &= \int_{\Gamma_{\text{Im}}} \frac{d^2\tau}{(\text{Im } \tau)^{14}} \left(e^{-4\pi\text{Im } \tau} + \dots \right) \\
 &= \int_{1-i\infty}^{1+i\infty} \frac{dy}{y^{14}} e^{-4\pi y} = \frac{i 2^{27} \pi^{14}}{13!}
 \end{aligned}$$

Subleading terms disappear in the limit $\text{Im } \tau \rightarrow \infty$



Baikov representation

Sum over
massive states

Integral over on-
shell phase space

Gram determinant

Sum over polarizations,
Explicit formula via generating function,
Double copies from open string

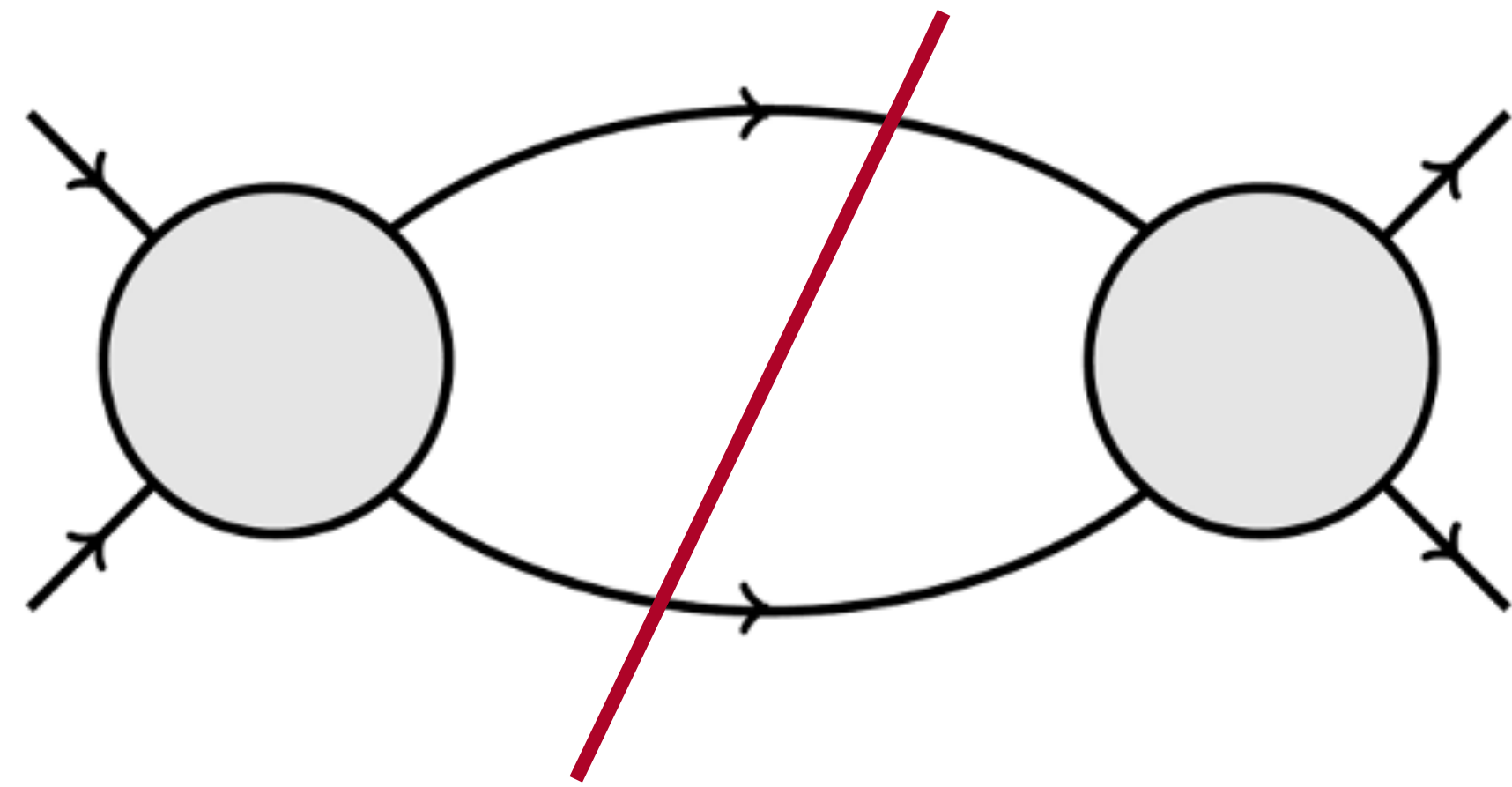
$$\text{Im } A = \frac{16\pi s^4}{15\sqrt{stu}}$$

$$\sum_{\sqrt{m_D} + \sqrt{m_U} \leq \sqrt{s}}$$

$$\int dt_L dt_R$$

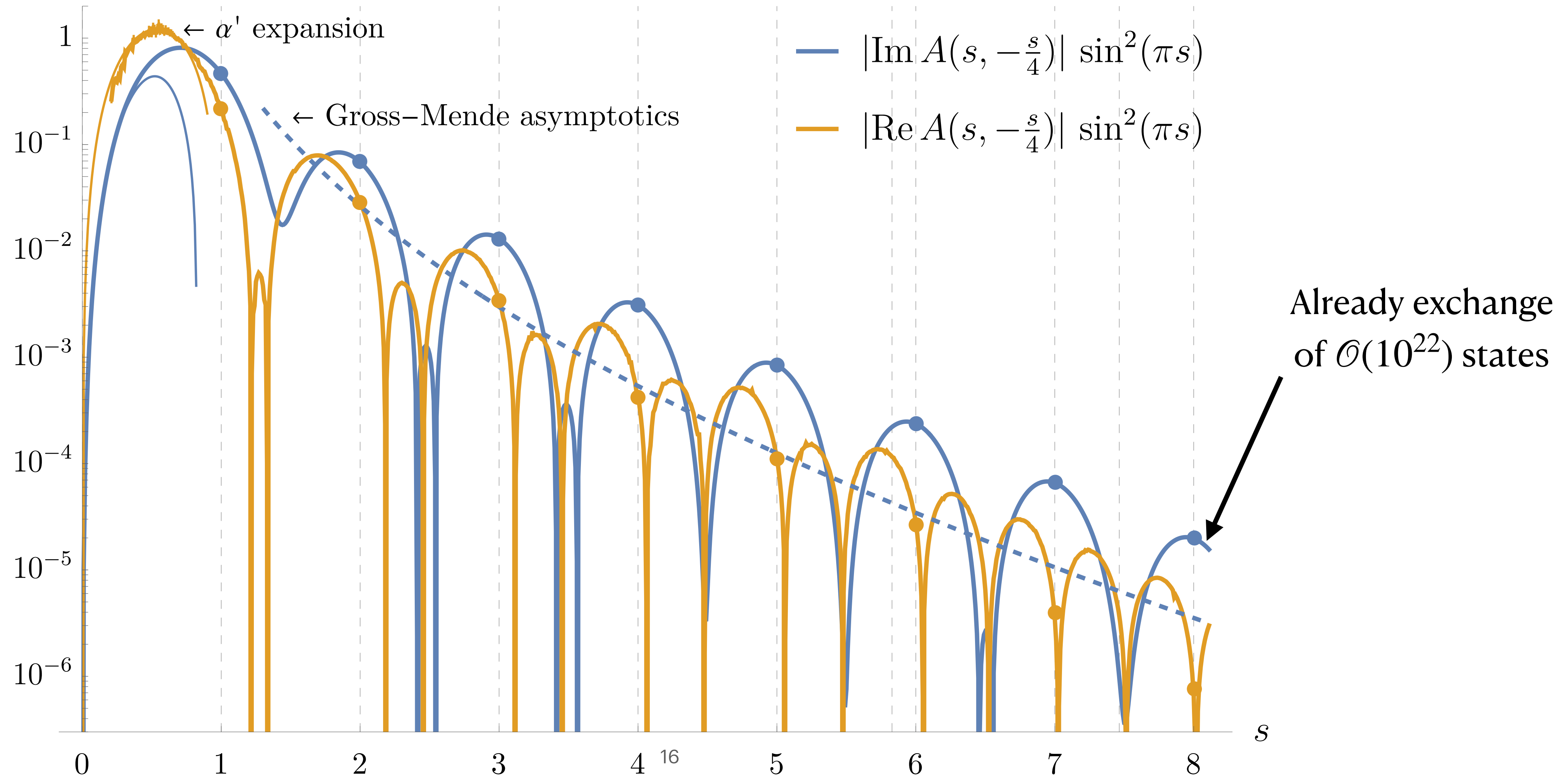
$$P_{m_D, m_U}^{\frac{5}{2}} Q_{m_D, m_U}^2$$

$$\times \frac{\Gamma(-s)\Gamma(-t_L)\Gamma(-u_L)}{\Gamma(1+s)\Gamma(1+t_L)\Gamma(1+u_L)} \times \frac{\Gamma(-s)\Gamma(-t_R)\Gamma(-u_R)}{\Gamma(1+s)\Gamma(1+t_R)\Gamma(1+u_R)}$$



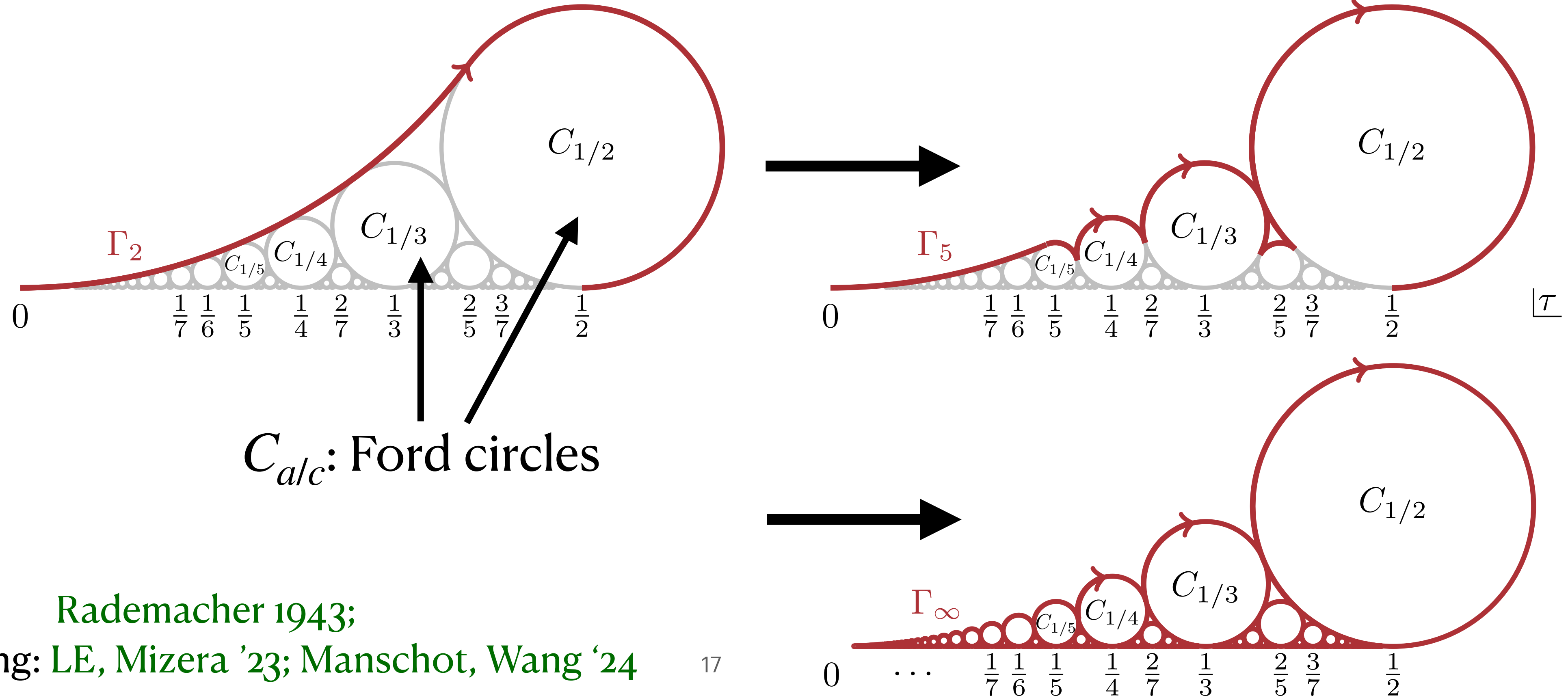
Massive Veneziano amplitudes

Gives the blue curve



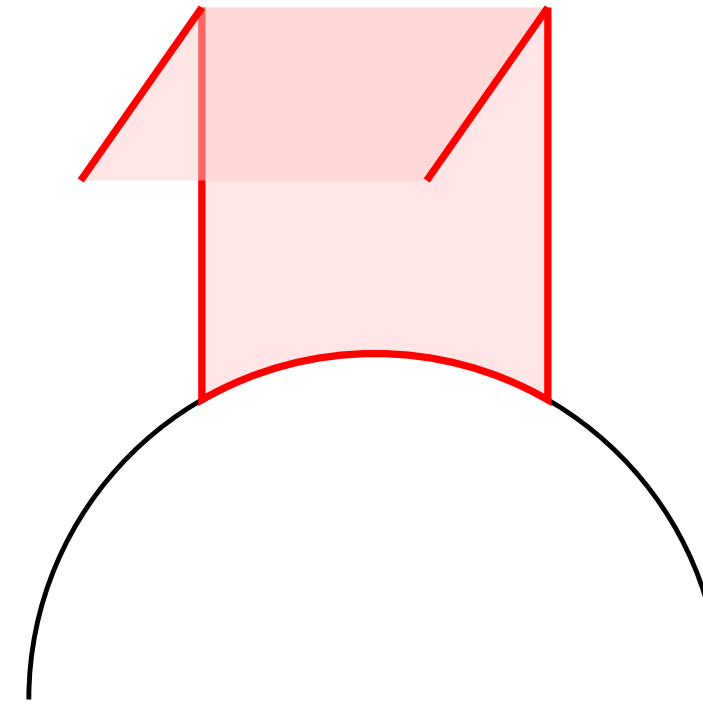
For the real part: Rademacher expansion

- Idea: deform the contour such that it is only sensitive to the cusps in moduli space
- For a contour integral in τ (open string): $\lfloor \tau$



Back to the toy example

$$Z = \int_{\Gamma} \frac{d^2\tau}{(\operatorname{Im} \tau)^{14} |\eta(\tau)^{24}|^2}$$



Evaluate it in terms of its behavior near the cusps in the complexified moduli space $(\tau, \tilde{\tau}) \in (\mathbb{H} \times \mathbb{H})/\mathrm{PSL}(2, \mathbb{Z})$ away from the real slice $\tilde{\tau} = -\bar{\tau}$:

Need a two-dimensional version of the Rademacher expansion!

Other approaches: Zagier '81, Lerche, Nilsson, Schellekens, Warner '88;
Angelantonj, Florakis, Pioline '10; Korpas, Manschot,
Moore, Nidaiev '19; Benjamin, Chang '22

A non-holomorphic Rademacher expansion

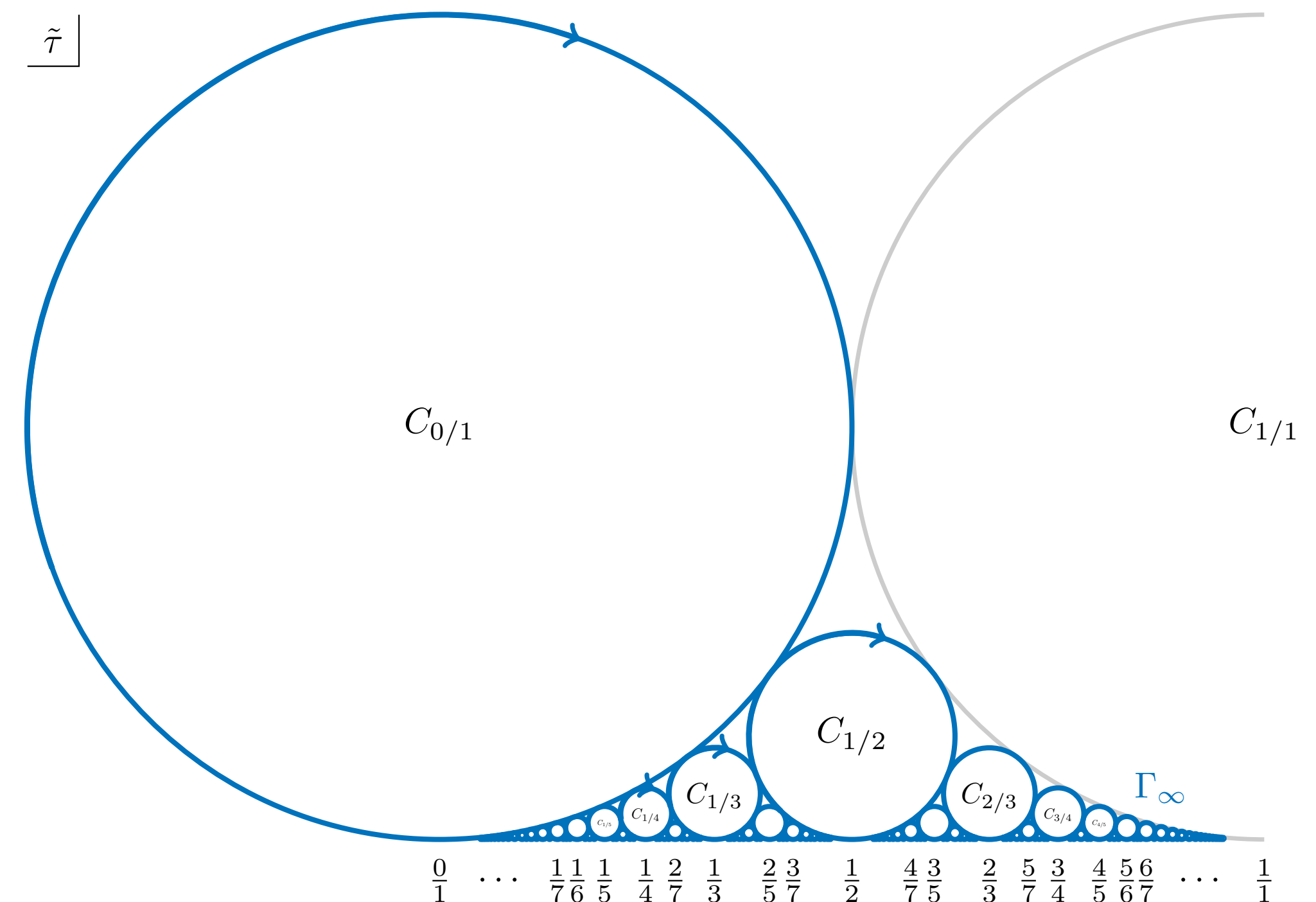
$$\operatorname{Re} \int_{\Gamma} d^2 \tau f(\tau, \tilde{\tau}) = \sum_{c=1}^{\infty} \sum_{a=0, (a,c)=1}^{c-1} \int_{\rightarrow} d\tau \int_{C_{a/c}} d\tilde{\tau} \left[\frac{1}{12i} \left(\tau - \tilde{\tau} + \frac{2a}{c} \right) + is(a, c) \right] f(\tau, \tilde{\tau})$$

Baccianti, Chandra, Hartman, LE, Mizera '25

$$s(a, c) = \sum_{k=1}^{c-1} \left(\left(\frac{k}{c} \right) \right) \left(\left(\frac{ak}{c} \right) \right) \quad \text{Dedekind sums}$$

$$((x)) = x - [x] - \frac{1}{2}$$

Proof: many contour deformations!



Rademacher expansion of the toy example

$$Z = \int_{\Gamma} \frac{d^2\tau}{(\operatorname{Im} \tau)^{14} |\eta(\tau)^{24}|^2}$$
$$= \frac{(4\pi)^{15}}{24 \cdot 13!} \sum_{c=1}^{\infty} \sum_{a=0, (a,c)=1}^{c-1} \frac{e^{2\pi i \frac{a+a^*}{c}}}{c^2} \left[12ic s(a, c) J_{13}\left(\frac{4\pi}{c}\right) + J_{12}\left(\frac{4\pi}{c}\right) - J_{14}\left(\frac{4\pi}{c}\right) \right] + \frac{(4\pi)^{14} i}{4 \cdot 13!} .$$

Works for any modular invariant function provided it satisfies a boundedness condition ensuring that the sum converges (analogous to negative weight for the holomorphic Rademacher expansion)

Closed string four-point function

$$A(s, t) = \sum_{\substack{\text{irreducible} \\ \text{fractions} \\ 0 \leq \frac{a}{c} < 1}} \sum_{\substack{\text{windings} \\ n_L, n_D, n_R, n_U \in \{0, \dots, c-1\} \\ n_L + n_D + n_R + n_U = -a - 1 \bmod c}} A_{a/c}^{n_L, n_D, n_R, n_U}(s, t)$$

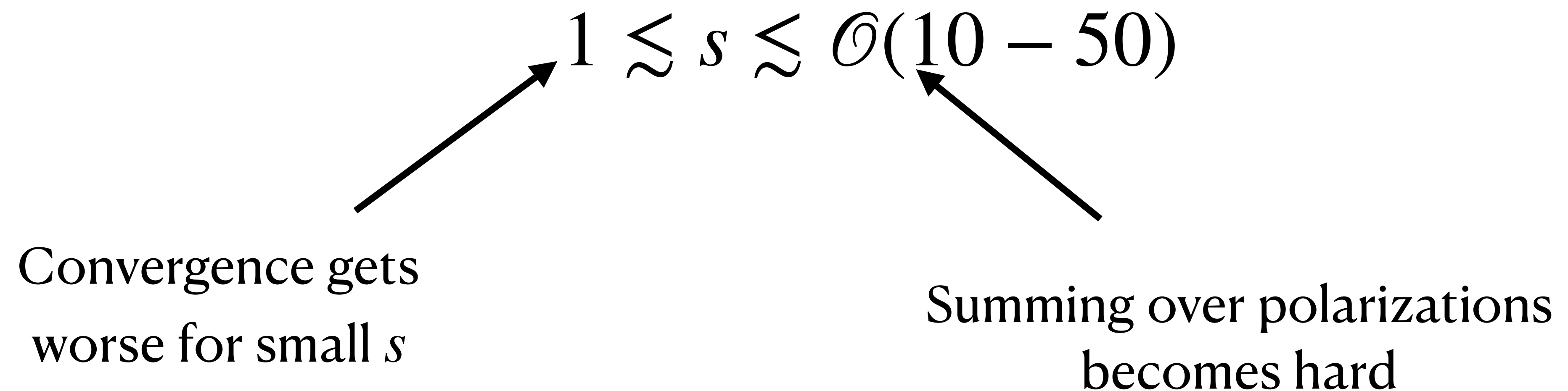
Sum over Ford circles

Additional sum because of z Integration

Each contribution has a Baikov-like representation with additional phases

Convergence

- Convergent representation of the amplitude
- Practical for

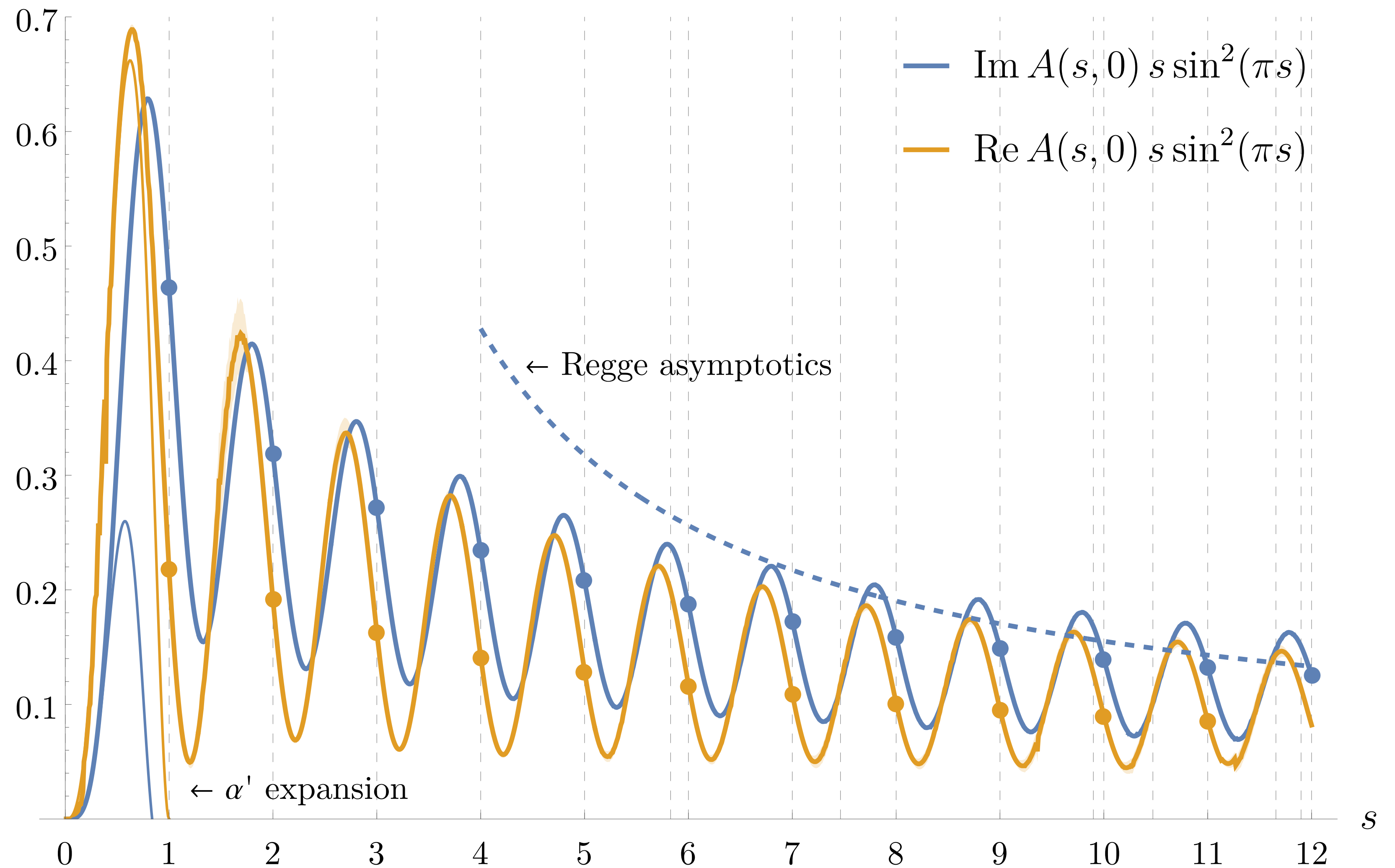


- For small s : direct numerical integration/ α' expansion

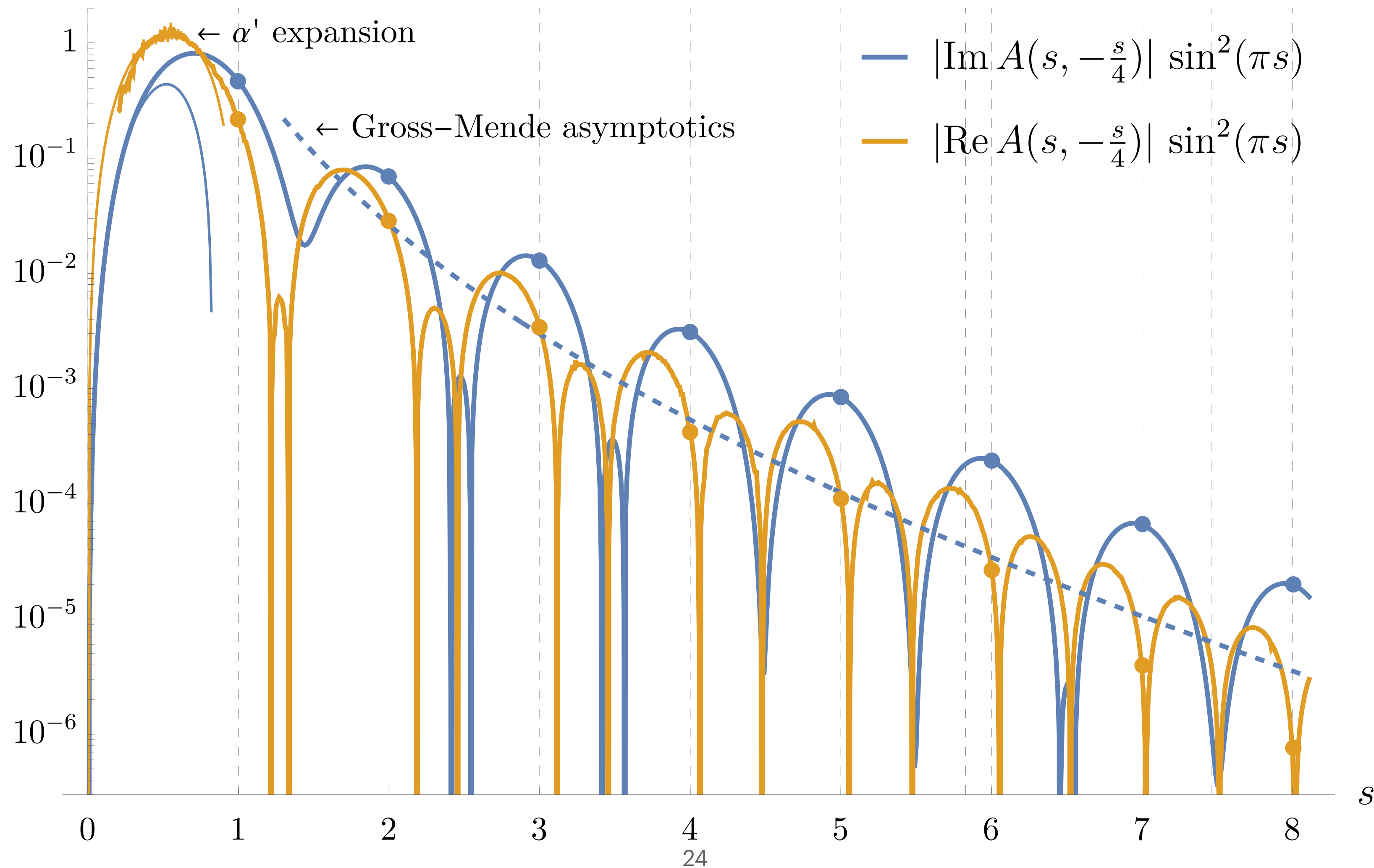
Forward limit (closed string)

- Amplitude mostly imaginary
- Asymptotics understood analytically (error of order $(\log s)^{-1}$)
- consistent with Regge physics

Amati, Ciafaloni, Veneziano '88,
Sundborg '88



Fixed angle scattering @ 60 degrees (closed string)



III. High-energy limit

The Gross-Mende argument

- String theory is most interesting at high energies Gross, Mende '87
- At high energies ($s \rightarrow \infty$ with scattering angle fixed), amplitudes simplify at every loop order. The integrand has the form

$$\int_{\mathcal{M}_{g,4}} C A^{-s} B^{-t}$$

- So the integral can be done via saddle point approximation:

$$\int_{\mathcal{M}_{g,4}} C A^{-s} B^{-t} \sim \sum_{\text{saddles}} \frac{C_* A_*^{-s} B_*^{-t}}{\sqrt{\det \text{Hess}}}$$

What are all the saddles?

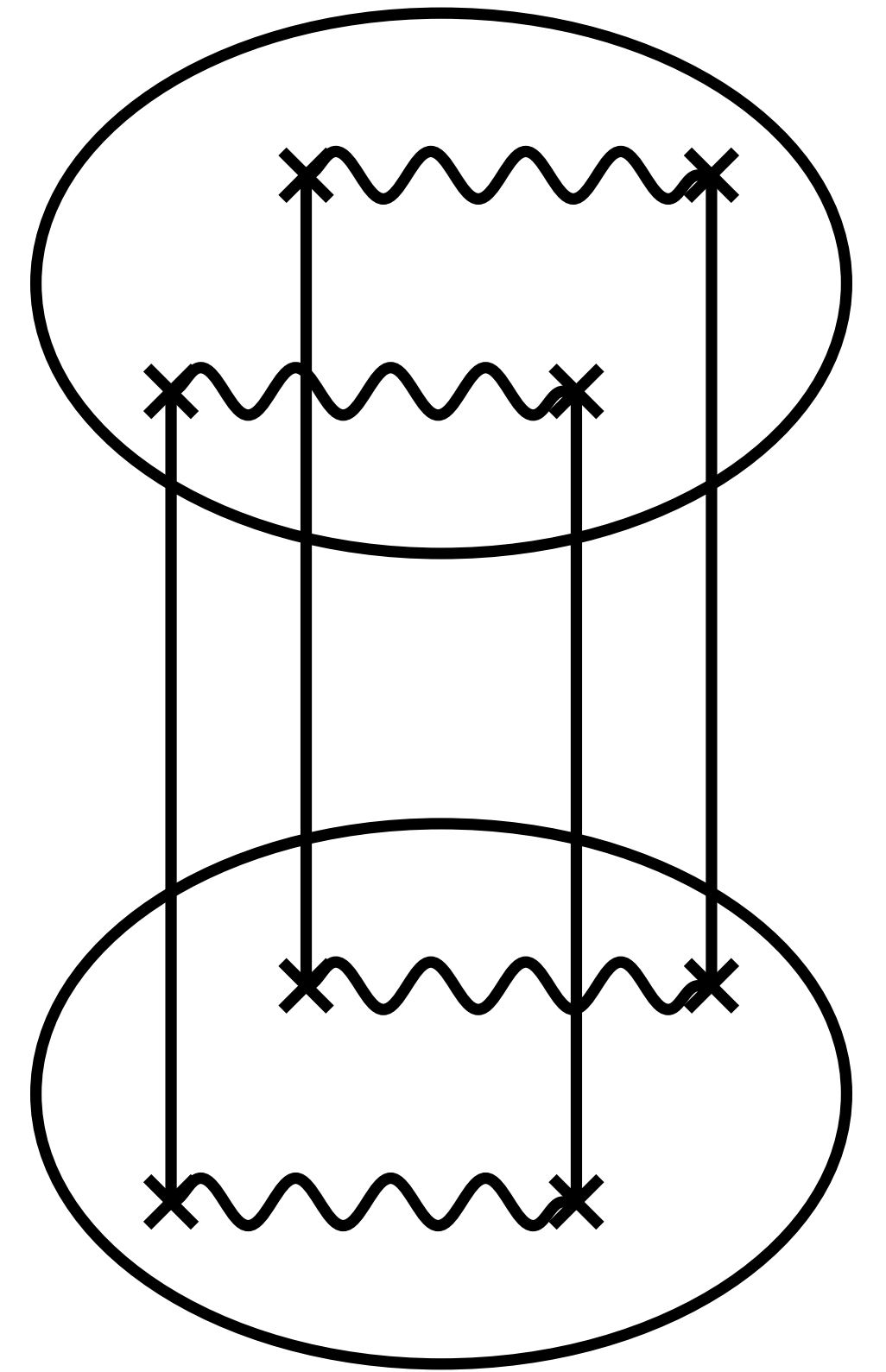
- Tree-level: $\partial_z \log(|z|^{-2s} |1 - z|^{-2t}) = 0 \implies z_* = -\frac{s}{u}$

- Gross and Mende identified a set of saddles in $\mathcal{M}_{g,4}$ obtained as a covering surface of the tree-level saddle

$$S_* = \frac{1}{g+1} S_{\text{tree}} = \frac{1}{g+1} (s \log(s) + t \log(-t) + u \log(-u))$$

$$A_g \sim s^{1-g} e^{-\frac{s}{g+1}} f(\theta)$$

- Spacetime perspective: worldsheet looks like a tree-level worldsheet wrapped $g+1$ times around itself

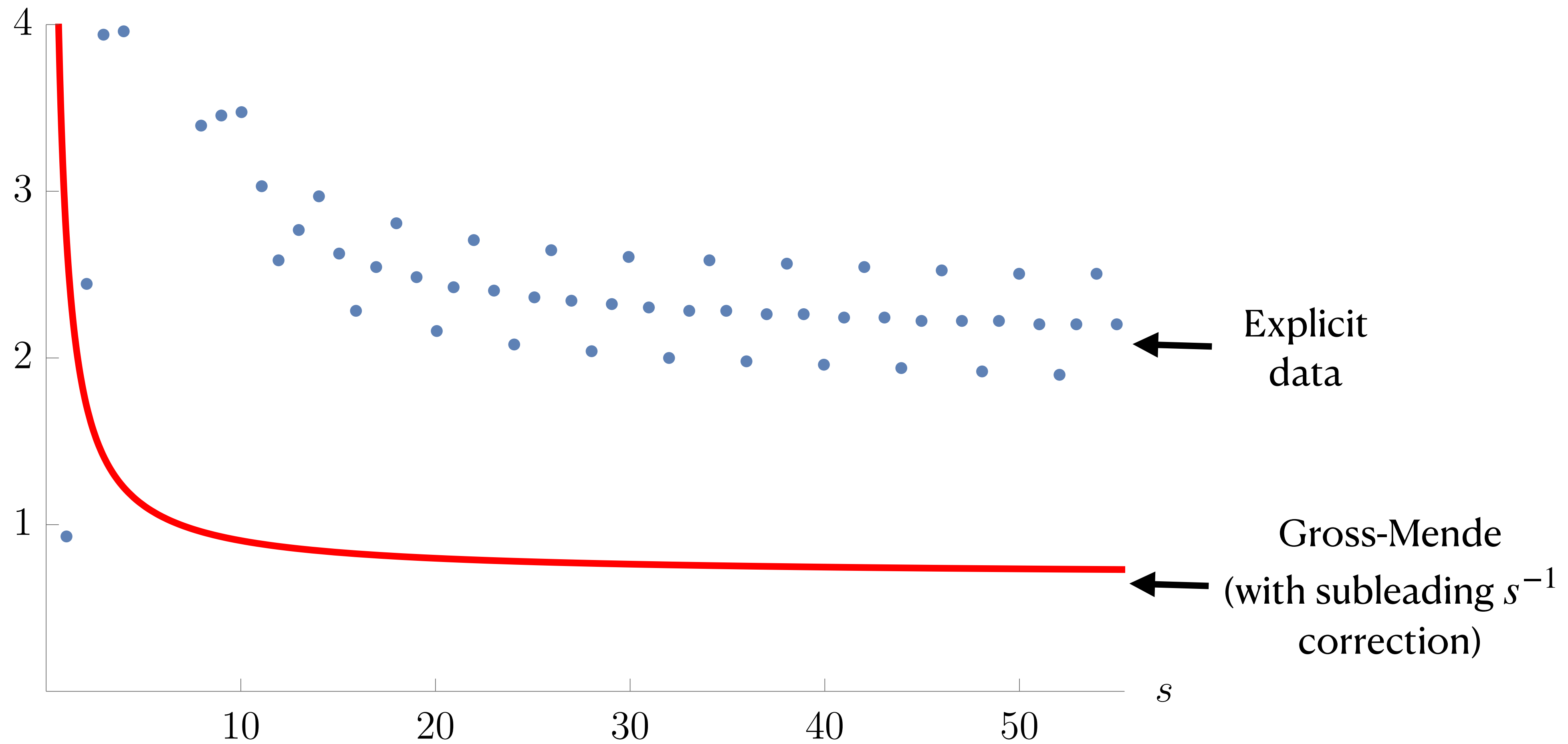


Challenges

- The contour is actually naturally defined in $\mathcal{M}_{g,4}^{\mathbb{C}}$, so we should also search for complex saddles
- Are these all the most dominant saddles?
 - Extensive numerical search at $g = 1$: yes, but
 - There are also saddles with the same $\text{Re } S_*$ not on $\mathcal{M}_{g,4}$: Lorentzian saddles
 - Unknown for $g \geq 2$
- What are their multiplicities?
- Computing the Hessian becomes very hard already at $g = 2$

Let's test it: imaginary part at 90 degrees

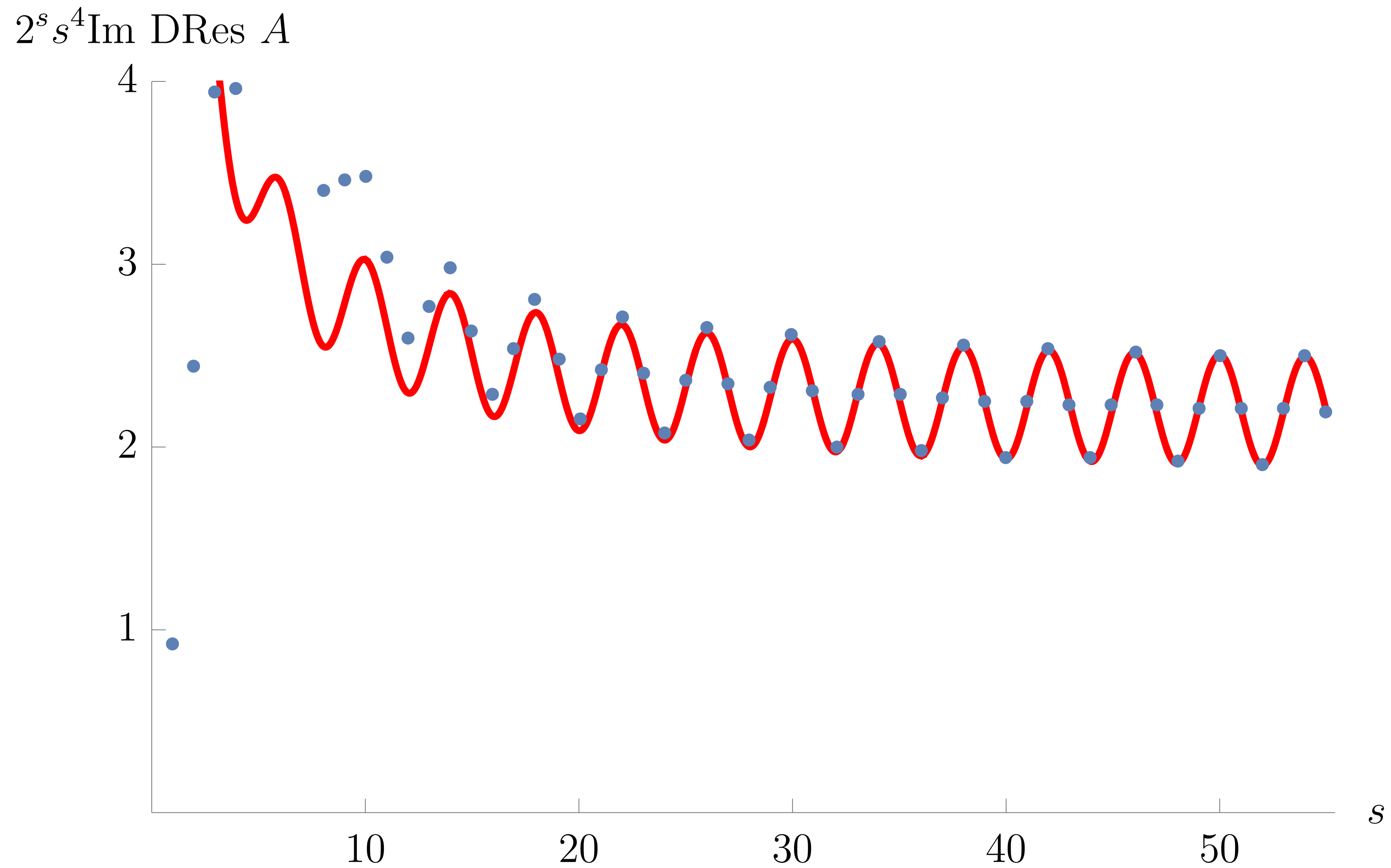
$2^s s^4 \text{Im DRes } A$



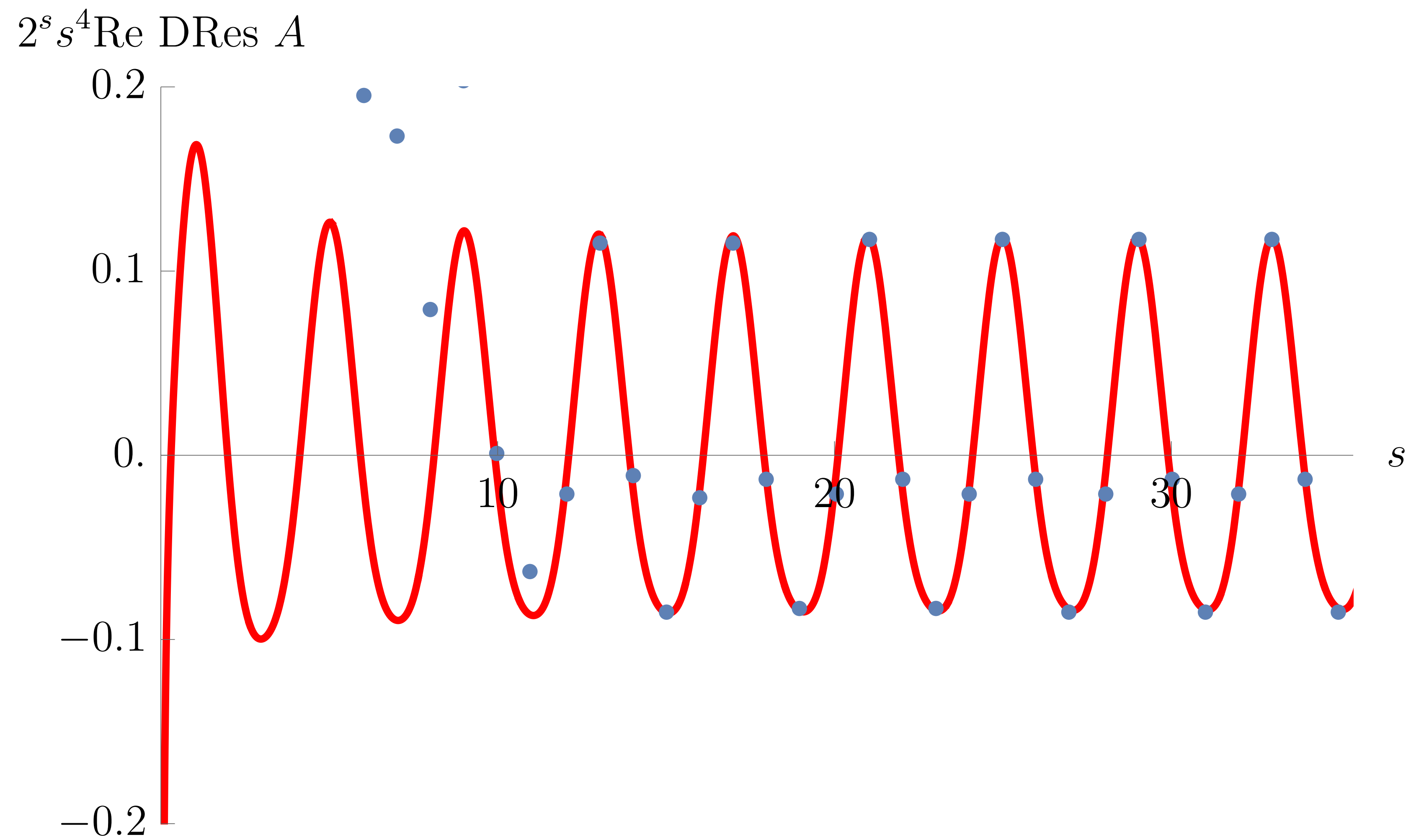
Lorentzian saddles contribute

- From the numerical data, we can conclusively see that Lorentzian saddles contribute
- Multiplicity of the Gross-Mende saddle = 3.
- The simplest has $\bar{\tau} = -\tau$ instead of them being complex conjugates.
- Including them gives a much better fit
- At the moment: saddle multiplicities are determined “experimentally”

With Lorentzian saddle



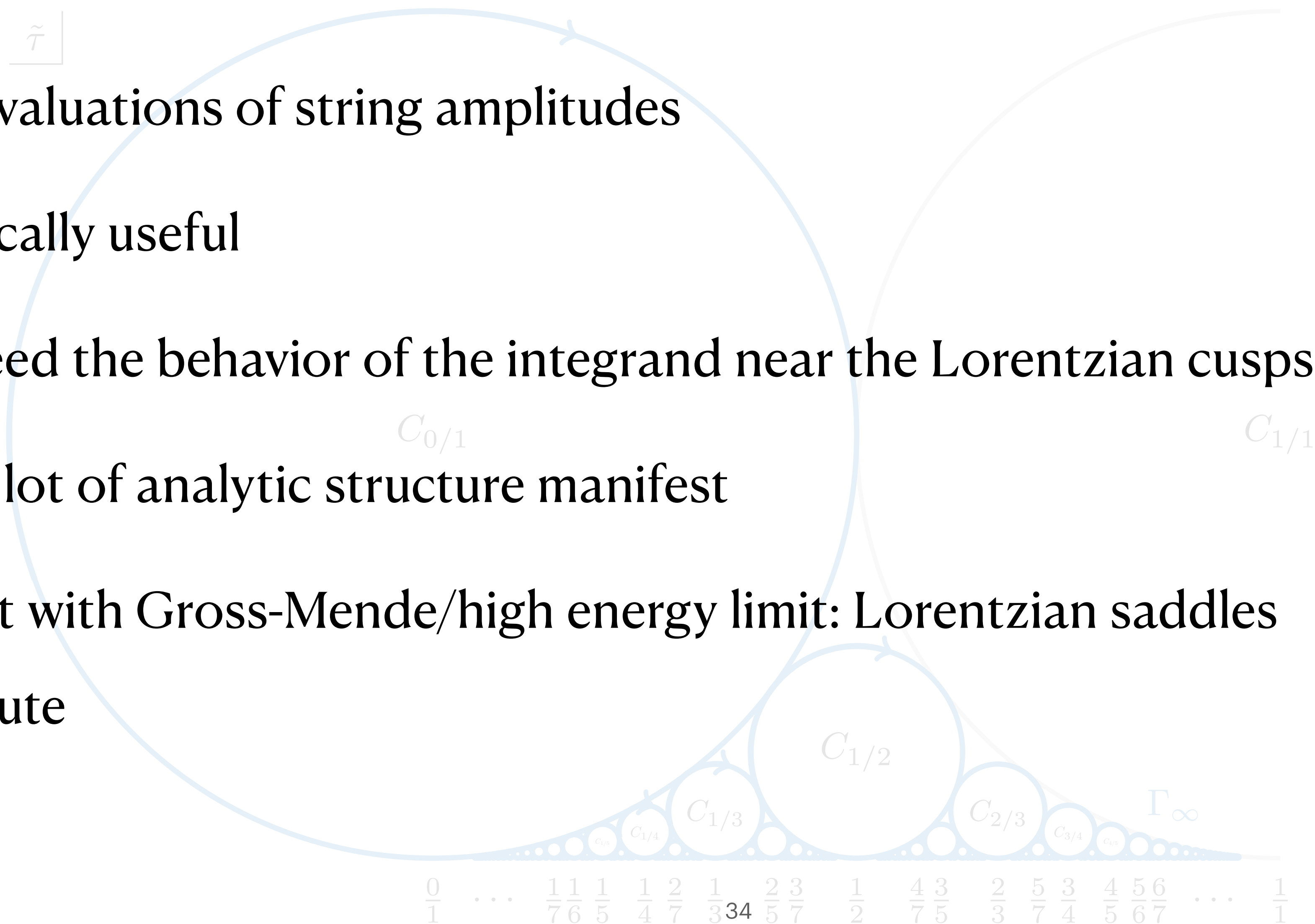
Works also for the real part!



Summary

Contour deformation methods are powerful!

- Exact evaluations of string amplitudes
- Numerically useful
- Only need the behavior of the integrand near the Lorentzian cusps
- Make a lot of analytic structure manifest
- Contact with Gross-Mende/high energy limit: Lorentzian saddles contribute

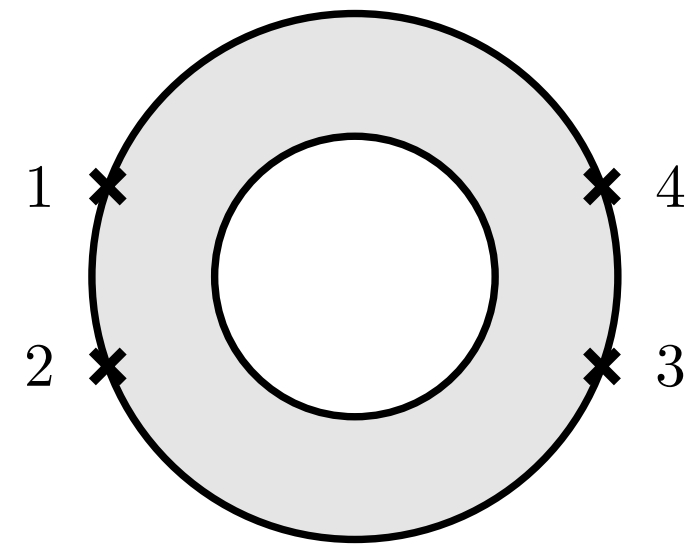


Thank you!

Backup slides

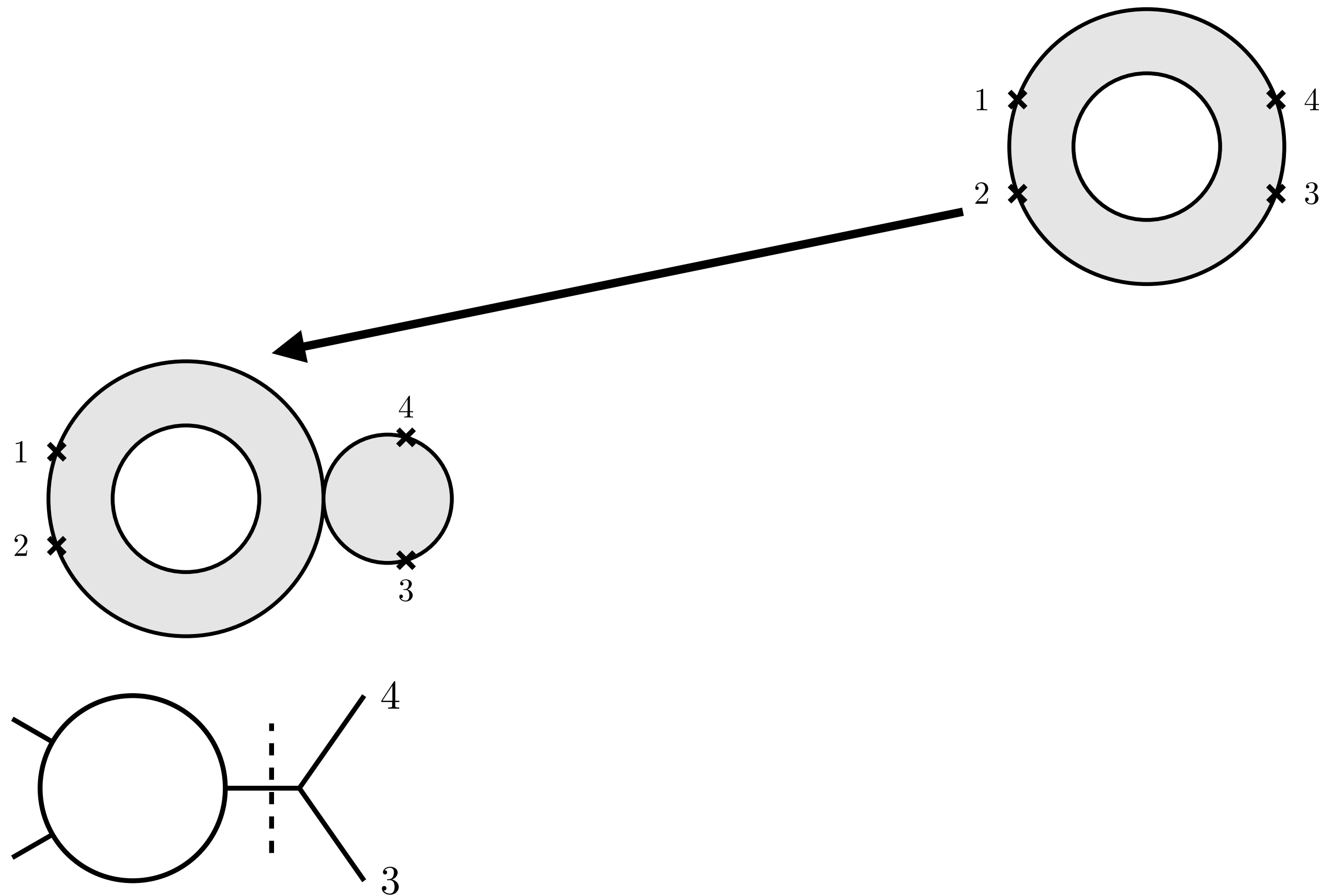
Degenerations of the worldsheet

- Modify contour of integration near a degeneration of the worldsheet



Degenerations of the worldsheet

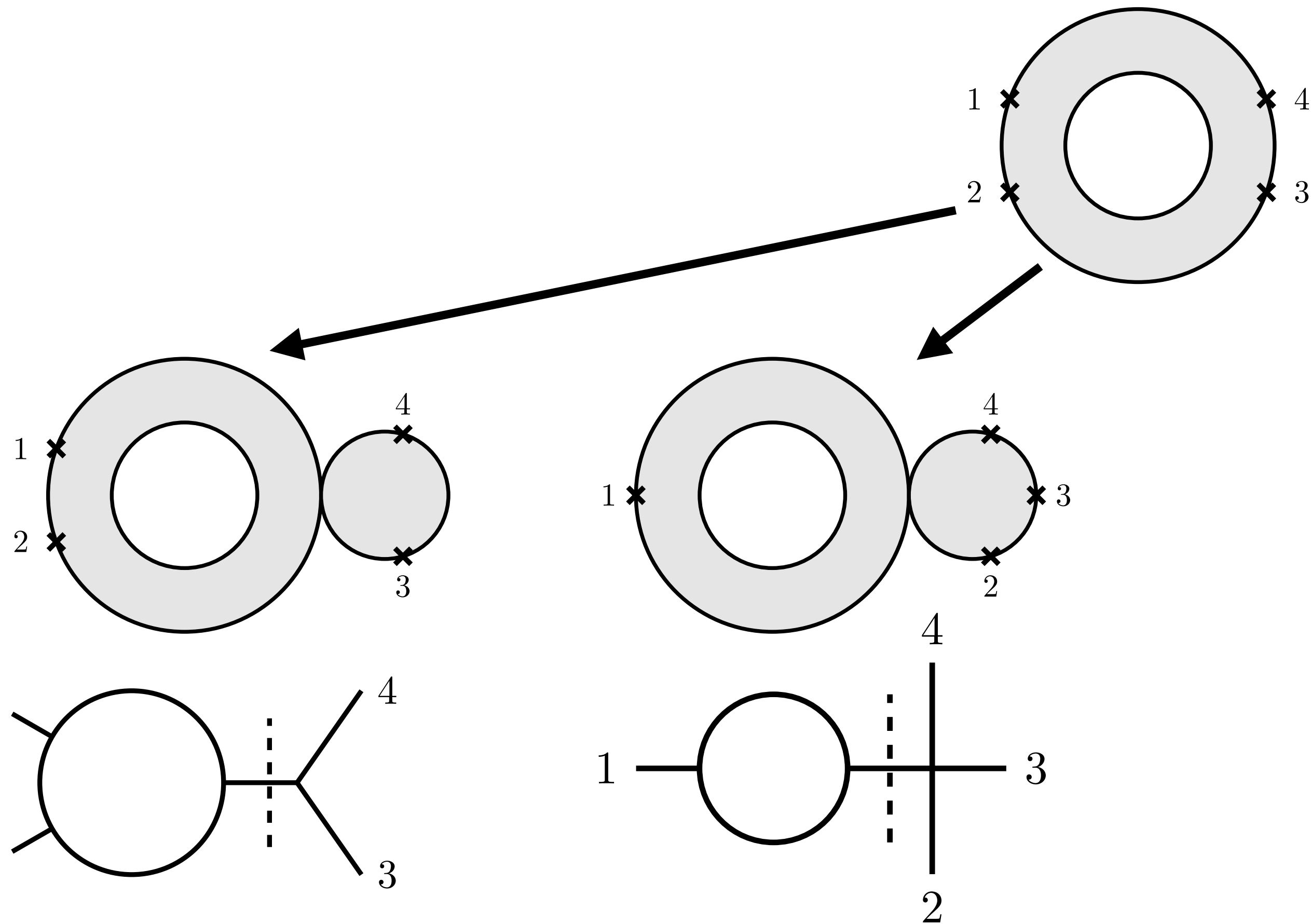
- Modify contour of integration near a degeneration of the worldsheet



Massive poles

Degenerations of the worldsheet

- Modify contour of integration near a degeneration of the worldsheet

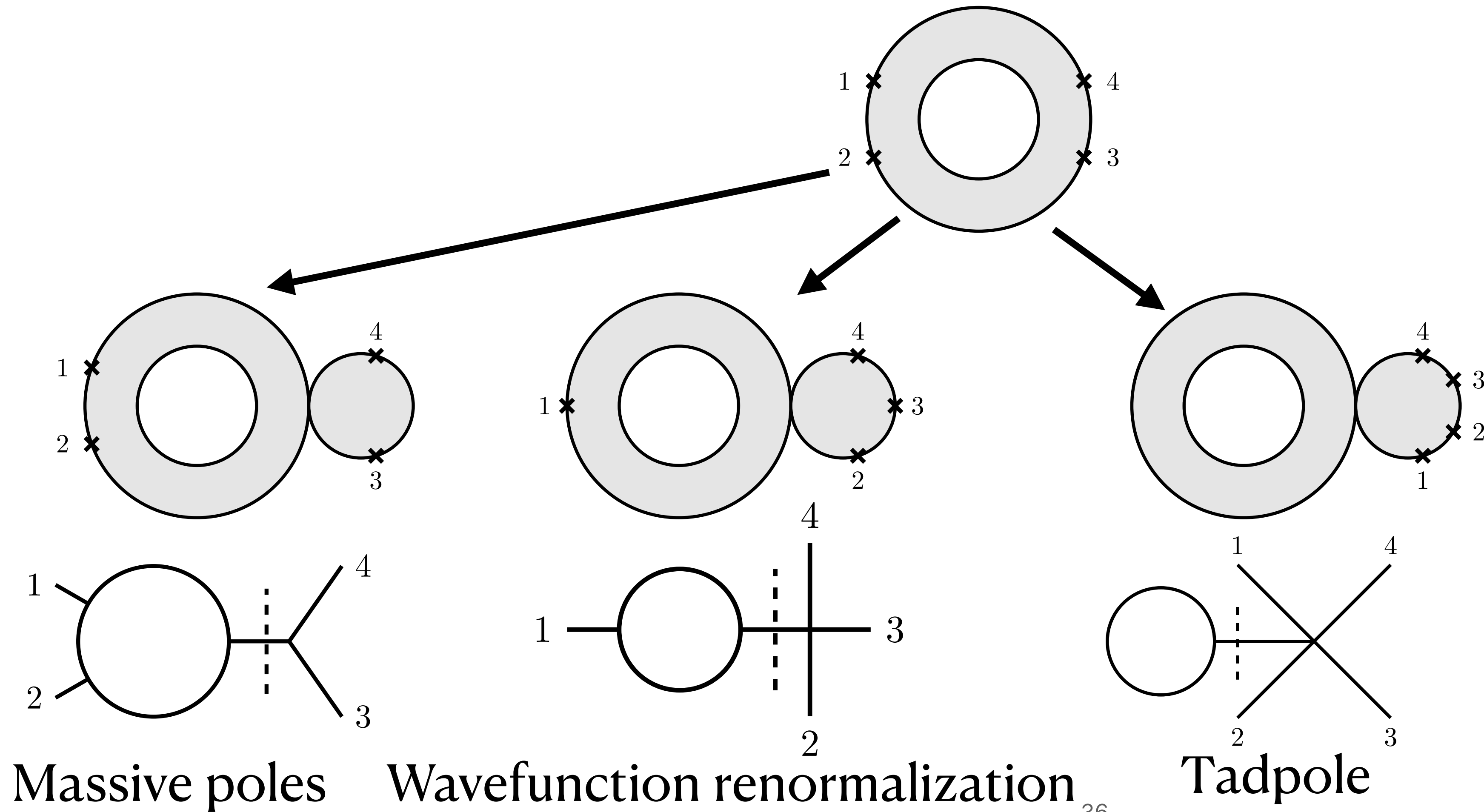


Massive poles

Wavefunction renormalization

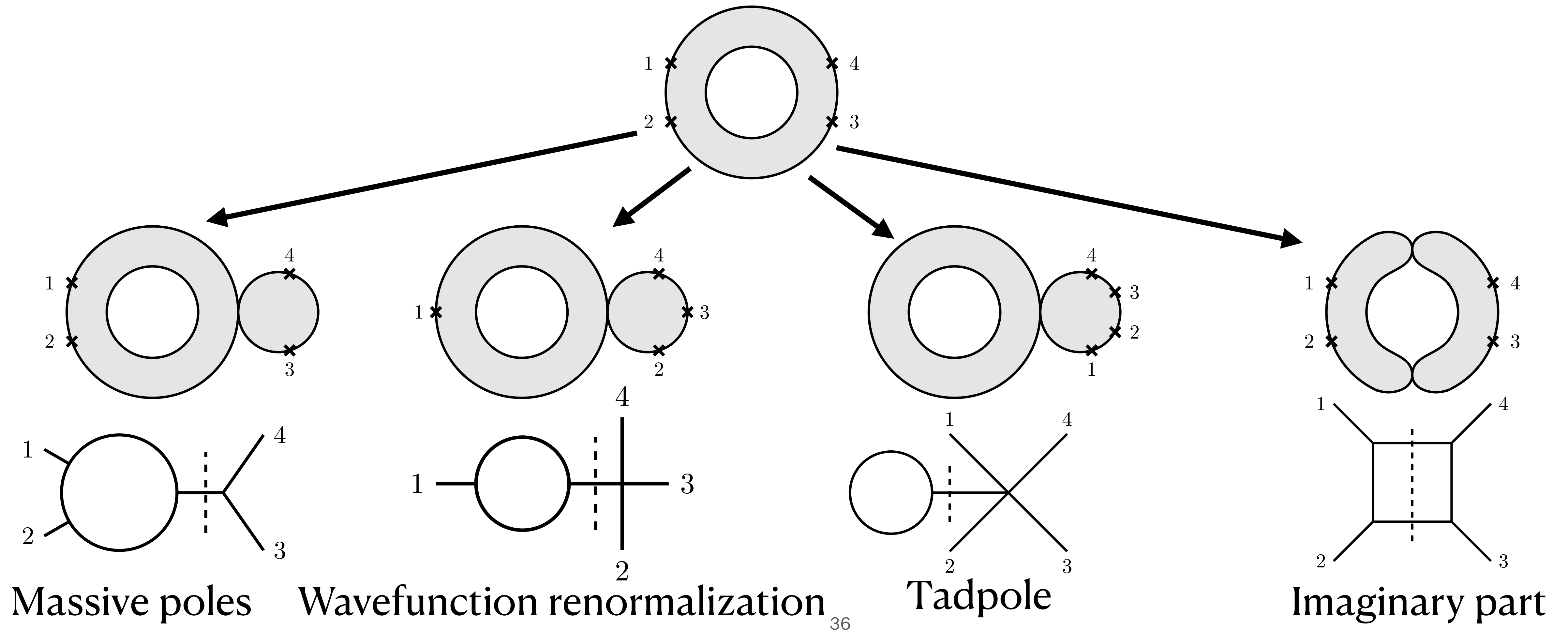
Degenerations of the worldsheet

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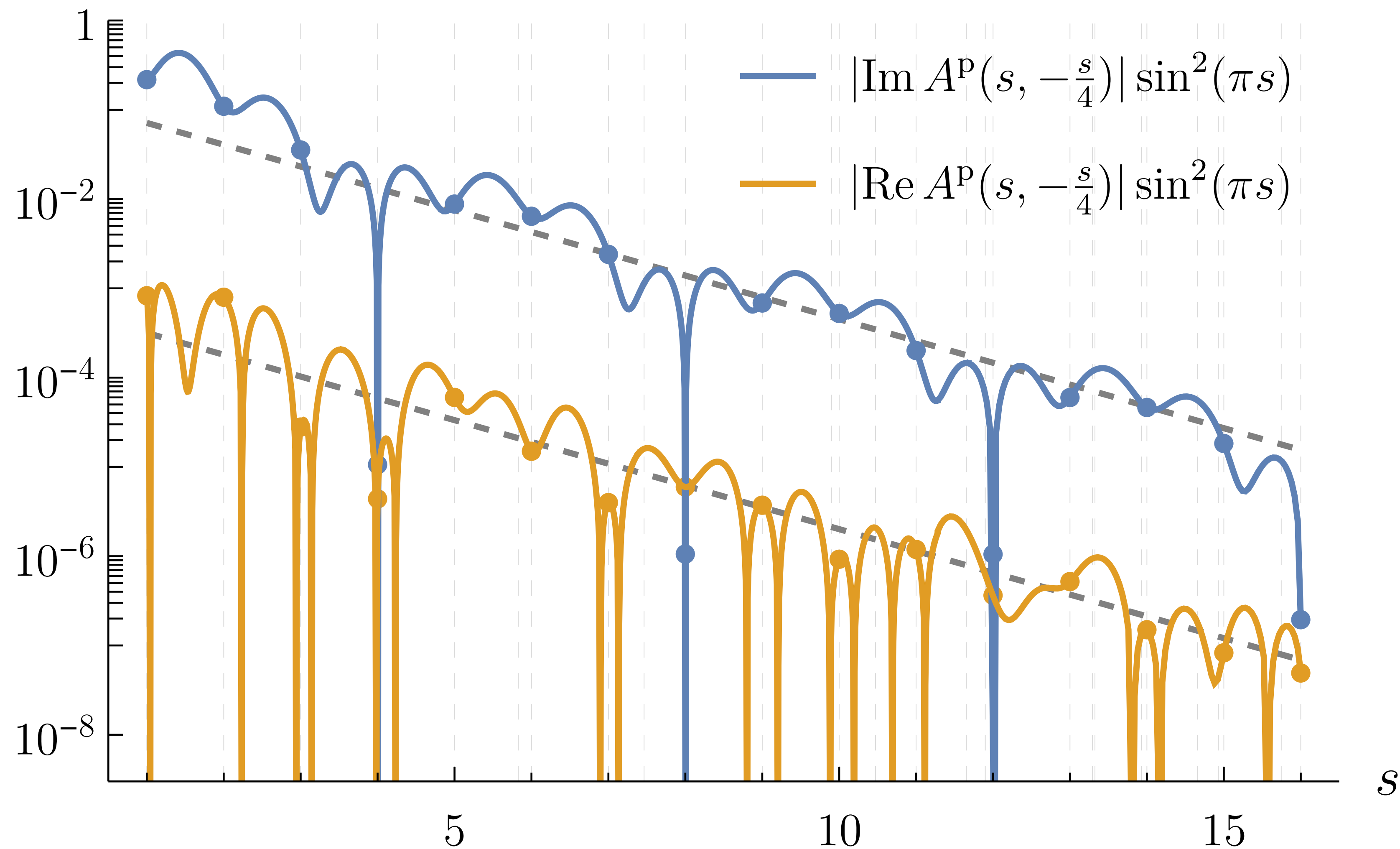


Degenerations of the worldsheet

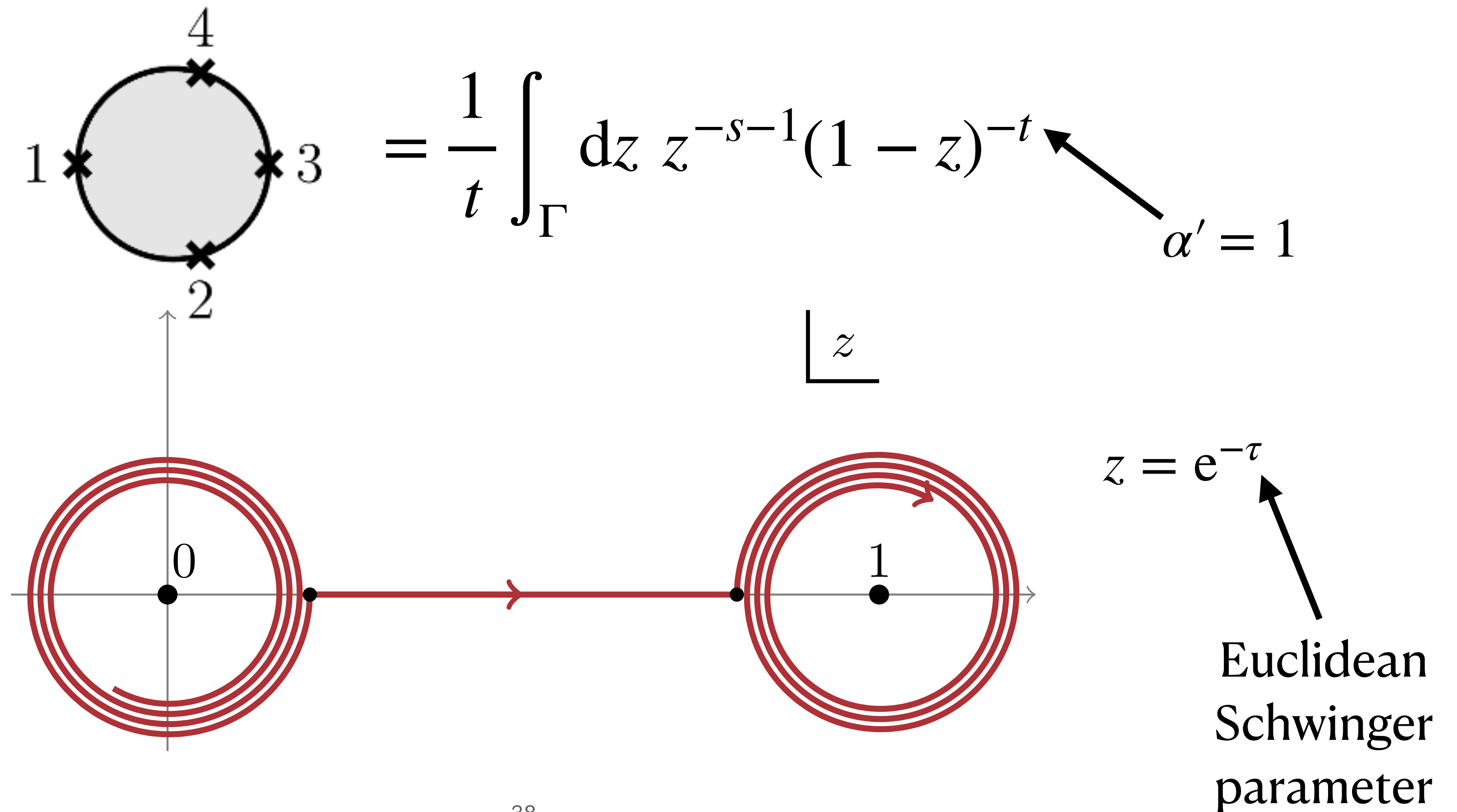
- Modify contour of integration near a degeneration of the worldsheet



Type I one-loop amplitude in 10d @ 60 degrees



The tree-level amplitude



Structure of a single term

Double copy structure:

$$\begin{aligned} \text{Re } A_{a/c}^{n_L, n_D, n_R, n_U} &\sim \frac{1}{stuc^{\frac{7}{2}}} \sum_{\sqrt{m_D} + \sqrt{m_U} \leq \sqrt{s}, \sqrt{\tilde{m}_D} + \sqrt{\tilde{m}_U} \leq \sqrt{s}} \int_{P>0} dt_L dt_R \int_{\tilde{P}>0} d\tilde{t}_L d\tilde{t}_R \mathcal{Q}_{m_D, m_U} \mathcal{Q}_{\tilde{m}_D, \tilde{m}_U} (P\tilde{P})^{\frac{5}{4}} \\ &\times \left[\sqrt{\frac{\tilde{P}}{P}} \left(J_{\frac{3}{2}} \left(\frac{4\pi\sqrt{P\tilde{P}}}{c} \right) - J_{\frac{7}{2}} \left(\frac{4\pi\sqrt{P\tilde{P}}}{c} \right) \right) + 12ic s(a, c) J_{\frac{5}{2}} \left(\frac{4\pi\sqrt{P\tilde{P}}}{c} \right) \right] A_{\text{massiveV-S}}(s, t_L) A_{\text{massiveV-S}}(s, t_R) \times \text{phases} \end{aligned}$$

Angular dependence

