Integral Identities from Symmetry Breaking of Conformal Defects

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Overview

- Defects are ubiquitous in physics and in theory:
  - Boundaries and interfaces
  - Impurities: the condo effect.
  - Wilson and ’t Hooft lines.
  - Cosmic strings.
  - $\ldots$
- Break spacetime and possibly also flavour symmetries (and SUSY).
- Defect is of dimension p in d dimensional spacetime.

Displacement and tilt

- QFT has special operators $T_{\mu\nu}$, $J_{\mu a}$ (and $j_{\mu\alpha}$) for these symmetries.
  - Correlation functions capture central charges and anomalies.
- They induce special operators constrained to defects

$$\partial^\mu T_{\mu\nu} = \delta^{d-p}(x_\perp) P^r_\nu \mathbb{D}_r \qquad \partial^\mu J_{\mu a} = \delta^{d-p}(x_\perp) P^i_a t_i$$

$\mathbb{D}_r$ and t_i are operators restricted to the defect.

Displacement and tilt

- Examples:
  - Wilson loop in $\mathcal{N}=4$ SYM:

$$\mathcal{W} = \mathrm{Tr}\,\mathcal{P}e^{\oint d\tau (iA_\mu \dot{x}^\mu + w^a \phi_a)} \quad \Rightarrow \quad \begin{cases} \mathbb{D}_r \approx iF_{r\mu} + w^a D_r \phi_a \\ t_i \approx |w|\phi_i \end{cases}$$

  - Magnetic line in $O(N)$ model:

$$\mathcal{W} = e^{\oint d\tau\, w^a \phi_a(x(\tau))} \quad \Rightarrow \quad \begin{cases} \mathbb{D}_r \approx w^a \partial_r \phi_a \\ t_i \approx |w|\phi_i \end{cases}$$

- More precisely

$$\mathbb{D}_r(\tau) = \frac{\delta \mathcal{W}}{\delta x^r(\tau)} \qquad t_i(\tau) = \frac{\delta \mathcal{W}}{\delta w^i(\tau)}$$

Main question

What are the consequences of the broken global symmetries on the defect?

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What are the consequences of the broken global symmetries on the defect?

These are captured by integrated correlators involving displacements and tilts!

Non-linearly Realized Global Symmetry

- Consider a defect $\mathcal{W}$ which breaks a global symmetry G down to a subgroup H. In the general case, we can describe the symmetry-breaking by $w \in G/H$. We expect that for any $g \in G$:

$$Z[w] = Z[L_g(w)]$$

  where

$$L_g : \mathfrak{h}^\perp \to \mathfrak{h}^\perp$$

  is the left action of G on G/H, expressed in terms of $\mathfrak{h}^\perp$.
- Crucially, L_g depends on the renormalization scheme. However, to get the scheme-independent properties, let us keep agnostic about the scheme for now.

Non-linearly Realized Global Symmetry

$$Z[w] = Z[L_g(w)]$$

- Enough to impose infinitesimally in g and expand in w to all orders. Let

$$L_{e^\lambda}(w) = w + \sum_{n=0}^{\infty} \frac{1}{n!} l_n(\lambda; w, \dots, w) + O(\lambda^2)$$

- We have

$$l_0(\lambda) = \lambda^\perp, \quad l_1(\lambda; w) = \text{scheme-dependent}, \ \cdots$$

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- Then for the flavour symmetry

$$\sum_{k=0}^{n} \binom{n}{k} \langle t(l_k(\lambda; w, \cdots, w)) t(w)^{n-k} \rangle_c = 0\,, \quad \forall n \geq 0$$

where $t(w) \equiv \int d^p\tau\, w^i(\tau) t_i(\tau)$.

Non-linearly Realized Global Symmetry

$$\sum_{k=0}^{n} \binom{n}{k} \langle t(l_k(\lambda; w, \cdots, w)) t(w)^{n-k} \rangle_c = 0$$

- For a given choice of the renormalization scheme these equations determine l_n with $n > 0$.
- Eliminating l_n from these equations one obtains non-trivial scheme-independent identities.
- Extra identities from l_n's having to represent the Lie algebra of G

$$[l^i(\lambda_1, w)\partial_i, l^j(\lambda_2, w)\partial_j] = -l^i([\lambda_1, \lambda_2], w)\partial_i$$

$$l(\lambda, w) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} l_n(\lambda; w, \cdots, w)$$

Non-linearly Realized Global Symmetry

$$\sum_{k=0}^{n} \binom{n}{k} \langle t(l_k(\lambda; w, \cdots, w)) t(w)^{n-k} \rangle_c = 0$$

- $n = 0$

$$\langle t(\lambda) \rangle = 0$$

Explicitly,

$$\lambda^i \int d^p \tau \langle t_i(\tau) \rangle = 0$$

Non-linearly Realized Global Symmetry

- $n = 1$

$$\langle t(\lambda) t(w) \rangle = 0$$

Explicitly,

$$\lambda^i \int d^p\tau_1 d^p\tau_2 w^j(\tau_2) \langle t_i(\tau_1) t_j(\tau_2) \rangle = 0$$

Taking $\partial/\partial\lambda$, $\partial/\partial w$,

$$\int d^p\tau_1 \langle t_i(\tau_1) t_j(\tau_2) \rangle = 0$$

For flat defects the two-point function has the form

$$\langle t_i(\tau_1) t_j(\tau_2) \rangle = \frac{g_{ij}}{|\tau_1 - \tau_2|^{2p}}$$

The integral identity is satisfied for any definition of $\langle tt \rangle$ at $\tau_1 = \tau_2$.

Non-linearly Realized Global Symmetry

- $n = 2$

$$\langle t(\lambda)t(w)^2\rangle_c + 2\langle t(l_1(\lambda; w)t(w)\rangle_c = 0$$

After taking variational derivatives

$$\int d^p\tau_1 \langle t_i(\tau_1)t_j(\tau_2)t_k(\tau_3)\rangle + l_1^m(e_i; e_j)\langle t_m(\tau_2)t_k(\tau_3)\rangle$$
$$+ l_1^m(e_i; e_k)\langle t_j(\tau_2)t_m(\tau_3)\rangle = 0$$

The three-point function takes the form

$$\langle t_i(\tau_1)t_j(\tau_2)t_k(\tau_3)\rangle = \frac{d_{ijk}}{|\tau_1 - \tau_2|^p|\tau_2 - \tau_3|^p|\tau_1 - \tau_3|^p}$$

Non-linearly Realized Global Symmetry

- $n = 2$

  One can check that for any definition at coincident points

  $$\int d^p\tau_1 \langle t_i(\tau_1) t_j(\tau_2) t_k(\tau_3)\rangle \ni d_{ijk} \log|\tau_2 - \tau_3| \times \cdots$$

  Therefore, $d_{ijk} = 0$. The rest of $n = 2$ equations with

  $$l_1(\lambda; w) - l_1(w; \lambda) = [\lambda, w]^\perp$$

  determines $l_1(w; \lambda)$

  $$l_1(\lambda_1; \lambda_2) = \frac{1}{2}\Big([\lambda_1, \lambda_2]^\perp - (\mathrm{ad}^\perp(\lambda_1))^* \lambda_2 - (\mathrm{ad}^\perp(\lambda_2))^* \lambda_1\Big)$$

Non-linearly Realized Global Symmetry

- $n = 3$

$$\langle t(\lambda) t(w)^3 \rangle_c + 3 \langle t(l_2(\lambda; w, w)) t(w) \rangle_c = 0$$

After taking derivatives we find

$$\begin{aligned} &\int d^p\tau_1 \, \langle t_i(\tau_1) t_j(\tau_2) t_k(\tau_3) t_l(\tau_4) \rangle_c \\ &+ \delta^p(\tau_3 - \tau_4) \langle t_j(\tau_2) t_m(\tau_4) \rangle l_2^m(e_i; e_k, e_l) + \cdots \\ &= 0 \end{aligned}$$

Setting $\tau_2 = 0, \tau_3 = 1, \tau_4 = \infty$, we get

$$\int d^p\tau \, \langle t_i(\tau) t_j(0) t_k(1) t_l(\infty) \rangle_c = 0$$

plus a determination of $l_2(\lambda; w, w) = \cdots$.

Application: 3d $\mathcal{N} = 4$ Chern-Simon-matter theories

- Analytic bootstrap determines the correlator at strong coupling up to two parameters [Pozzi, Trancanelli'24]

$$
\begin{aligned}
&\left\langle\!\left\langle t(\tau_1)\bar{t}(\tau_2)\bar{t}(\tau_3)t(\tau_4) \right\rangle\!\right\rangle_c \\
=&\frac{4\eta}{\tau_{12}^2\tau_{34}^2}\left(\chi - \tfrac{(3-\chi)\chi^2}{(1-\chi)}\log|\chi| + (1-\chi)^2\log|1-\chi|\right) \\
+&\frac{4\epsilon}{\tau_{12}^2\tau_{34}^2}\left(\tfrac{\chi}{2} - \chi^2 - 1 - \tfrac{(1-\chi)^2(\chi^2+\chi+1)}{\chi}\log|1-\chi| - \chi^2(1-\chi)\log|\chi|\right)
\end{aligned}
$$

- Explicit integration

$$
\int d\tau \left\langle\!\left\langle t(0)\bar{t}(\tau)\bar{t}(1)t(\infty) \right\rangle\!\right\rangle_c = 8\pi^2\eta
$$

- Conclusion $\Rightarrow \eta = 0$, so same answer as ABJM [Bianchi, Bliard, Forini, Griguolo, Seminara'20].

Non-linearly Realized Global Symmetry

Together with

$$
\begin{aligned}
& l_2(\lambda; w, w) - l_2(w; \lambda, w) \\
& = l_1([\lambda, w], w] - l_1(\lambda; l_1(w; w)) + l_1(w; l_1(\lambda, w))
\end{aligned}
$$

we recover the integrated identity for the curvature of G/H

$$
2 \int_{-\infty}^{+\infty} d\tau \, \log|\tau| \langle t_i(1) t_j(\tau) t_k(0) t_l(\infty) \rangle_c = R_{ijkl}
$$

[Kutasov'89] [Friedan, Konechny'12] [Drukker, Kong, Sakkas'22]

Non-linearly Realized Global Symmetry

- The equation

$$\sum_{k=0}^{n} \binom{n}{k} \langle t(l_k(\lambda; w, \cdots, w)) t(w)^{n-k} \rangle_c = 0$$

  produces new integral identities for $(n+1)$-point correlation functions for general n.
- We verified $n = 5$ identity for $\langle t^6 \rangle$ in $O(N)$ model to the leading order in ε-expansion.

Generalization

Basic (rough) statement

$$\int d^2\chi\, P(\chi) \log|\chi| \left\langle\!\left\langle \mathbb{O}_1(\hat{e})\mathbb{O}_2(\chi) O(0) O'(\infty) \right\rangle\!\right\rangle_c$$
$$= \left\langle\!\left\langle [Q_1, Q_2] \circ O(0)\, O'(\infty) \right\rangle\!\right\rangle$$

- $\mathbb{O}_{1,2}$ are operators arising from broken global symmetries $Q_{1,2}$: spacetime, flavour or also SUSY.
- $O(0)$ and $O'(\infty)$ are any operators on the defect.
- $P(\chi)$ is a polynomial.
- This is a very nontrivial relation between integrated 4-pt functions and 2-pt functions.

Broken Conformal Symmetry

- When Q are the broken conformal symmetry generators, the operators $\mathbb{O}$ arising from the symmetry-breaking are the displacement operator $\mathbb{D}_\mu(\tau)$.

$$R^n_{\mu\nu\rho\sigma} \equiv \frac{1}{2}\mathrm{vol}S^{p-1}\int d^p\tau \log|\tau| f_n(\tau)\langle \mathbb{D}_\rho(0)\mathbb{D}_\mu(\tau)\mathbb{D}_\nu(\hat{e}_1)\mathbb{D}_\sigma(\infty)\rangle_c,$$

$$\underset{p>1}{=} 2^{-p}\mathrm{vol}S^{p-1}\mathrm{vol}S^{p-2}$$

$$\int d^2z|z-\bar{z}|^{p-2}\log|z| f_n(z,\bar{z})\langle \mathbb{D}_\mu(0)\mathbb{D}_\mu(z,\bar{z})\mathbb{D}_\nu(1)\mathbb{D}_\sigma(\infty)\rangle_c$$

$$\underset{p=1}{=} \int d\tau \log|\tau| f_n(\tau)\langle \mathbb{D}_\mu(0)\mathbb{D}_\mu(\tau)\mathbb{D}_\nu(1)\mathbb{D}_\sigma(\infty)\rangle_c$$ [Gabai, Sever, Zhong'25]

$$f_n(\tau) \equiv \begin{cases} 1, & n=0 \\ \tau^1 = \frac{1}{2}(z+\bar{z}), & n=1 \\ \tau^2 = z\bar{z}, & n=2 \end{cases}$$

Anomalies

- The condition $Z[w] = Z[L_{e^\lambda}(w)]$ is too naive.
- Instead, we can consider a more general one

$$Z[w] = e^{\mathcal{A}[\lambda,w]+O(\lambda^2)} Z[L_{e^\lambda}(w)]$$

$$\mathcal{A}[\lambda, w] = \text{local functional of } w\text{, linear in } \lambda$$

- $\mathcal{A}$ defines a representation of $\mathfrak{g}$
  $\Rightarrow$ Wess-Zumino consistency conditions:

$$\pi(\lambda_1)\mathcal{A}[\lambda_2; w] - \pi(\lambda_2)\mathcal{A}[\lambda_1; w] - \mathcal{A}[[\lambda_1, \lambda_2]; w] = 0$$

  where $\pi(\lambda) = -\int d^p\tau\, l^i(\lambda; w)\frac{\delta}{\delta w^i(\tau)}$.

Example: Line defect in the dual of $AdS_3 \times S^3 \times T^4$

- Explicit bootstrap and holographic calculations give nonzero three-point functions for the tilts [Bliard, Correa, Lagares, Landea'24]

$$\langle\!\langle t_i(\tau_1) t_j(\tau_2) t_k(\tau_3) \rangle\!\rangle_c = \frac{\sin\lambda}{\sqrt{2\pi g}} \frac{\epsilon_{ijk}}{\tau_{12}\tau_{13}\tau_{23}}$$

- Here tilts arise from breaking $SU(2) \to 1$.
- One can consider an anomaly term:

$$\mathcal{A} = \int d\tau\, \epsilon_{ijk} \lambda^i w^j \partial_\tau w^k \left(\frac{12}{w^2} - 6 \frac{\cot \frac{|w|}{2}}{|w|} \right)$$

Example: Line defect in the dual of $AdS_3 \times S^3 \times T^4$

- Z is not a function of w but a section of a $\mathbb{C}$-line bundle over $LSU(2)$.
- $\mathbb{C}$-line bundles are classified by their Chern class $H^2(LSU(2), \mathbb{Z}) = \mathbb{Z}$
- Solution: $\frac{\sin\lambda}{\sqrt{2\pi g}} \in \frac{\mathbb{Z}}{16\pi^3}$.
- Consistent with flux quantisation of string theory on S^3.

Summary

As we are always told:
Symmetries are important.

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Symmetries are important.

Regular symmetries of QFT lead to nontrivial identities for correlation functions with tilts, displacements and broken SUSY operators.

Thank you!