Holographic duality from Howe duality: Chern-Simons gravity as an ensemble of code CFTs

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25 August 2025

Error-correcting codes

Modular forms

Narain CFT

Quantum information

Holography

Representation theory

Chern–Simons TQFT

Finite fields

1-form symmetry

Howe duality

Based on [2504.08724] with Brian McPeak and Anatoly Dymarsky

Overview

Recently, the factorisation puzzle in holography has motivated the idea of **ensemble averaging** [Saad, Shenker, Stanford 2019]. It is important to exhibit explicit instances of this proposal, and understand better why they work

Narain ensemble	Code CFT ensemble
[Afkhami-Jeddi et al 2020; Maloney, Witten 2020] $\int Z(\mu,\Omega)d\mu = \frac{\mathcal{E}_{\frac{n}{2},\frac{n}{2}}(\Omega)}{\Phi(\Omega)^n}$	
Continuous Narain moduli space, Zamolodchikov metric	
Infinite sum over modular images $\frac{\mathcal{E}_{\frac{n}{2},\frac{n}{2}}(\Omega)}{\Phi(\Omega)^n} = \sum_{\gamma\in\Gamma\backslash Sp(2g,\mathbb{Z})} \chi_{\text{vac}}^{U(1)}(\gamma\Omega)$	
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[Afkhami-Jeddi et al 2020; Maloney, Witten 2020] $\int Z(\mu,\Omega)d\mu = \frac{\mathcal{E}_{\frac{n}{2},\frac{n}{2}}(\Omega)}{\Phi(\Omega)^n}$	$\frac{1}{N}\sum_i Z_{\mathrm{CFT}_i}(\Omega) = \sum_{\gamma\in\Gamma\backslash Sp(2g)} \psi_{\vec{0}}(\gamma\Omega)$
Continuous Narain moduli space, Zamolodchikov metric	Discrete ensemble, equal weights
Infinite sum over modular images $\frac{\mathcal{E}_{\frac{n}{2},\frac{n}{2}}(\Omega)}{\Phi(\Omega)^n} = \sum_{\gamma\in\Gamma\backslash Sp(2g,\mathbb{Z})} \chi^{U(1)}_{\mathrm{vac}}(\gamma\Omega)$	Finite sum over modular images [JH, McPeak 2022; Aharony, Dymarsky, Shapere 2023]
Equality follows from Siegel–Weil formula	Equality follows from **Howe duality** [Dymarsky, JH, McPeak 2025]

Error-correcting codes

Code = collection of (binary/p-ary) vectors

E.g. {(00000000), (00001111), (00110011), (00111100),

(01010101), (01011010), (01100110), (01101001),

(10010110), (10011001), (10100101), (10101010),

(11000011), (11001100), (11110000), (11111111)}

Studied since the 1950's (Voyager mission, classification, good codes $\leftrightarrow$ large distance/"gap," Siegel modular forms, . . .)

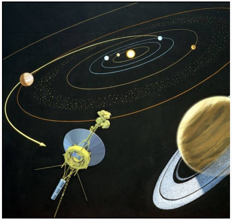


Image: NASA

Two important aspects:

- Codes can be used to construct CFTs [Dolan, Goddard, Montague 1990]

  Code $\to$ lattice $\to$ CFT

- Codes come in natural ensembles $\Rightarrow$ Averaging over codes [Pless, Sloane 1975]

CLASSIFICATION AND ENUMERATION OF SELF-DUAL CODES 319

4. THE SUM OF ALL WEIGHT ENUMERATORS

From the results of Section 3 it follows easily that (a) the sum of the weight enumerators of all self-dual codes of length n is (for n even)

$$\prod_{j=1}^{(n/2)-2} (2^j + 1) \cdot \left[2^{(n/2)-1}(1 + x^n) + \sum_{2|i} \binom{n}{i} x^i\right];$$

and (b) the corresponding sum when the weights are divisible by 4 is (for n divisible by 8)

$$\prod_{j=0}^{(n/2)-3} (2^j + 1) \cdot \left[2^{(n/2)-2}(1 + x^n) + \sum_{4|i} \binom{n}{i} x^i\right].$$

Code CFTs

Today consider codes over $\mathbb{F}_p$

Definition: $\mathcal{C} = \{v | v \in \mathbb{F}_p^{2n}\}$

Code defines a (Lorentzian) lattice by embedding $\mathbb{F}_p^{2n}$ in $\mathbb{Z}^{2n}$ $\Rightarrow$ Code CFTs are discrete points in Narain moduli space

Enumerator polynomial

$$\Psi_{\mathcal{C}} = \sum_{v \in \mathcal{C}} \psi_v \in \mathcal{H} = \mathrm{span}\left\{\psi_v | v \in \{0, \ldots, p-1\}^{2n}\right\}, \quad \dim \mathcal{H} = p^{2n}$$

becomes CFT partition function under the substitution

$$\psi_v \mapsto \Psi_v(\tau, \xi, \bar{\xi}) = \frac{1}{|\eta(\tau)|^2} \sum_{n,m} e^{i\pi(\tau p_L^2 - \bar{\tau} p_R^2) + 2\pi i (p_L \xi - p_R \bar{\xi}) + \frac{\pi}{2\tau_2}(\xi^2 + \bar{\xi}^2)}$$

Ψ_v non-holomorphic conformal blocks of abelian CS theory [Gukov et al 2004]

$$p_{L,R} = \sqrt{\frac{p}{2}} \left(\frac{1}{r}(n + \alpha/p) \pm r(m + \beta/p)\right), \quad v = (\alpha, \beta)$$

Ensembles of codes

Number of even self-dual codes grows with n and p

n	# codes ($p = 3$)	# codes ($p = 5$)
1	2	2
2	8	12
3	80	312
4	2240	39312
5	183680	24609312
6	44817920	76928709312
7	32717081600	1202088011709312
8	71584974540800	939143280028017093l2
9	469740602936729600	36685378290422420501709312
10	9246374028206585446400	71651166158859580464821501709312

At small n, p, possible to list all even self-dual codes and compute average explicitly

Example: $n = 1$, $p = 3$:

$$\Psi_{\mathcal{C}_1} = \psi_{00} + \psi_{01} + \psi_{02}, \qquad \Psi_{\mathcal{C}_2} = \psi_{00} + \psi_{10} + \psi_{20}$$

Average: $\overline{\Psi} = \psi_{00} + \frac{1}{2}(\psi_{01} + \psi_{02} + \psi_{10} + \psi_{20})$

Next: recover this by a "holographic" computation

Averaging over codes

Code average formulas can be rewritten as a sum of modular images of $\psi_{\vec{0}}$

[JH, McPeak 2022; Aharony, Dymarsky, Shapere 2023]

Modular invariance [MacWilliams 1962]: $\Psi_{\mathcal{C}}$ invariant under

$$S: \ \psi_v \mapsto \frac{1}{p^n} \sum_{v'} e^{-\frac{2\pi i}{p} v\eta v'} \psi_{v'} \qquad T: \ \psi_v \mapsto e^{\frac{i\pi}{p} v\eta v} \psi_v \qquad \eta = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$$

Example $n=1$, $p=3$. Seed ψ_{00}

$$\begin{aligned}
\overline{\Psi} &= \frac{1}{2}\left(\psi_{00} + S\psi_{00} + TS\psi_{00} + T^2 S\psi_{00}\right) \\
&= \frac{1}{2}\Big(\psi_{00} + \frac{1}{3}[\psi_{00} + \psi_{01} + \psi_{02} + \psi_{10} + \psi_{11} + \psi_{12} + \psi_{20} + \psi_{21} + \psi_{22}] \\
&\quad + \frac{1}{3}[\psi_{00} + \psi_{01} + \psi_{02} + \psi_{10} + e^{2\pi i/3}\psi_{11} + e^{4\pi i/3}\psi_{12} + \psi_{20} + e^{4\pi i/3}\psi_{21} + e^{2\pi i/3}\psi_{22}] \\
&\quad + \frac{1}{3}[\psi_{00} + \psi_{01} + \psi_{02} + \psi_{10} + e^{4\pi i/3}\psi_{11} + e^{2\pi i/3}\psi_{12} + \psi_{20} + e^{2\pi i/3}\psi_{21} + e^{4\pi i/3}\psi_{22}]\Big) \\
&= \psi_{00} + \frac{1}{2}(\psi_{01} + \psi_{02} + \psi_{10} + \psi_{20}) = \frac{1}{2}(\Psi_1 + \Psi_2)
\end{aligned}$$

At genus g: $Sp(2g)$ images of $\psi_{\vec{0},\dots,\vec{0}}$ (40 terms at $g=2$)

Average at any length and genus

By explicit computation, we can check

$$\text{Ensemble average} = \text{sum over modular images}$$

case by case in n, p and g. But can we prove it in general?

- Everything takes place inside finite-dimensional vector space
$$\mathcal{H}^{(g)} = \text{span}\left\{\psi_{v_1 \cdots v_g} \,\middle|\, v_i \in \{0, \ldots, p-1\}^{2n}\right\}, \qquad \dim \mathcal{H}^{(g)} = p^{2gn}$$
- Two finite groups act on $\mathcal{H}^{(g)}$
  - $Sp(2g, \mathbb{F}_p)$ modular transformations
  - $O(n, n, \mathbb{F}_p)$: $\psi_{v_1, \ldots v_g} \mapsto \psi_{hv_1, \ldots hv_g}$. Acts transitively on the set of codes

Rewrite equality in a more symmetric form

$$\frac{1}{N'} \sum_{h \in O(n,n)} U_h \Psi_{\mathcal{C}_0} \stackrel{?}{=} \sum_{\gamma \in Sp(2g)} U_\gamma \psi_{\vec{0}}$$

If we can show that there is a unique one-dimensional subspace of $\mathcal{H}^{(g)}$ invariant under $Sp \times O$, then it follows that the two sides are proportional

Howe duality

Howe duality/theta correspondence [Howe 1973, ...] explains certain structure of paired representations. Ingredients:

- Reductive dual pair $G \times H \subset \mathbf{G}$ mutually centralising. For us $Sp(2g) \times O(2n) \subset Sp(4gn)$
- ω oscillator representation: a certain "minimal" representation of $Sp(4gn)$: For us $\omega \cong \mathcal{H}^{(g)}$

Central statement: *Irreps parametrised by joint label*

$$\omega|_{G\times H} = \sum_i (\rho_i)_G \otimes (\pi_i)_H$$

Similar to Schur–Weyl duality

$$(\mathbb{C}^n)^{\otimes N} = \sum_\lambda (r_\lambda)_{S_N} \otimes (s_\lambda)_{GL_n}$$

Example: $\mathbb{C}^n \otimes \mathbb{C}^n = S^2 + \Lambda^2, \qquad M_{\mu\nu} = M_{(\mu\nu)} + M_{[\mu\nu]}$

Outline of proof

We would like to prove

$$\frac{1}{N'} \sum_{h \in O(n,n)} U_h \Psi_{\mathcal{C}_0} \overset{?}{=} \sum_{\gamma \in Sp(2g)} U_\gamma \psi_{\vec{0}} \tag{1}$$

Let us consider Howe duality with $H = O(n, n, \mathbb{F}_p)$, $G = Sp(2g, \mathbb{F}_p)$ and $\omega \cong \mathcal{H}^{(g)}$

$$\omega|_{Sp \times O} = \sum_i (\rho_i)_{Sp} \otimes (\pi_i)_O = \mathbf{1}_{Sp} \otimes \mathbf{1}_O + (\text{non-invariant terms}) \tag{2}$$

$\Rightarrow$ One-dimensional vector space invariant under $Sp \times O$, must contain both sides of (1)

Precise statements require **Howe duality over finite fields**. Structure of (2) follows from results conjectured in [Aubert, Michel, Rouquier 1996] and proven in [Pan 2024; Ma, Qiu, Zou 2024].

Vol. 83, No. 2 DUKE MATHEMATICAL JOURNAL © May 1996

CORRESPONDANCE DE HOWE POUR LES GROUPES RÉDUCTIFS SUR LES CORPS FINIS

ANNE-MARIE AUBERT, JEAN MICHEL ET RAPHAËL ROUQUIER

Soit (G, G') une paire de sous-groupes d'un groupe symplectique $\mathbf{Sp}_{2n}(\mathbb{F}_q)$ (où $\mathbb{F}_q$ est un corps fini de caractéristique p impaire) dont chacun est le centralisateur de l'autre dans $\mathbf{Sp}_{2n}(\mathbb{F}_q)$. Suivant R. Howe (voir [H1] et [H2]), il y a une correspondance entre représentations de G et de G' donnée par la représentation de Weil ω du groupe symplectique $\mathbf{Sp}_{2n}(q) := \mathbf{Sp}_{2n}(\mathbb{F}_q)$.

Plus précisément, si $(G_m, G'_{m'})$ est une paire réductive duale irréductible dans $\mathbf{Sp}_{2n}(q)$, i.e., l'une des paires $(\mathbf{Sp}_{2m}(q), \mathbf{O}^{\pm}_{2m'}(q))$ (avec $n = 2mm'$), $(\mathbf{Sp}_{2m}(q), \mathbf{O}_{2m'+1}(q))$ (avec $n = m(2m'+1)$), $(\mathbf{U}_m(q), \mathbf{U}_{m'}(q))$, $(\mathbf{GL}_m(q), \mathbf{GL}_{m'}(q))$ (avec $n = mm'$), la restriction de ω à $G_m.G'_{m'}$ définit une application entre groupes de Grothendieck de représentations complexes $\mathscr{R}(G_m) \to \mathscr{R}(G'_{m'})$ que nous appellerons correspondance de Howe.

Intermittent summary

- We have investigated "holography by averaging" following the Narain ensemble blueprint:

$$\text{ensemble average} = \text{sum over modular images}$$

- We exhibited explicit instances using the ensemble of error-correcting codes
- We proved equality using Howe duality over finite fields

$$\mathcal{H}^{(g)} = \mathbf{1}_{Sp} \otimes \mathbf{1}_{O} + \dots$$

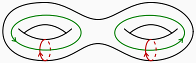
$\Rightarrow$ *both sides of equality must lie in one-dimensional vector space invariant under $Sp \times O$*

Next: What does this have to do with "Chern–Simons gravity"?

Interpretation as CS gravity

How to construct Chern–Simons gravity?

- Consider level-p abelian CS theory on B, $\partial B = \Sigma_g$
- Hilbert space $\mathcal{H}^{(g)} = \text{span}\, \Psi_{v_1,\dots v_g}$, v_i charges of Wilson lines on non-contractible cycles



- This TQFT has a global $\mathbb{Z}_p \times \mathbb{Z}_p$ 1-form symmetry carried by the Wilson lines
- "No global symmetry:" gauge away symmetry (maximal non-anomalous subgroup) [Benini, Copetti, Di Pietro 2022]. $\Psi = \sum_v \Psi_v$

Maximal non-anomalous subgroup $\leftrightarrow$ even self-dual code $\mathcal{C}$

- No unique choice: Average over all gaugings/codes with equal weight
$$\frac{1}{N}\sum_{\mathcal{C}} \Psi_{\mathcal{C}}$$ [Barbar, Dymarsky, Shapere 2023]

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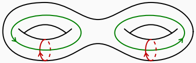

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Alternatively, one could follow the logic

- Gravity implies a sum over topologies
- Sum over all B with $\partial B = \Sigma_g$ (handlebodies)
- Sum reorganises to sum over modular images of fixed topology (c.f. [Maloney, Witten 2007])

$$\sum_{\gamma} Z_{B_0}(\gamma\Omega)$$

$$Z_{B_0}(\Omega) = \Psi_{\vec{0},\ldots\vec{0}}(\Omega)$$

Summary and outlook

Summary

- We have seen an explicit instance of holography by averaging

$$\text{ensemble average} = \text{sum over modular images} \tag{1}$$

- The ensemble arises naturally by gauging maximal subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ 1-form symmetry of abelian CS theory

$$\text{sum over gaugings} = \text{sum over topologies}$$

  Thus (1) represents equality of two dual pictures of "Chern–Simons gravity"

- We proved (1) using Howe duality over finite fields

$$\mathcal{H}^{(g)} = \mathbf{1}_{Sp} \otimes \mathbf{1}_{O} + \dots$$

Outlook

- Different manifestations of Howe duality: Siegel–Weil formulas, quantum information (ω=qudit Hilbert space), spherical potential $\psi_{nlm} = R_{nl}(r) Y_{lm}(\theta, \phi)$
  [Ashwinkumar et al 2021], [Gross, Nezami, Walter 2017], [Rowe et al 2012; Basile et al 2020]
- Study other observables in ensemble averaging

$$\langle \mathcal{O}_i \cdots \rangle_{\text{theories containing } \mathcal{O}_i} \stackrel{?}{=} \frac{\delta}{\delta \xi_i} \cdots \sum_{\gamma} U_\gamma(\Psi_{vac} + \Psi_{O_i})$$

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Thank you for your attention!

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Howe duality