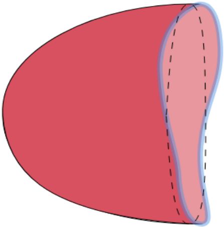
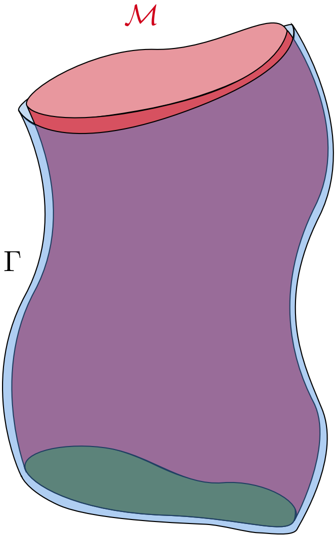
Finite boundaries in (Euclidean) gravity

Andrew Svesko
Department of Mathematics
King's College London



Eurostrings 2025, Nordita

[hep-th/2504.14003] – Galante, **Maneerat**, Svesko
JHEP 01 (2025) 120 – Aguilar-Gutierrez, Svesko, Visser
JHEP 01 (2023) 054 – Banihashemi, Jacobson, Svesko, Visser
JHEP 08 (2022) 075 – Svesko, Verheijden, Verlinde, Visser



Phenomenology

- Numerical relativity
- Membrane paradigm

Thermodynamics

- worldline observers
- dS thermo
- BH thermo

Holography

- Renormalization
- Braneworlds
- $T\overline{T}$

$$\textbf{(1)}\ \text{Dirichlet: Fix} \quad h_{\mu\nu}|_\Gamma$$

$$\textbf{(2)}\ \text{Neumann: Fix} \quad K_{\mu\nu}|_\Gamma$$

Generically, ill-posed in (3+1)-dimensional GR

- Euclidean (Anderson, '09; Witten, '18)
- Lorentzian (An, Anderson, '21)
- Λ-independent (Anninos, Galante, Maneerat, '23, '24)
- Special cases allowed (An, Anderson, '25)

$$\textbf{(3)}\ \text{Conformal: Fix}$$

$$[h_{\mu\nu}]|_\Gamma \quad \text{and} \quad K|_\Gamma$$

- 1-parameter extension (Liu, Santos, Wiseman, '24)

Finite boundaries in Euclidean gravity

- **Dirichlet thermodynamics**
  - dS thermodynamics defined
  - Minus sign in the first law

$$\boxed{\delta E_{\mathrm{BY}} = T_{\mathrm{Tol}}\delta S - P\delta A_{\mathcal{S}}}$$

- **Conformal thermodynamics**
  - Thermally stable cosmological systems
  - New universal scalings near-extremality
  - 2D description $\neq$ JT-gravity; effective 1D action

$$\boxed{I_{\mathrm{bdy}} = -\frac{1}{8\pi}\int_{\partial\mathfrak{m}} du \left[\mathrm{Sch}(t(u),u) - \frac{\delta K}{2}\Gamma'(u)^2 - \mathrm{Sch}(\Gamma(u),u)\right]}$$

Schwarzian coupled to 1D quantum gravity

- **Outlook**

Related developments

- Timelike boundaries in GR Brown, York...Whiting; Friedrich, Nagy; Anderson; Bredberg, Strominger; Anninos, et al; Andrade, et al; Sarbach,Tiglio; Marolf, Rangamani; Fournadavlos, Smulevici; An, Anderson; Witten; Odak, Speziale; Anninos, Galante, Maneerat; Liu, Santos, Wiseman; Liu, Reall, Santos, Wiseman
- $T\overline{T}$ holography, multi-trace deformations Aharony, Berkooz, Silverstein; Witten; Smirnov, Zamolodchikov; Cavaglia et al; Dubovsky et al; Freidel; McGough, Mezei, Verlinde; Kraus, Liu, Marolf; Taylor; Hartman et al; Dong et al, De Luca, Silverstein, Torroba; Guica; Gross et al; Shyam; Aguilar-Gutierrez, Svesko, Visser; Georgescu et al; Parvizi et al...
- Thermodynamics/statistics of dS Anninos, Hartnoll, Hofman; Anninos, Anous, Bredberg, Ng; Svesko, Verheijden, Verlinde, Visser; Draper; Farkas; Banihashemi, Jacobson; Banihashemi, Jacobson, Svesko, Visser; Susskind; Shaghoulian; Anninos, Harris; Anninos, Galante, Maneerat; Maldacena; Banihashemi, Shaghoulian, Shashi; Galante, Maneerat, Svesko; Philcox, Silverstein; Torroba...
- Microscopic models, observers/algebras Banks et al; Anninos et al, Narovlansky, Verlinde; Susskind, Rahman; Chandrasekaran et al, Bahiru et al; Cotler, Jensen; Antonini et al, Collier et al; Trivedi et al; Ivo et al...

Dirichlet thermodynamics

Black hole thermodynamics

Bekenstein proposed BHs have entropy. Hawking found they radiate

$$\boxed{S_{\text{BH}} = \frac{A}{4G} \sim M^2 \,, \quad T_{\text{H}} = \frac{\kappa}{2\pi} \sim M^{-1}}$$

- Euc. PI derivation (Gibbons-Hawking '77)

$$\mathcal{Z}_{\text{can}}(\beta) \approx e^{-I_{\text{E}}^{\text{on-shell}}} \,, \quad I_{\text{E}}^{\text{on-shell}} = \beta M - S_{\text{BH}}$$

- Negative heat capacity, thermal instability

$$C \equiv \beta^2 \partial_\beta^2 \log \mathcal{Z} < 0$$

- Canonical ensemble ill-defined
- Appeal to microcanonical ensemble (Hawking, '79); (Brown-York, '93)

Black hole in a box

Sch. BH of mass M in a box of radius $r = \mathfrak{r}$ (York, '86)

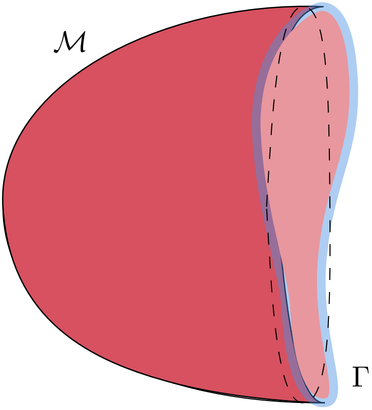
- **Dirichlet** BCs on Γ
- Canonical ensemble:

$$\boxed{\text{Fix } \beta_{\text{Tol}} \text{ and } A_{\mathcal{S}}}$$

$$\beta_{\text{Tol}} = \beta_{\text{H}}\sqrt{1 - \frac{2MG}{\mathfrak{r}}}$$

$$E_{\text{BY}} = \mathfrak{r} - \mathfrak{r}\sqrt{1 - \frac{2MG}{\mathfrak{r}}}$$

$$S = \frac{\text{Area(horizon)}}{4G}$$



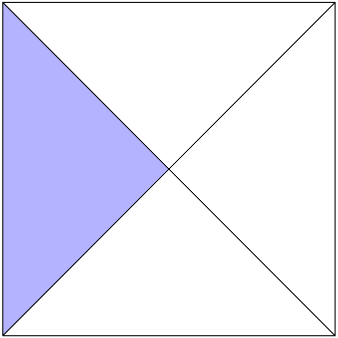
- $C > 0$ for large-BH, high-T_{Tol}

Quasi-local first law: $\boxed{\delta E_{\text{BY}} = T_{\text{Tol}}\delta S - P\delta A_{\mathcal{S}}}$

de Sitter static patch

(Chawakorn Maneerat)

$\mathcal{I}^+$



$\mathcal{I}^-$

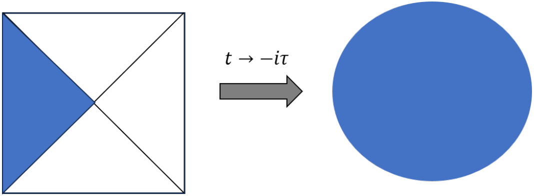
$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

- cosmological horizon at $r = L$

Cosmological horizon has thermal attributes (Gibbons-Hawking '77)

$$\boxed{S_{\mathrm{GH}} = \frac{A_c}{4G}\,, \quad T_{\mathrm{GH}} = \frac{\kappa_c}{2\pi}}$$

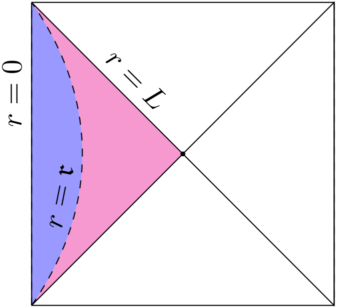
- Euc. PI derivation



$$\mathcal{Z}_{\mathrm{can}} \approx e^{-I_{\mathrm{E}}^{\text{on-shell}}}\,, \quad I_{\mathrm{E}}^{\text{on-shell}} = -S_{\mathrm{GH}}$$

- No boundary to define ensembles
- Minus sign in 'first law'

$$\delta E_m = -\frac{\kappa_c}{8\pi G}\delta A_c \overset{?}{\Leftrightarrow} \delta E = T_{\mathrm{GH}}\delta S_{\mathrm{GH}}$$



Canonical ensemble: (Hayward, '90; Huang, Wang, '01; Draper, Farkas, '22; Baniheshemi, Jacobson, '22)

$$\boxed{T = T_{\mathrm{Tol}}(r=\mathfrak{r}) = \frac{\kappa}{2\pi N(\mathfrak{r})}\,, \quad A_{\mathcal{S}} \equiv \int_{\mathcal{S}} d^2x\sqrt{\sigma}}$$

$$E_{\mathrm{BY}} = -\frac{k\mathfrak{r}^2}{2G}\,, \quad S = S_{\mathrm{GH}}$$

Microscopic interpretation using "T$\overline{\mathrm{T}}$"-formalism (Gorbenko...'18; Coleman...;Shyam; Silverstein, '22; Batra...; Silverstein, Torroba, '24)

Quasi-local first law: (Baniheshemi, Jacobson, Svesko, Visser '23)

$$\boxed{\delta E_{\mathrm{BY}} = \delta E_m + T_{\mathrm{Tol}} \delta S_{\mathrm{GH}}}$$

- E_m (outside) matter energy and quasi-local Brown-York energy

$$\delta E_m = \int_\Sigma \delta T_\mu^{\ \nu} \xi^\mu d\Sigma_\nu , \qquad E_{\mathrm{BY}} = -\frac{1}{8\pi G} \oint_{\mathcal{S}} d^2x \sqrt{\sigma} k$$

- As $\mathfrak{r} \to 0$, then $E_{\mathrm{BY}} \to 0$ and $T_{\mathrm{Tol}} \to T_{\mathrm{GH}}$

$$\boxed{\delta E_m = -T_{\mathrm{GH}} \delta S_{\mathrm{GH}}}$$

Minus sign resolution: (Baniheshemi, Jacobson, Svesko, Visser '23)

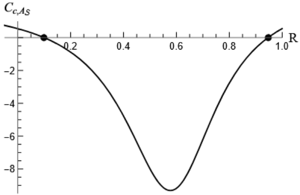
> Mistaken interpretation of E_m referring to total internal energy thermalized due to being beyond the horizon

- Matter in equilibrium: $\delta S_{\mathrm{gen}}|_{E_{\mathrm{BY}}} = 0$; dS vac. maximally mixed

Conformal thermodynamics

Two motivations to consider other BCs

- Cosmological patch thermally unstable: (Svesko et al, '22; Banihashemi, Jacobson...'22; Anninos, Harris, '22; Aguilar-Gutierrez, Svesko, Visser, '25)

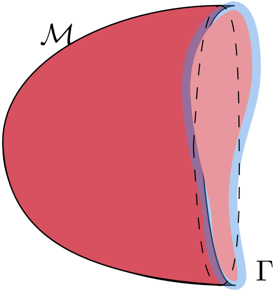


- Generally ill-posed
  - *Euclidean* (Anderson, '09; Witten, '18); Dirichlet BC not elliptic; does not lead to sensible perturbative expansion about classical soln.
  - *Lorentzian* (An, Anderson, '21) generic $h_{\mu\nu}$ existence & uniqueness issues for IBVP
    - **Existence:** $\mathcal{R}^{(3)} + K^2 - K^2_{\mu\nu} - 2\Lambda|_\Gamma = 0$
    - **non-uniqueness:** *flat* bdrys have infinite class of physical diffeos
  - **Exception:** (Witten, '18; Anninos, Galante, Maneerat, '23, '24; An, Anderson, '25) "$K_{\mu\nu}$"/$T^{\text{BY}}_{\mu\nu}$ has same sign as bdry metric $h_{\mu\nu}$

Conformal BCs in Euclidean gravity

$$\text{Fix:}\quad [h_{\mu\nu}]|_\Gamma\,,\qquad K|_\Gamma$$

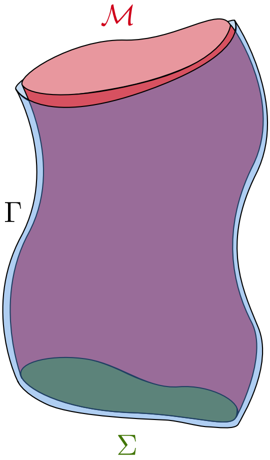
$$I = I_{\rm EH} + I_{\rm bdy}$$



$$I_{\rm bdy} = -\frac{1}{8\pi G_d(d-1)}\int_\Gamma d^{d-1}y\sqrt{h}K$$

$$T^{\rm CBC}_{\mu\nu} = -\frac{1}{8\pi G_d}\left(K_{\mu\nu} - \frac{1}{(d-1)}K h_{\mu\nu}\right)$$

- Proposed by York for *spacelike* surfaces (York, '72)
- Same BCs as in AdS/CFT; an "AdS box"
- Fluid/gravity (Bredberg, Strominger, '11; Anninos, Anous, Bredberg, Ng, '11)
- Well-posed in Euclidean (Anderson, '09; Witten, '18)
- 1-parameter extension (Liu, Santos, Wiseman, '24)



Status on proofs of well-posed:

- Linearized proof about any Einstein background (An, Anderson, '25)
- Spherical bdys: growing modes for large angular momentum $\boxed{\omega \sim \ell + i\ell^{1/3}}$ (Anninos, Galante, Maneerat, '23, '24; Liu, Santos, Wiseman, '24)
- Violation of Hadamard criterion (continuity solns w/ initial data) (Liu, Reall, Santos, Wiseman, '25)
- Click for Damian's talk!

Question: Why do finite conformal bdys have growing modes?

WIP: Anninos, Galante, Georgescu, Maneerat, Svesko

Conformal thermal ensembles

- Gibbons-Hawking prescription:

$$\mathcal{Z}_{\text{can}} \sim \sum_{g^*_{\mu\nu}} e^{-I_{\text{E}}^{\text{on-shell}}[g^*_{\mu\nu}]}$$

- BCs characterize ensemble

$$\boxed{\text{Dirichlet: } (\beta, \mathfrak{r}) \Longrightarrow \text{Conformal: } (\tilde{\beta}, K)\,, \quad \tilde{\beta} = \beta/\mathfrak{r}}$$

- Conformal thermodynamics

$$\boxed{E \equiv -\partial_{\tilde{\beta}}|_K \log \mathcal{Z}\,, \quad S \equiv (1-\tilde{\beta}\partial_{\tilde{\beta}})|_K \log \mathcal{Z}\,, \quad C_K \equiv \tilde{\beta}^2 \partial^2_{\tilde{\beta}}|_K \log \mathcal{Z}}$$

Ex: Sch. BH (Anninos, Galante, Maneerat, '23; Banihashemi, Shaghoulian, Shashi, '24)

$$E = E_{\text{BY}}^{\text{Conf.}}\,, \quad S = \frac{A_{\text{BH}}}{4G}\,, \quad dE = \tilde{T}dS + VdK$$

Highlights

- *Stable* cosmic patches (Anninos, Galante, Maneerat, '24)
- High-T limit, entropy as CFT_{d-1} (Anninos, Galante, Maneerat, '23, '24)

$$\boxed{S(\tilde{\beta}, K) = \frac{N_{\text{d.o.f.}}(K)}{\tilde{\beta}^{d-2}}}$$

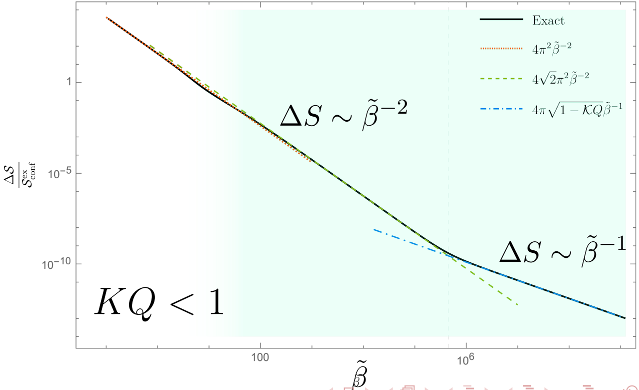
$$N_{\text{d.o.f.}} = \begin{cases} \frac{\Omega_{d-2} L^{d-2}}{4G_d} \left(\frac{4\pi}{(d-1)^2}\right)^{d-2} \left(\sqrt{K^2L^2 + (d-1)^2} - KL\right)^{d-2}, & \Lambda > 0 \\ \frac{\Omega_{d-2}(2\pi)^{d-2}}{4G_d K^{d-2}}, & \Lambda = 0 \\ \frac{\Omega_{d-2} L^{d-2}}{4G_d} \left(\frac{4\pi}{(d-1)^2}\right)^{d-2} \left(KL - \sqrt{K^2L^2 - (d-1)^2}\right)^{d-2}, & \Lambda < 0 \end{cases}$$

- *Local* QFT dual?
- AdS_3: (Allameh, Shaghoulian, '25)

- New scalings for S at low-temperatures (Galante, Maneerat, Svesko, '25)

$$\boxed{\tilde{\beta} = \tilde{\beta}_{\text{ex}}(K, Q) \sim (K^2Q^2-1)^{-1/2}}$$

Near-extremal limit is double scaled: $(\tilde{\beta} \to \infty\ , \quad KQ \to 1)$

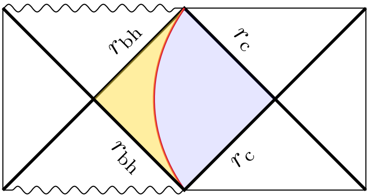


Near-extremality: JT is not enough

- Extremal/Nariai

$$ds^2_{\rm NH} \approx (\text{A})\text{dS}_2 \times S^2$$

- Away from Ext/Nariai $\Rightarrow$ (A)dS$_2$ JT
- Finite Dirichlet bdy (Svesko, Verheijden, Verlinde, Visser, '22; Anninos, Harris, '22; Aguilar-Gutierrez, Svesko, Visser, '25)



2D conformal ensembles: (Galante, Maneerat, Svesko, '25)

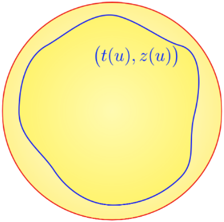
$$\boxed{\text{CBC:}\ \ \Phi^{-\alpha}\sqrt{h}\ ,\ \Phi^{-\alpha}\left(\Phi K + \alpha\, n^\mu \nabla_\mu \Phi\right)}$$

$$I = \int_{\mathfrak{m}} d^2x \sqrt{g}\,(\Phi R + U(\Phi)) + 2\int_\Gamma dy\sqrt{h}\left(\frac{2\alpha-1}{2\alpha}\Phi K - \frac{1}{2}\partial_n \Phi\right)$$

Away from extremality: $\boxed{U(\Phi) = \gamma_1 \Phi + \gamma_2 \Phi^2 + \ldots}$

- Spherically symm. with dynamical bdys:

$$ds^2|_{\partial\mathfrak{m}} = e^{2\omega(x^m)}(du^2 + \mathfrak{r}^2 d\Omega_{d-2}^2), \quad K|_{\partial\mathfrak{m}} = \text{const.}$$



- **Ex:** AdS$_d$, $K = \frac{d-1}{L} + \delta K$

$$\partial_u^2\omega + \frac{\delta K}{L} e^{2\omega} = -\frac{(d-3)}{2\mathfrak{r}^2}[\mathfrak{r}^2(\partial_u\omega)^2 - 1]$$

- 1D Liouville Eqn in $d = 3$ (Galante, Maneerat, Svesko, '25)

1D effective action: (Galante, Maneerat, Svesko, '25)

Follow procedure to derive Schwarzian via dim. reduction of AdS$_3$

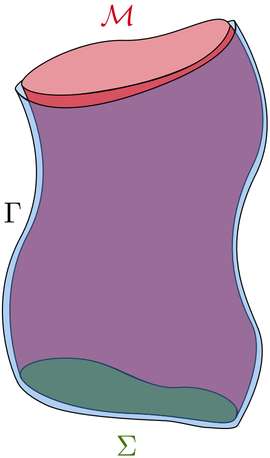
$$I_{\text{bdy}} = -\frac{1}{8\pi}\int_{\partial\mathfrak{m}} du \left[\mathrm{S}(t(u), u) - \frac{\delta K}{2}\Gamma'(u)^2 - \mathrm{S}(\Gamma(u), u)\right], \quad \Gamma \sim \int du e^{\omega(u)}$$

$$\boxed{\text{Schwarzian coupled to 1D quantum gravity}}$$

(Anninos, Galante, Hofman '18; Anninos, Hofman, Vitouladitis, '21)

- ***Finite boundaries***
  - Finite Bdys far reaching
  - CBCs invite new physics
- ***Future directions***
  - Microscopics? $T\bar{T}$? (Coleman, Shyam, '21; Allameh, Shaghoulian, '25)
  - 1-loop quantum corrections?
  - Posedness/stability of screens? (Liu Reall, Santos, Wiseman, '25; WIP)

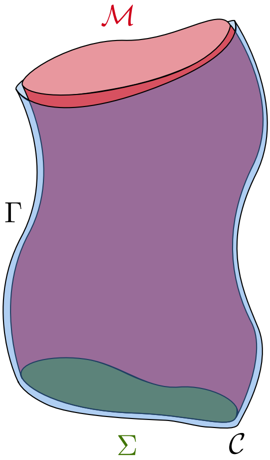
Thank you!



Initial conditions on Σ: $(g_{ij}^{\Sigma}, \mathcal{K}_{ij})$

Dirichlet problem:

- Initial value problem (IVP) well-posed in GR (Choquet, Bruhat, '52)
- IBVP is w/Dirichlet BCs well-posed for scalar, Yang-Mills (Sarbach, Tiglio '12)
- Particular Dirichlet BCs are well-posed (positive definite BY stress-tensor (An, Anderson, '25)



Initial conditions on Σ: $(g_{ij}^{\Sigma}, \mathcal{K}_{ij})$

IBVP w/ CBCs well-posed

- Conjecture: CBCs on Γ (An, Anderson, '21)
- Specify corner data (Liu, Santos, Wiseman; An, Anderson, '25)

Schwarzschild de Sitter black hole:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2\,, \quad f(r) = 1 - \frac{r^2}{L^2} - \frac{2MG_4}{r}$$

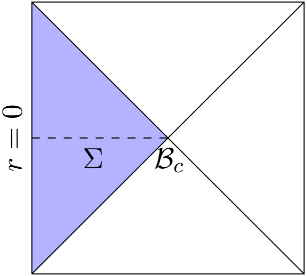
- For $0 < M < M_{\rm N}$, $f(r)$ has two roots, $r_h < r_c$
- On-shell action? (Morvan, van der Schaar, Visser, '22)
- $M_{\rm N}$ – *Nariai limit*; $r_h = r_c$
- Largest 'extremal' black hole to fit inside cosmological horizon
- Nariai geometry:

$$\boxed{\mathrm{NSdS}_d = \mathrm{dS}_2 \times S_{d-2}}$$

What theory describes low energy dynamics?

(i) *'First law of cosmological horizons'*

$$\delta E_m = \int_\Sigma \delta T_\mu^{\ \nu} \xi^\mu d\Sigma_\nu = -\frac{\kappa_c}{8\pi G}\delta A_c$$



- vacuum restricted to SP is *thermal*; entropy of SP is *reduced*

$$\boxed{\delta E_m = -T_{\rm GH}\delta S_{\rm GH}}$$

- $S_{\rm GH}$ attributed to region *outside*

(ii) *Relevant energy E lies behind horizon* (Spradlin, et. al. '01)

$$\delta E_m = \delta(-E) = -T_{\rm GH}\delta S_{\rm GH}$$

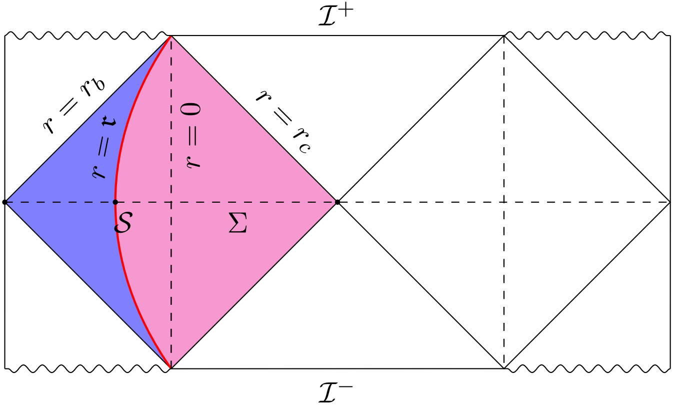
- ξ past pointing outside ⇒ 'energy' is opposite than normal

(iii) *Temperature of SP is* ***negative***, (Klemm, Vanzo '04)

$$\delta E_m = (-T_{\rm GH})\delta S_{\rm GH} = T\delta S_{\rm GH}$$

- dS has largest mass black hole; but Killing temp. is *positive*

de Sitter black holes



- **BH system:** $r \in [r_b, \mathfrak{r}]$; **CH system:** $r \in [\mathfrak{r}, r_c]$

$$\boxed{\delta E_{\mathrm{BY}} = \frac{\kappa_c}{8\pi G N}\delta A_c - P_c \delta A_{\mathcal{S}} + \delta E_m}$$

- Near-Nariai: dS_2-JT gravity (Svesko, Verheijden, Verlinde, Visser '22)