

Lightcone and modular bootstraps

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based on 2412.06907 and 2505.02897
work with Sridip Pal and Jiaxin Qiao

What is the lightcone bootstrap?

Consider a CFT four-point function in $d > 2$:

$$\langle \phi(0) \phi(z, \bar{z}) \phi(1) \phi(\infty) \rangle = \frac{1}{(z\bar{z})^{\Delta_\phi}} g(z, \bar{z}), \text{ with}$$

$$g(z, \bar{z}) = 1 + \sum_P \lambda_P \underbrace{g_{\Delta_P}^{(\mathcal{D}_P)}(z, \bar{z})}_{\text{conf. block}} = 1 + \sum_k c_k \underbrace{z^{h_k} \bar{z}^{\bar{h}_k}}_{\text{"toy" conf. block}}$$

* OPE: converges for $|z|, |\bar{z}| < 1$

* unitarity: $c_k, h_k, \bar{h}_k > 0$

* locality: $\bar{h}_k - h_k \in \mathbb{Z}$

* crossing: $g(z, \bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} g(1-z, 1-\bar{z})$

Now take the lightcone limit $\bar{z} \rightarrow 1$:

$$g(z, \bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} \times \left(1 + \mathcal{O}((1-\bar{z})^{\bar{h}_{\min}}) \right)$$

strictly positive

Q: what does this imply for $(c_k, h_k, \bar{h}_k)$?

$$\begin{aligned} h &= \frac{1}{2}(\Delta - \mathcal{J}) \\ \bar{h} &= \frac{1}{2}(\Delta + \mathcal{J}) \end{aligned}$$

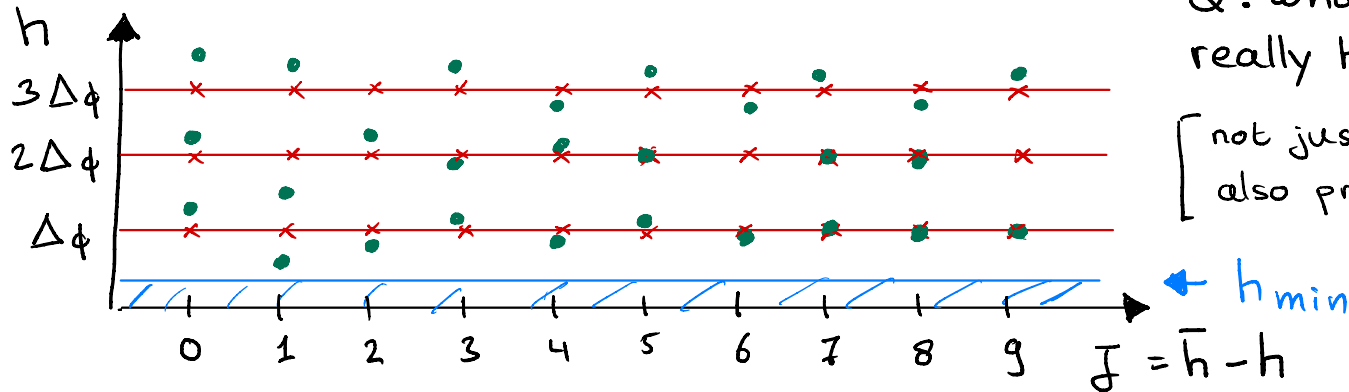
What can we expect?

$$1 + \sum_k \underbrace{c_k z^{h_k} \bar{z}^{\bar{h}_k}}_{\text{green wavy}} = \left(\frac{z \bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta\phi} \times (1 + \mathcal{O}((1-\bar{z})^{h_{\min}}))$$

Imprecise analysis:

- $\bar{z} \rightarrow 1$ corresponds to $\bar{h}_k \rightarrow \infty$, h_k finite
 - $\left(\frac{z \bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta\phi} = \sum_{m,n} \frac{(\Delta\phi)_m (\Delta\phi)_n}{m!n!} \underbrace{z^{\Delta\phi+m} \bar{z}^{\Delta\phi+n}}_{\text{red wavy}}$
- $\Rightarrow$ " green wavy $\rightarrow$ red wavy as $\bar{h}_k \rightarrow \infty$ "

Scenarios (for the spectrum):



Q: what is really happening?

[not just rigor, also precision!]

Outline

- 0 Introduction
- 1 History & motivation
- 2 Main result
- 3 Modular bootstrap

History and motivation

* 2012: "same" analysis for conformal blocks $[z^h \bar{z}^{\bar{h}} \rightsquigarrow G_{\Delta}^{(J)}(z, \bar{z})]$

[Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov]

→ many followups!

→ no precise results until 2022! [Pal, Qiao, Rychkov (2022)]

→ can we do better? [and compare with e.g. numerics]

* 2017: "analyticity in spin" [Caron-Huot]

→ many followups!

→ reliance on benign "LLSS" which remains unproven.

* 2022: "QFT in AdS instead of LSZ" [BvR, Zhao]

→ dream: prove e.g. Froissart bound from CFT axioms.

→ need rigorous control as $\Delta_{\phi} \rightarrow \infty$

↑ from $m^2 R^2 = \Delta_{\phi}(\Delta_{\phi} - d)$

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 - Tauberian theorem
 - Vitali's theorem
 - Dominated convergence theorem
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Tauberian theorems

(first used for CFTs in [Pappadopulo, Rychkov, Espin, Rattazzi (2012)])

$$\sum_k x^k = \frac{1}{1-x}, \text{ but also:}$$

$$\underline{\text{thm}} \quad \sum_k d_k x^k \underset{x \rightarrow 1}{\sim} \frac{1}{1-x} \Rightarrow \sum_k d_k \underset{\Delta \rightarrow \infty}{\sim} \sum_k^{\Delta} 1 = \Delta \quad \text{if } d_k \geq 0$$

-
- need $\sum_k^{\Delta} : \sum_k 2^k x^{2^k} \underset{x \rightarrow 1}{\sim} \frac{1}{1-x}$
 - $\exists$ estimates of error terms
 - many generalizations : powers, exponentials, complex vars, ...

$$\underline{\text{thm}} \quad \sum_k d_k x^{\bar{h}_k} \underset{x \rightarrow 1}{\sim} \frac{1}{(1-x)^{\Delta}} \Rightarrow \sum_{\bar{h}_k \leq \Delta} d_k \underset{\Delta \rightarrow \infty}{\sim} \sum_{k \leq \Delta} \frac{(\Delta)_k}{k!} \sim \frac{\Delta^{\Delta}}{\Gamma(\Delta+1)}$$

if $d_k \geq 0$

Toy lightcone bootstrap again

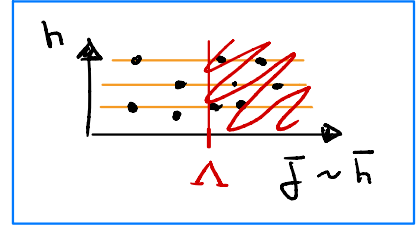
We know:

$$1 + \sum_k c_k z^{h_k} \bar{z}^{\bar{h}_k} \underset{\bar{z} \rightarrow 1}{\sim} \left(\frac{z \bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi}$$

and the Tauberian theorem then says:

$$\sum_{h_k \leq \Delta} c_k z^{h_k} \underset{\Delta \rightarrow \infty}{\sim} \left(\frac{z}{1-z} \right)^{\Delta_\phi} \frac{\Delta^{\Delta_\phi}}{\Gamma(\Delta_\phi + 1)}$$

pointwise for $0 < z < 1$ (where $c_k z^{h_k} > 0$).



Problem: how to trade z for h ?

- cannot expand around $z=0$ (even though it looks OK)
- second Tauberian theorem as $z \rightarrow 1$ loses information

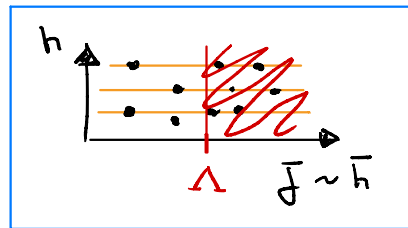
Toy lightcone bootstrap again

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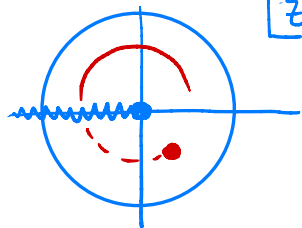
and the Tauberian theorem then says:

$$\sum_{\bar{h}_k \leq \Delta} c_k z^{h_k} \underset{\Delta \rightarrow \infty}{\sim} \left(\frac{z}{1-z} \right)^{\Delta_\phi} \frac{\Delta^{\Delta_\phi}}{\Gamma(\Delta_\phi + 1)}$$



initially pointwise for $0 < z < 1$ (where $p_z(\bar{h}) \geq 0$),
but in fact in a much bigger domain:

$\mathbb{D}$ if $z = re^{i\theta}$ (so $z^h = r^h e^{i\theta h}$) then
also for $0 < r < 1$ and any $\theta \in \mathbb{R}$.



(proof uses $|r^h e^{i\theta h}| < |r^h|$ and
Vitali's theorem in complex analysis)

So far we know that, with $z = re^{i\theta}$

$$\sum_{h_k \leq \Delta} c_k r^{h_k} e^{i\theta h_k} \rightsquigarrow \frac{\Delta^{\Delta\phi}}{\Gamma(\Delta\phi+1)} \frac{r^{\Delta\phi} e^{i\theta\Delta\phi}}{(1 - re^{i\theta})^{\Delta\phi}} \quad \text{for all } 0 < r < 1 \text{ and } \theta \in \mathbb{R}.$$

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Inverse Fourier:

$$\sum_{h_k \in \Delta} c_k \delta(h - h_k) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{e^{-i\theta h}}{r^h} \sum_{h_k \in \Delta} c_k r^{h_k} e^{i\theta h_k}$$

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$$Q: \sum_{h_k \leq \Delta} c_k \delta(h - h_k) \underset{\Delta \rightarrow \infty}{\stackrel{?}{\sim}} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{e^{-i\theta h}}{r^h} \frac{\Delta^{\Delta\phi}}{\Gamma(\Delta\phi+1)} \frac{r^{\Delta\phi} e^{i\theta\Delta\phi}}{(1 - re^{i\theta})^{\Delta\phi}} = (\text{known})$$

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A: no, because "lim $\int \neq \int \lim$ " in general.

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Inverse Fourier with weight function $\varphi(h)$:

$$\begin{aligned} \sum_{h_k \in \Delta} c_k \varphi(h - h_k) &= \int dh \varphi(h) \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{e^{-i\theta h}}{r^h} \sum_{h_k \in \Delta} c_k r^{h_k} e^{i\theta h_k} \\ &= \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left[\int dh \varphi(h) \frac{e^{-i\theta h}}{r^h} \right] \sum_{h_k \in \Delta} c_k r^{h_k} e^{i\theta h_k} \end{aligned}$$

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Q: $\sum_{h_k \leq \Delta} c_k \varphi(h - h_k) \stackrel{?}{\rightsquigarrow} (\text{known})$ at least for some φ ?

So far we know that, with $z = re^{i\theta}$

$$\sum_{h_k \in \Delta} c_k r^{h_k} e^{i\theta h_k} \underset{\Delta \rightarrow \infty}{\rightsquigarrow} \frac{\Delta^{\Delta\phi}}{\Gamma(\Delta\phi+1)} \frac{r^{\Delta\phi} e^{i\theta \Delta\phi}}{(1 - re^{i\theta})^{\Delta\phi}} \text{ for all } 0 < r < 1 \text{ and } \theta \in \mathbb{R}.$$

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Q: $\sum_{h_k \in \Delta} c_k \varphi(h - h_k) \underset{\Delta \rightarrow \infty}{\rightsquigarrow} ?$ (known) at least for some φ ?

A: Yes, by dominated convergence theorem and provided

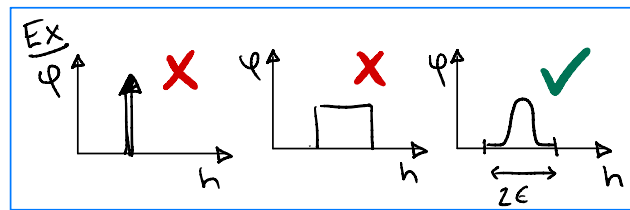
$$\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left| \int dh \varphi(h) \frac{e^{-i\theta h}}{r^h} \right| < \infty.$$

Toy lightcone bootstrap conclusion

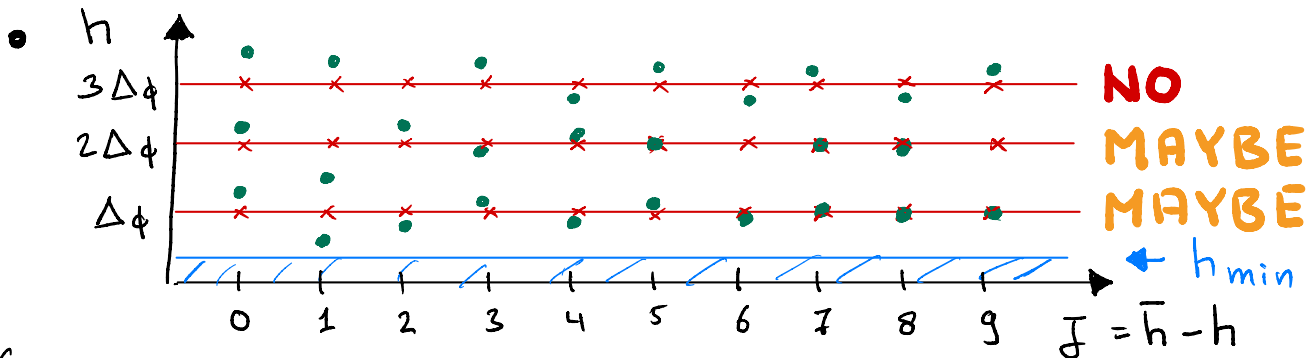
Using 3 theorems we obtained our main theorem:

$$\sum_k c_k z^{h_k} \bar{z}^{\bar{h}_k} \underset{\bar{z} \rightarrow 1}{\sim} \left(\frac{z \bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} \Rightarrow \sum_k c_k \varphi(h_k) \underset{\bar{h}_k \leq \Delta}{\sim} \frac{\Lambda^{\Delta_\phi}}{\Gamma(\Delta_\phi + 1)} \sum_{m=0}^{\infty} \frac{(\Delta_\phi)_m}{m!} \varphi(\Delta_\phi + m)$$

provided $\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left| \int dh \varphi(h) \frac{e^{-i\theta h}}{r^h} \right| < \infty.$



- $\forall \epsilon > 0, m \in \mathbb{N}, \exists \infty$ -ly many terms with $h \in (\Delta_\phi + m - \epsilon, \Delta_\phi + m + \epsilon).$



(Same result for conformal blocks in [BvR (2024)].)

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More or less averaging

Our result: $\sum_k c_k z^{h_k} \bar{z}^{\bar{h}_k} \underset{\bar{z} \rightarrow 1}{\hookrightarrow} (\text{known})$ implies, for certain $\varphi(h)$,

$$\Rightarrow \underbrace{\int d\bar{h} \int dh \varphi(h)}_{\text{"averaging"}} \left[\underbrace{\sum_k c_k \delta(h-h_k) \delta(\bar{h}-\bar{h}_k)}_{\text{"OPE density"}} \right] \underset{\Lambda \rightarrow \infty}{\hookrightarrow} (\text{known})$$

but is this the best we can do?

Q: how much averaging is actually needed?

- e.g.
- can we replace $\lim_{\Lambda \rightarrow \infty} \sum_{\mathcal{F}}^{\Lambda} \dots$ with just $\lim_{\mathcal{F} \rightarrow \infty} \dots$?
 - can $\varphi(h)$ be $\mathcal{F}$ -dependent?
 - rate of convergence? subleading corrections?

Modular bootstrap

"non-rational 2d CFT"

- $c > 1$
- twist gap: $\Delta - J \geq 2T_{\text{gap}}$ except for Vir. id. module
- nice vacuum
- modular invariance:

$$Z(\beta_L, \beta_R) = Z(\beta_L + 2\pi i, \beta_R - 2\pi i) = Z\left(\frac{4\pi^2}{\beta_L}, \frac{4\pi^2}{\beta_R}\right)$$

([Antunes, Behan (2022, 2024)] and [Antunes, Rong (2025)] for possible examples.)

Q: What does

$$Z(\beta_L, \beta_R) \underset{\beta_R \rightarrow 0}{\hookrightarrow} Z_{\text{vac}}\left(\frac{4\pi^2}{\beta_L}, \frac{4\pi^2}{\beta_R}\right) \quad (\text{known})$$

imply for the spectrum?

Modular bootstrap result

thm [PQVR (2025)] in any such 2d CFT,

$$\underbrace{\int dh \varphi_{\mathcal{F}}(h)}_{\text{averaging with } \mathcal{F}\text{-dep test fn. (see paper)}} \underbrace{p_{\mathcal{F}}(\mathcal{F}+2h)}_{\text{Vir. primary density}} \underbrace{=}_{\mathcal{F} \rightarrow \infty} \underbrace{(\text{known}) \left(1 + \mathcal{O}\left(1/\mathcal{F}^N\right)\right)}_{\substack{\forall N > 0. \\ \text{rate of convergence}}}$$

averaging
with $\mathcal{F}$ -dep
test fn.
(see paper)

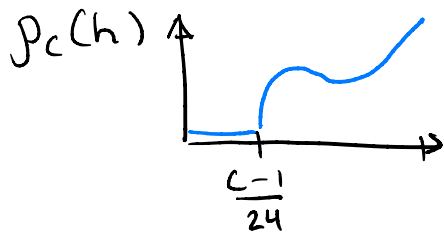
Vir. primary
density

no
sum!

$$p_{\mathcal{F}}(\Delta) = \sum_k \delta(\Delta - \Delta_k) \delta_{\mathcal{F}, \mathcal{F}_k}$$

(see paper)

where (known) is given by Cardy density $e^{\frac{4\pi\sqrt{\frac{c-1}{24}}\mathcal{F}}{\sqrt{2\mathcal{F}}}} p_c(h)$



Corollaries

- spectrum becomes dense at large $\mathcal{F}$.
- "no spin can misbehave"
- max gap bounds cf. [Mz (2019), GP, MP (2020)]

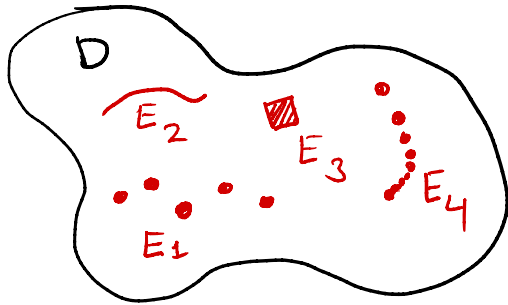
Thank you!

Vitali's theorem

ex $\{f_n(z)\}_{n=1,2,\dots}$ sequence of fns, holomorphic for $|z| < 1$
such that $\lim_{n \rightarrow \infty} f_n(z) = 0(z)$ pointwise for real z .

Q: what happens around the origin?

thm $\{f_n(z)\}_{n=1,2,\dots}$ sequence of fns, holomorphic for $z \in D \subseteq \mathbb{C}$,
(locally) bounded $\sup_n |f_n(z)| < \infty$, and s.t. $\lim_{n \rightarrow \infty} f_n(z)$ exists
pointwise on $E \subseteq D$. ^{n, z} If E has an accumulation point in D ,
then $\lim_{n \rightarrow \infty} f_n(z) = f(z)$ everywhere on D , ucc,
and $f(z)$ is holomorphic.



ex; continued: $f_n(z)$ cannot stay bounded.