The importance of being Exact: integrability, gauge, black holes physics and their correspondence

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Elementary, not many citations (too many): apology!

Synopsis and **Motivation**

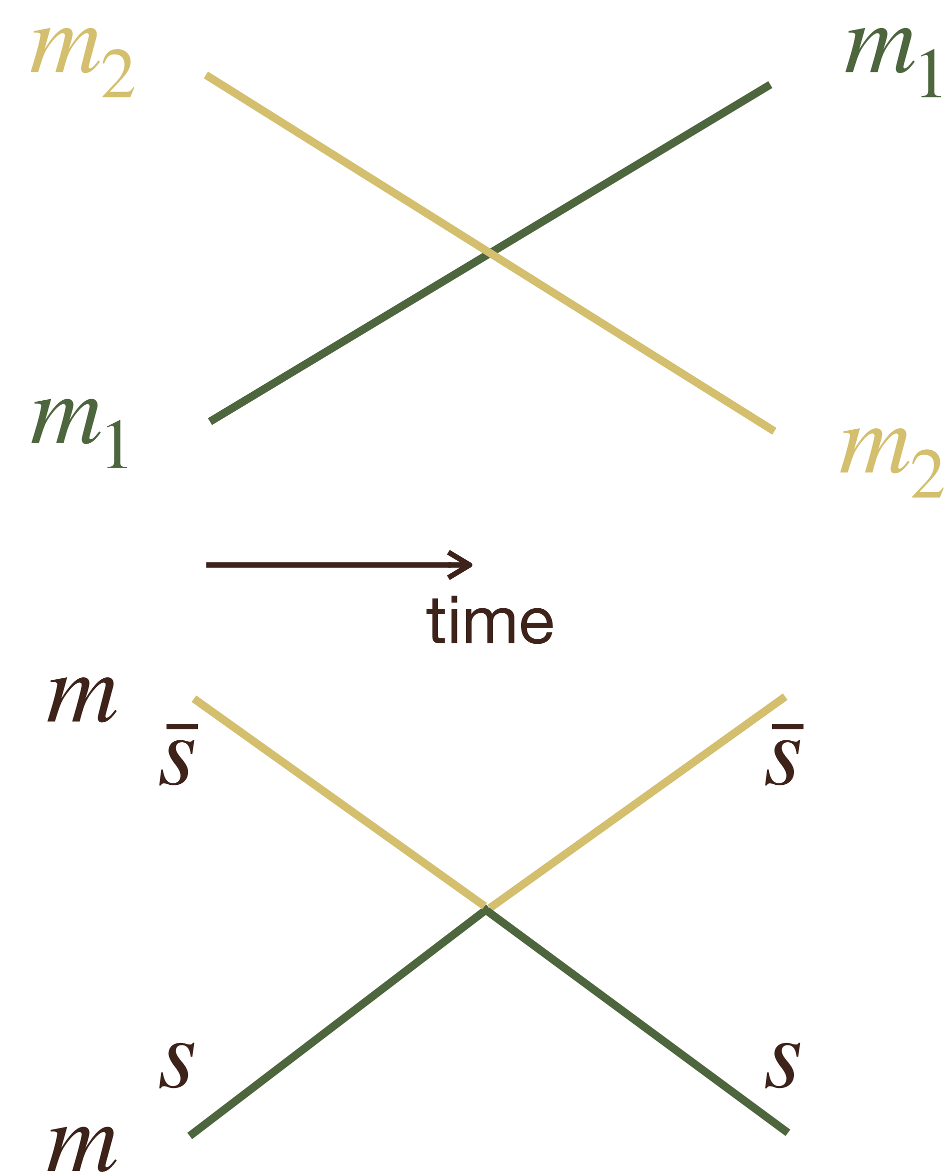
Quantum Integrability

from **Integral** and **Functional equations** to **classical ODEs** (Lax **integrability**) in **gauge theories** (and **BH**)

in the papers

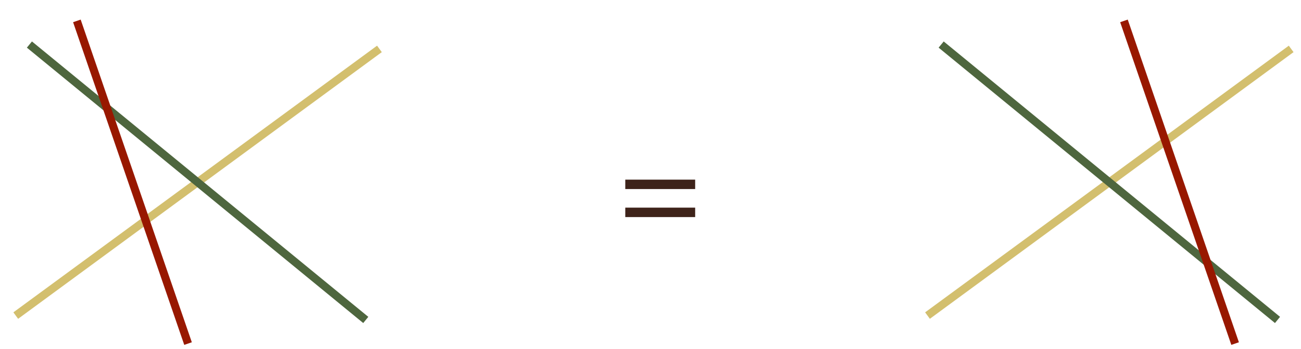
Scattering theory in 1+1d:
Two body scattering

- **Infinite charges** implies particles maintain their rapidity θ_i:
  $p_i = m_i \sinh\theta_i \quad E_i = m_i \cosh\theta_i$
  if $m_1 \neq m_2$: $S = e^{i\delta}$ phase shift.
- If equal masses, possible exchange of internal degrees of freedom: $s + \bar{s} \to s + \bar{s}$ or $s + \bar{s} \to \bar{s} + s$ in the quantum Sine-Gordon, i.e. non-diagonal scattering.



Three body scattering

- **Infinite charges** implies the scattering is independent of the order or associative: Yang-Baxter relation



- The general n-body scattering matrix **factorises** into two-body processes.

$$A_{Sh-G} = \int d^2x\,[(\partial_a\phi)^2 + \mu\cosh(b\phi)]$$

Sinh-Gordon S-matrix

- Two-body S-matrix

$$S(\theta) = \frac{\sinh\theta - i\sin\pi p}{\sinh\theta + i\sin\pi p}$$

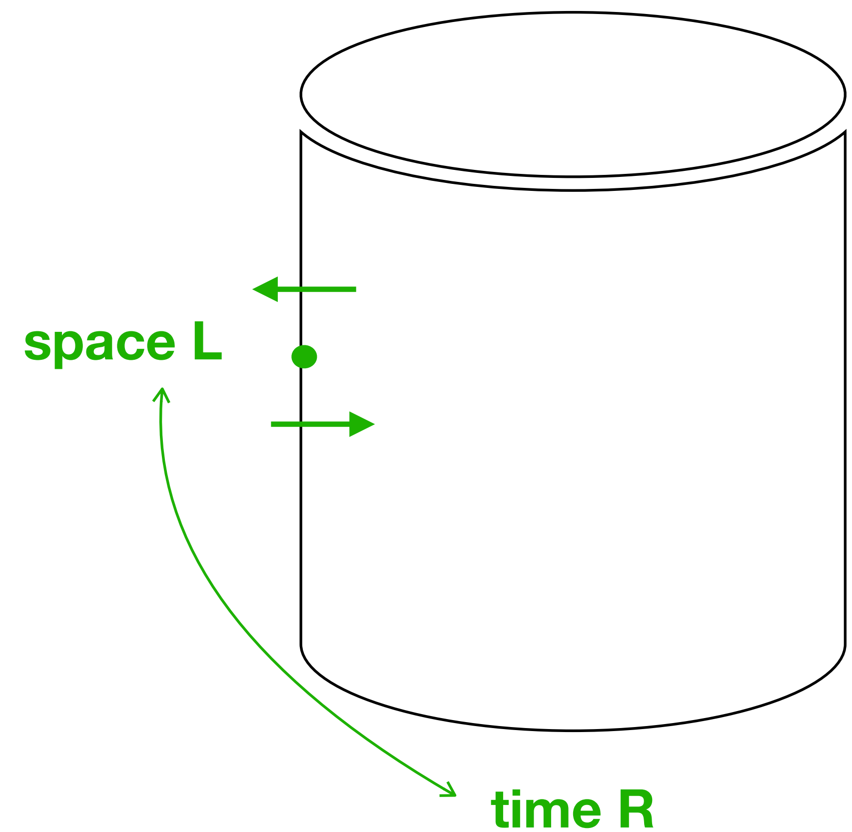
with notation for the coupling constant:

$p = b/(b + 1/b)$.

- This is an on-shell information or infinite size.
- How to compute finite size energy (off-shell)?

Thermodynamic Bethe Ansatz

Mirror theory (**space**↔**time**):
Bethe eqs. (derived via analytic continuation) in space $L \to \infty$ become exact!



Finite R no longer a problem: Thermodynamics/statistical mechanics at temperature T=1/R (Yang-Yang) gives the minimal **free energy** $f_{min}(L)$ so that

$$Z = \exp[-LRf_{min}(R)] + \ldots = \exp[-LE_0] + \ldots$$

Thermodynamic Bethe Ansatz

- Upon equating the two exponents of Z
$$E_0(R) = Rf_{min}(R)$$
- Ground state energy (Sinh-Gordon in the present case) given by thermodynamic free energy.
- Relativistic models: $S_{mirror} = S$ of original theory, because of difference of rapidities.
- Minimising a thermodynamic functional ⇒ Non-Linear TBA equations, whose SOLUTION gives the ENERGY

Sinh-Gordon: TBA

- Ground state energy $E_0(R) = -\frac{m}{2\pi}\int_{-\infty}^{+\infty} d\theta \cosh\theta \ln[1 + e^{-\epsilon(\theta)}]$

- <u>TBA EQUATION</u> for the pseudoenergy (ratio of particle densities)

$$\epsilon(\theta) = mR\cosh\theta - \int_{-\infty}^{+\infty} d\theta' \varphi(\theta - \theta')\ln[1 + e^{-\epsilon(\theta')}]$$

  describe the scattering $\varphi = \delta' = -i\frac{d}{d\theta}\ln S$.

- It has only ONE solution, easy to find numerically (by recursion) $\epsilon(\theta) = R(m\cosh\theta) + \ldots \rightarrow$ <u>**relativistic particle**</u>, and analytically in some regimes.

Sinh-Gordon: Y-system

- TBA equation $a = 1 - 2p$ (coupling)

$$\varepsilon(\theta) = mR\cosh\theta - \int_{-\infty}^{\infty}\left[\frac{1}{\cosh(\theta-\theta'+ia\pi/2)} + \frac{1}{\cosh(\theta-\theta'-ia\pi/2)}\right]\ln\left[1+e^{-\varepsilon(\theta')}\right]\frac{d\theta'}{2\pi}$$

⇒(not the reverse: driving term disappears) $\varepsilon(\theta) = -\ln Y(\theta)$

- FUNCTIONAL EQUATION: Y-system

$$Y(\theta+i\pi/2)Y(\theta-i\pi/2) = \Big(1+Y(\theta+ia\pi/2)\Big)\Big(1+Y(\theta-ia\pi/2)\Big)$$

- Both functional and integral equations constraint form and expansions (in conserved charges) of solutions: **<u>strength and power of quantum integrability</u>**

Introducing Ordinary Differential equations (ODE)

- Foreword: important observation of derivation of functional equations from **polynomial potentials** in Schrödinger eq. for minimal CFTs: ODE→IM correspondence. Dorey,Tateo;BLZ;Dunning;Suzuki;DF,Masoero ;….;Ito,Marino,Shu;….
- But need two irregular singularities, like in SW differential and HEUN EQ
- **Can we construct a GENERAL INVERSION THEORY? YES**, but requires time and arrives at TWO Lax Operators:MASSIVE TH. DF, M Rossi Gaiotto,Moore,Neitzke;Lukyanov,Zam
- Y beautiful, but not elementary (QQ-system, definition of integrability)
- Our rough and heuristic idea: QQ-system is **unitarity in QM**

$$r(\theta)r(\theta+i\pi)+t(\theta)t(\theta+i\pi)=1,\ \ k=ie^{\theta}$$

Help from

Seiberg-Witten theory,

i.e. *N*=2 susy (effective) gauge theory

N=2 gauge periods and ODEs

- Original **Seiberg-Witten** idea: **the prepotential is give by two periods of a differential**. Nekrasov-Shatashvili $\hbar$ instanton regularisation/deformation: **an ODE** quantises the differential. Moreover: **AGT correspondence**: level two null vector ODE. **Pure SU(2): Mathieu eq.** (periodic potential: quasi-periodic (Floquet) sols.)

$$-\frac{\hbar^2}{2}\frac{d^2}{dz^2}\psi(z) + [\Lambda^2 \cos z - u]\psi(z) = 0$$

Floquet exponent: $\psi_\pm(z+2\pi;a) = e^{\pm 2\pi i a}\psi_\pm(z;a)$

- **Quantum SW differential** $\mathscr{P}(z) = -i\dfrac{d}{dx}\ln\psi(z) \to$ **Quantum periods**

$$a(\hbar,u,\Lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{P}(z;\hbar,u,\Lambda)\,dz \implies u = u(a)\,, \qquad a_D(\hbar,u,\Lambda) = \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0}\mathscr{P}(z;\hbar,u,\Lambda)\,dz = \partial\mathscr{F}/\partial a \implies \mathscr{F}(\hbar,a,\Lambda)$$

Prepotential

- Focus on how to extract **FUNCTIONAL EQUATIONS**

N=2 gauge periods and ODE

- Thinking of **ODE/IM correspondence**, we go to confining potential: complex $z = -iy - \pi$

$$\left\{ -\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2 \right\}\psi(y) = 0$$ **Heun** DB-confluent eq.

- with **gauge/integrability** change of variable

$$\frac{\hbar}{\Lambda} = e^{-\theta}\,, \qquad \frac{u}{\Lambda^2} = \frac{P^2}{2e^{2\theta}}$$

- **Can we use this ODE for quantum sinh-Gordon?**

ODE/IM correspondence:

- Introducing b: Generalised Mathieu Equation (GME) $y \in \mathbb{C}$

$$\left\{ -\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2 \right\} \psi(y) = 0$$

Zamolodchikov; DF,Gregori

- 2 irr. sings.,guessed by some discrete symmetries

$$\Lambda_b : \theta \to \theta + i\pi\frac{b}{q} \qquad y \to y + \frac{2\pi i}{q} \quad ; \quad \Omega_b : \theta \to \theta + i\pi\frac{1}{bq} \qquad y \to y - \frac{2\pi i}{q}$$

Fundamental Ingredients

- <u>**Discrete Symmetries BROKEN**</u>: acting on a solution they generate a new one
- <u>**Fundamental ODE/IM basis**</u>: decaying solutions

$$V_0(y) \simeq \frac{1}{\sqrt{2}} \exp\left\{-\theta/2 + yb/4\right\} \exp\left\{-\frac{2}{b} e^{\theta - yb/2}\right\} \qquad y \to -\infty \ ;$$

$$U_0(y) \simeq \frac{1}{\sqrt{2}} \exp\left\{-\theta/2 - y/4b\right\} \exp\left\{-2b e^{\theta + y/2b}\right\} \qquad y \to +\infty$$

- From asymptotic, <u>new solutions</u> by 'rotations' $U_k = \Lambda_b^k U_0, \ V_k = \Omega_b^k V_0,$ while <u>invariant</u> $U_0 = \Omega_b^k U_0, \ V_0 = \Lambda_b^k V_0$

- Fundamental object Q-function=**Wronskian** of decaying solutions

$$Q(\theta, P^2) = W[U_0, V_0]$$

- Thanks to the Λ_b and <u>then</u> Ω_b symmetries

$$iV_0(y) = Q(\theta + i\pi p)U_0(y) - Q(\theta)U_1(y)$$
$$iV_1(y) = Q(\theta + i\pi)U_0(y) - Q(\theta + i\pi(1-p))U_1(y)$$

- That is **CONNEXION COEFFICIENT** $Q(\theta) \to$ **Stokes MONODROMY**: $T(\theta)$ or $Y(\theta)$

$T(\theta, P^2) = -iW[U_1, U_{-1}] = Q(\theta - i\pi p)Q(\theta + i\pi) - Q(\theta + i\pi(1-2p))Q(\theta + i\pi p),\ \tilde{T}(\theta, P^2) = iW[V_1, V_{-1}] = T(b \to 1/b)$

- Wronskian of both sides: **QQ-system**

$$1 = Q(\theta + i\pi)Q(\theta) - Q(\theta + i\pi(1-p))Q(\theta + i\pi p)$$

for AdS/CFT: Quantum Spectral Curve

The return of the Y-system

- Q is the most basic object which generates all the integrability structures from the QQ-system: e.g. the Y-system upon composition $Y(\theta) = Q(\theta + i\pi a/2)Q(\theta - i\pi a/2)$

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \Big(1 + Y(\theta + ia\pi/2)\Big)\Big(1 + Y(\theta - ia\pi/2)\Big)$$

- The same as Sinh-Gordon! Same solution Y?

- The Y-system inversion is **not unique**: take the log and invert the shift operator, but ZERO MODES

$$s * l = l(\theta + i\pi/2) + l(\theta - i\pi/2) \Rightarrow s^{-1} \sim \frac{1}{\cosh}$$

NO: massless limit $m \sim 0, \theta \sim +\infty \Rightarrow m \cosh\theta \sim e^{\theta}$

Liouville Field theory

- TBA equation has only one exponent:

$$\varepsilon(\theta) = \frac{8\sqrt{\pi^3}\,q}{\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})} e^{\theta} - \int_{-\infty}^{\infty} \left[\frac{1}{\cosh(\theta-\theta'+ia\pi/2)} + \frac{1}{\cosh(\theta-\theta'-ia\pi/2)}\right] \ln\left[1+e^{-\varepsilon(\theta')}\right] \frac{d\theta'}{2\pi}$$

- Thanks to this (kink form), finite size scaling can be computed exactly and gives

$$c = 1 + 6(b+b^{-1})^2 \quad \Delta = (c-1)/24 - P^2$$

T, Q and N=2 gauge periods

- The same equation of **Liouville at the self-dual point** $b = 1$

$$\left\{ -\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2 \right\}\psi(y) = 0 \qquad z = -iy - \pi$$

- upon **gauge/integrability** change of variable

$$\frac{\hbar}{\Lambda} = e^{-\theta}\,, \qquad \frac{u}{\Lambda^2} = \frac{P^2}{2e^{2\theta}}$$

- **New integrability/gauge Correspondence: identification**

integrability **transfer matrix** **gauge period**, Floquet exponent

$$T(\hbar, u, \Lambda) \equiv T(\theta, P^2) = iW[V_1, V_{-1}] = 2\cos\{2\pi a(\hbar, u, \Lambda)\}$$

integrability **Q-function** **dual gauge period**

$$Q(\hbar, u, \Lambda) \equiv Q(\theta, P^2) = \exp\{2\pi i a_D(\hbar, u, \Lambda)\}$$

- The fundamental relation of the theory: **QQ-SYSTEM**

integrability gauge

$$1+Q^2(\theta,P^2) = Q(\theta-i\pi/2,P^2)Q(\theta+i\pi/2,P^2) \Longleftrightarrow 1+Q^2(\theta,u) = Q(\theta-i\pi/2,-u)Q(\theta+i\pi/2,-u)$$

- From the gauge **Y-system** the **gauge TBA eqs.**

monopole

$$\varepsilon(\theta,u,\Lambda) = -4\pi i a_D^{(0)}(u,\Lambda)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta',-u,\Lambda)\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}$$

$$\varepsilon(\theta,-u,\Lambda) = -4\pi i a_D^{(0)}(-u,\Lambda)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta',u,\Lambda)\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}$$

$\Longleftrightarrow$ Unique integrability TBA eq. above

dyon, i.e. **strong coupling spectrum**

- Quantum integrability tells more: e.g. T→weak coupling spectrum (a electric period), inside **TQ-system** ($a/\hbar \to a\,,\ a_D/\hbar \to a_D$)

$$\overset{2\cos\{2\pi a\}\quad \exp\{2\pi i a_D\}}{T(\theta)Q(\theta)} = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$$

- +periodicity, gauge interpretation: **quantum Bilal-Ferrari relations**. In fact T and Q are generating functions $\hbar/\Lambda = e^{-\theta} \to 0$ (asymptotic expansion) of for **Conserved Charges** and **Quantum Periods** (zero order=Seiberg-Witten). Opposite to **instanton expansion** $\Lambda/\hbar = e^{\theta} \to 0$ (other charges).

ODE/IN Correspondence

- **Prepotential** $\mathcal{F} = pert.(1 - loop) + (\textbf{INstantons})$ from **Nekrasov Partition** via **Young diagrams** in instanton series (AGT correspondence)
- INstanton coupling constant $\Lambda/\hbar = e^{\theta}$ expansion
- Dual instanton period/prepotential $A_D = \partial\mathcal{F}/\partial a$ appears in the **Q-function**

SW geometry

ODE/**INstanton** correspondence: $Q(a, \Lambda/\hbar) = i\dfrac{\sinh A_D}{\sinh 2\pi i a}$

simplest formula of this kind ∀Heun eq.

- A way to see this: **solution of the QQ-system** equivalent to $\mathbf{A_D}(\theta + \mathbf{i}\pi/2) = \mathbf{A_D}(\theta) + \mathbf{2}\pi\mathbf{ia}$ (check the asymptotic $\theta \to \pm\infty$), **functional equation solved by prepotential**!

Unveiling Prepotential in ODE/IN

DF, Rossi

- Deeper understanding: **prepotential** i.e. **SW geometry** inside **Floquet basis**, +:

$$\psi_+(y+2\pi i;a) = e^{+2\pi i a}\psi_+(y;a)$$

- behaves at $y \to -\infty$ as

$$\psi_+(y;a) \simeq (const.)(e^{\frac{y}{4}}e^{2e^{\theta}e^{-\frac{y}{2}}})$$

- and **acquires a *scattering* phase** at $y \to +\infty$

$$\psi_+(y;a) \simeq (const.)\ e^{\varphi}\ e^{-\frac{y}{4}}e^{2e^{\theta}e^{\frac{y}{2}}}$$

- It can be proven via $\varphi\left(\theta+\frac{i\pi}{2},P\right) = \varphi(\theta,P) - 2\pi i a$

- $$A_D = \varphi \overset{A_D = \partial\mathscr{F}/\partial a}{\Rightarrow} \mathscr{F} = \mathscr{F}(\Lambda/\hbar, a/\hbar)$$

- Explicit **exact expression for φ**: an alternative computation to and of **instantons,** at all orders small $\dfrac{\Lambda}{\hbar} = e^{\theta}$, $\theta \to -\infty$ (and **beyond,** e.g. large instanton coupling $\theta \to +\infty$)

- Since $\varphi(\theta, P) = \displaystyle\int_{-\infty}^{+\infty} dy' \left(\Pi_+(y') - reg\,.\right)$, not so easy as $2\pi i a = \displaystyle\int_0^{2i\pi} dy' \Pi_+(y')$

  $= \dfrac{d \ln \psi_+}{dy'} \to$ **quantum momentum**

  non-compact: kink method: $y \to y \pm 2\theta$

- **LESSON** from gauge dual period $\varphi = -4a\theta + \displaystyle\sum_{n=0}^{+\infty} c_n e^{4n\theta}$: better expand

$$\Pi_+(y \mp 2\theta) = \sum_{n=0}^{+\infty} \Pi^{(n)}_{\gtrless}(y; P) e^{4n\theta}, \; y \gtrless \pm 2\theta$$ two different intervals of validity!

- **Check**: $\psi_+ = Z_{quiver}$, but ours partially re-summed (product of instanton couplings only)

  …Bianchi,Fucito,Morales;

  Bonelli,Iossa,Tanzini, …..

Decaying vs. Floquet sols.

- The **quasi periodic Floquet solutions** are the novelty in ODE/IN correspondence w.r.t. decaying ones of ODE/IM.
- **Change of basis (simple idea: cancel the dominant divergence):**
$$V_0(y) = \frac{\sqrt{2}e^{\frac{\theta}{2}}}{W[\psi_+,\psi_-]}\left[e^{-\frac{\varphi}{2}}\psi_-(y) - e^{\frac{\varphi}{2}}\psi_+(y)\right] ,$$
$$U_0(y) = \frac{\sqrt{2}e^{\frac{\theta}{2}}}{W[\psi_+,\psi_-]}\left[e^{-\frac{\varphi}{2}}\psi_+(y) - e^{\frac{\varphi}{2}}\psi_-(y)\right] \implies Q(a,\theta) = -4e^{\theta}\frac{\sinh\varphi(a,\theta)}{W[\psi_+,\psi_-]}$$
General formulae: same form $\forall$**Heun-like (H and confluences)** eqs.
- All V_0, U_0 **INGREDIENTS**: $W[\psi_+,\psi_-]$, φ, $\Pi_\pm = d/dy[\ln\psi_\pm(y)]$ computed
- Why and where **Floquet approximation** $U_0(y) \simeq C\psi_-(y), +e^{-\varphi} \sim e^{a\theta}$ non-perturbative
- Here $W[\psi_+,\psi_-] = -4e^{\theta}\sin 2\pi a \implies Q(a,\Lambda/\hbar) = \dfrac{\sinh\varphi}{\sinh 2\pi i a}$, $\Lambda/\hbar = e^{\theta}$

Regge-Wheeler, Teukolsky, Zerilli,.....

Perturbation of BH (simplified): scalar perturbation D3 stack brane

- Upon radial-angular separation of **Black Hole** wave-form

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1+\frac{M^4}{r^4}\right) - \frac{(l+2)^2-\frac{1}{4}}{r^2}\right]\phi = 0$$ radial Regge-Wheeler eq.

- Change of variables $r = Me^{-\frac{y}{2}}$ $\omega M = 2ie^{\theta}$ $P = \frac{1}{2}(l+2)$

  to bring it into the integrability form $\phi = e^{\frac{y}{4}}\psi$ **BH frequency**

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{y}+e^{-y}) + P^2\right]\psi = 0$$

- ODE/IM basis reproduce gravitational **BH boundary conditions.**

$U_0(r) \sim e^{i\omega L^2/r}$, $r \to 0$ $(y \to +\infty)$; $V_0(y) \sim e^{i\omega r}$, $r \to +\infty$ $(y \to -\infty)$

in: ingoing at horizon $r = 0$ **up**: outgoing at $r = \infty$

Regge-Wheeler, Teukolsky,....

Extensions to realistic cases

- $N_f = 2 \rightarrow$ Intersection of four stacks of D3 branes (extremal **Kerr BH**; equal charges: extremal **Reissner-Nöstrom BH**)

$$\frac{d^2\phi}{dr^2} + \left[-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \omega^2 \sum_{k=0}^{4} \frac{\Sigma_k}{r^k} \right] \phi = 0$$

- which becomes in integrability form **confluent Heun eq**

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-2y}) + 2e^{\theta}(M_1 e^y + M_2 e^{-y}) + P^2\right]\psi = 0$$

- **fully new forms of Y-systems and TBA equations**

Regge-Wheeler, Teukolsky

- $N_f = 3 \rightarrow$ **confluent Heun Schwarzshild**, Kerr both radial and angular

- $N_f = 4 \rightarrow$ Heun eq. AdS BH ($N_f = 4$): **Heun and all confluences**. Quiver gauge theories.....

Approach by gauge instantons: Atsuda, Grassi, Hatsuda; Bianchi, Di Russo, Fucito, Morales, Russo, Poghossian; Arnaudo, Bonelli, Iossa, Tanzini,......

Quasinormal modes=Bethe roots

- Imposing the BH boundary conditions on

$$iV_0(y) = Q(\theta + i\pi/2)U_0(y) - Q(\theta)U_1(y)$$

- Proper **eigen-frequencies of the back hole**

$$Q(\theta_n) = 0$$

- **Integrability Thermodynamic Bethe Ansatz equation**

$$\ln Q(\theta) = -\frac{8\sqrt{\pi^3}}{\Gamma^2(\frac{1}{4})}e^{\theta} + \int_{-\infty}^{\infty} \frac{\ln\left[1 + Q^2(\theta')\right]}{\cosh(\theta - \theta')}\frac{d\theta'}{2\pi}$$

- Sort of solution up to quadratures. Important: **Q** is the full **spectral determinant** **(Bethe roots=QNMs are only the zeroes)**.

- ODE/IM fundamental Wronskian $Q(\theta, P^2) = W[U_0, V_0]$ is the same as the **gravitational** PDE solution ⇒more info (wave-function, space-time solution, etc.): applications to

BH and Gravitational waves?

Conclusions and some perspectives

- **Painlevé/gauge (NS) theory correspondence→Floquet.**
- **Many exact results for gauge SU(2) theories with matter.**
- **Thorough application to BH physics and Cosmology ?**
- **Gauge prepotential 'solves' particular ODE/IM**: what about general ODE/IM? Integrabilty community is unsatisfied……
- Extension to more complicated higher rank gauge theories (by now only pure SU(3)).
- NS limit $\epsilon_1 = \hbar, \ \epsilon_2 = 0 \rightarrow$ ODE/IM description: $\epsilon_2 \neq 0$ quantum ODE/IM? q-TBA? Similarly about classical string in *N*=4 SYM for null-polygonal Wls.
- Mathieu ODE is level 2 null vector equation, but our Liouville field theory is not AGT: meaning of $b \neq 1$?

Thank you!