$T\bar{T}$-like flows and links to string theory

EUROSTRINGS 2025 - Stockholm

Roberto Tateo

Istituto Nazionale di Fisica Nucleare
INFN TORINO
Sezione di Torino

UNIVERSITÀ DEGLI STUDI DI TORINO
ALMA UNIVERSITAS TAURINENSIS

Reviews

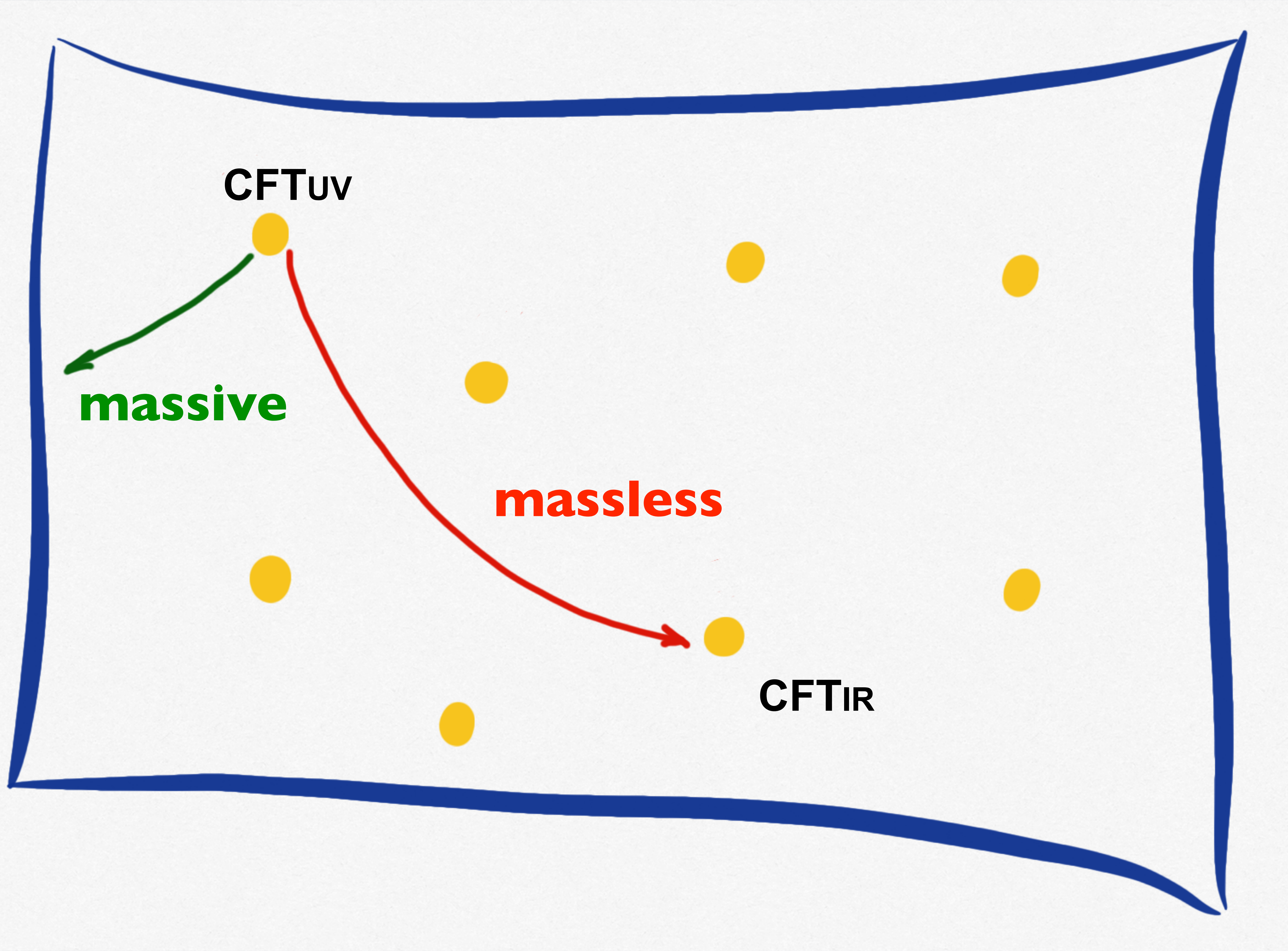
- *M. Guica, "$T\bar{T}$ deformations and holography." CERN winter school on supergravity, strings and gauge theory (2020).*

- *A. Giveon, "Comments on $T\bar{T}$, $J\bar{T}$ and String Theory", arXiv:1903.06883.*

- *Y. Jiang, "A pedagogical review on solvable irrelevant deformations of 2D quantum field theory." Communications in Theoretical Physics 73.5 (2021): 057201.*

- *S. He, Yi Li, H. Ouyang, Y. Sun, "$T\bar{T}$ Deformation: Introduction and Some Recent Advances." arXiv:2503.09997.*

Integrability and the early emergence of the $T\bar{T}$ operator



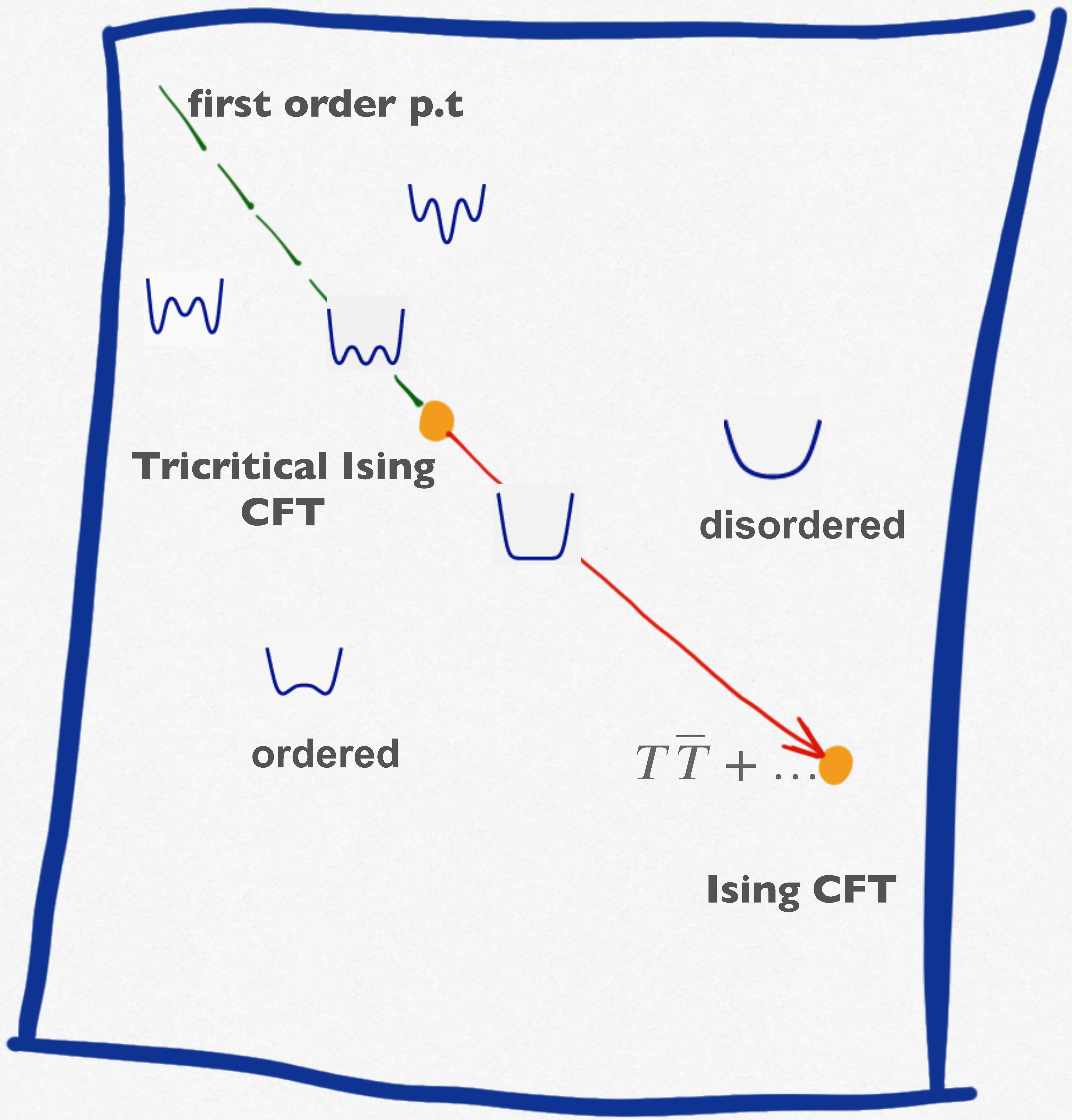
Massive and Massless integrable CFT perturbations:

Exact S-matrix

Finite-Size spectrum
(Thermodynamic Bethe Ansatz)

Correlation Functions
(Form-Factors)

(Zamolodchikov^2-Lukyanov-Bazhanov, Martins, Ravanini, Dorey, Martins, Mussardo, Delfino, Takacs, Watts, Fendley, Saleur)



Massless integrable CFT perturbations:

Sometimes $T\bar{T}$ is the leading attracting operators in the IR

In a CFT

$$T_{xx} = -T_{yy} = -\frac{1}{2\pi}(\bar{T}+T)$$

$$T_{yx} = T_{xy} = \frac{i}{2\pi}(\bar{T}-T)$$

and

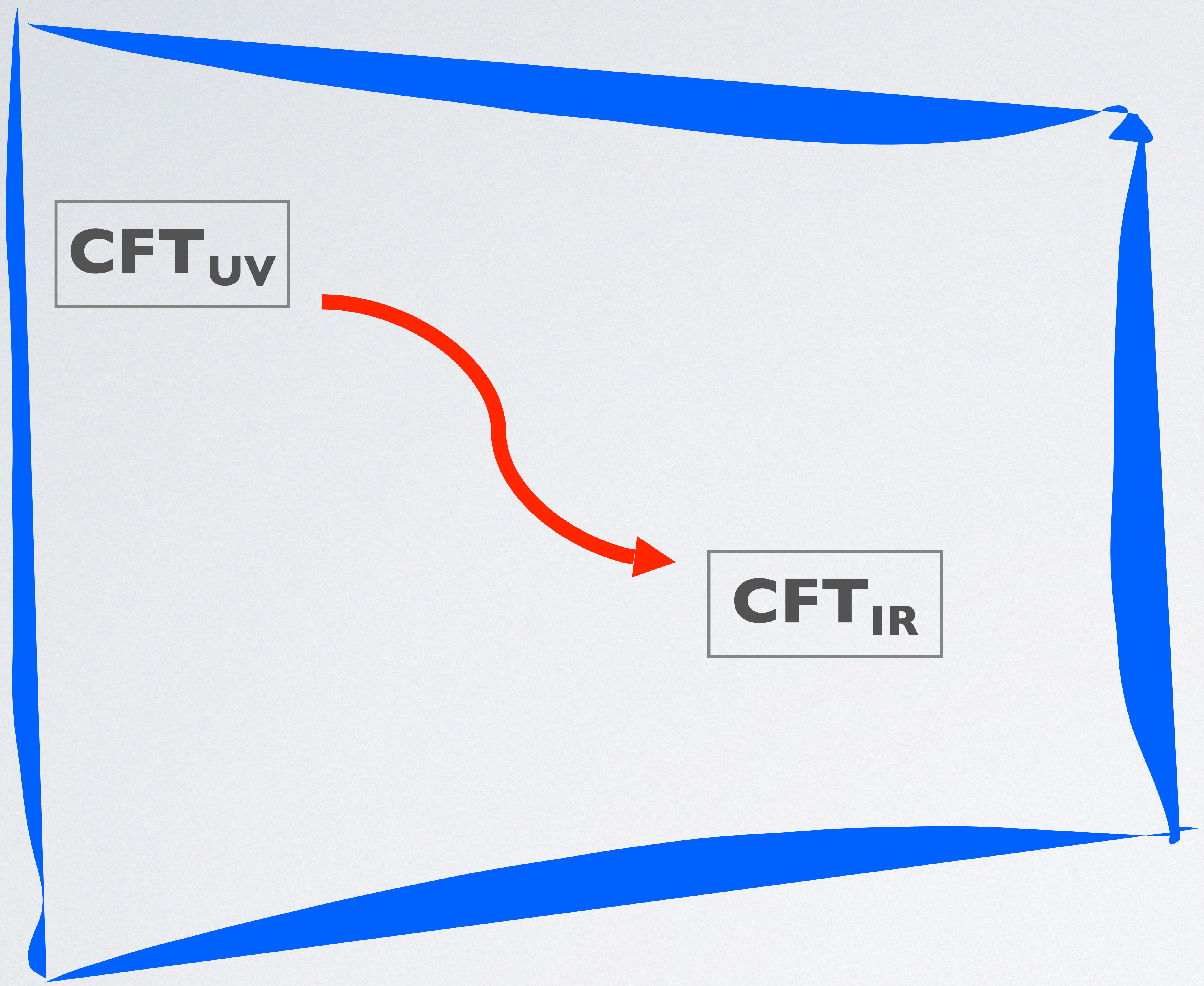
$$T\overline{T}(z,\overline{z}) = T(z)\overline{T}(\overline{z})$$

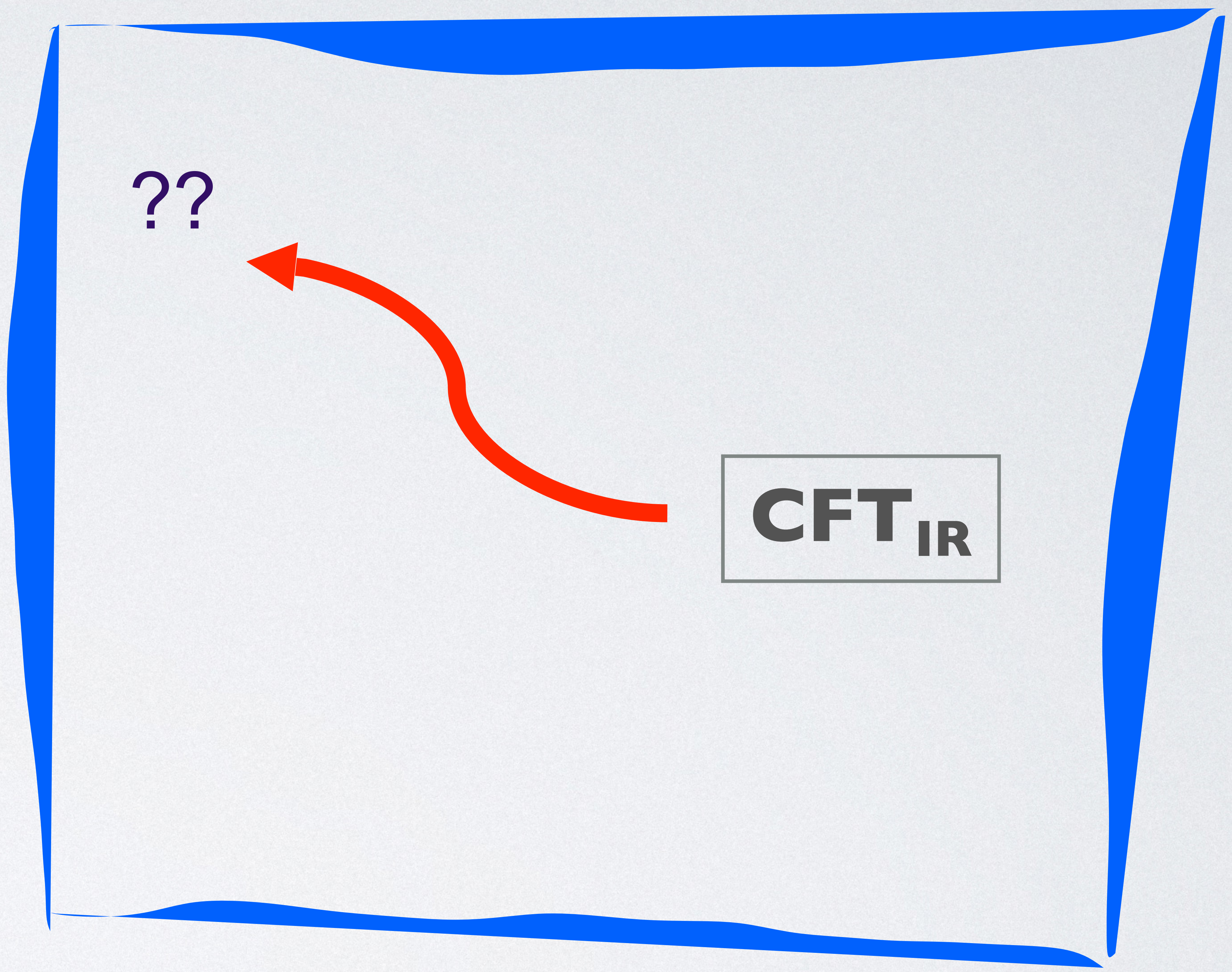
Recent interest one massless flows related to non-invertible symmetries:

$$\mathscr{M}(kq+I,q) \longrightarrow \mathscr{M}(kq-I,q)$$

(Katsevich-Klebanov-Sun '23, Tanaka-Nakayama '24)

Can we reverse the renormalisation group trajectory?





Let us try with the $T\overline{T}$ perturbation …

We need the correct definition of $T\bar{T}$ outside a CFT fixed point:

$$T_{xx} = -\frac{1}{2\pi}\left(\bar{T} + T - 2\Theta\right), \quad T_{yy} = \frac{1}{2\pi}\left(\bar{T} + T + 2\Theta\right), \quad T_{xy} = \frac{i}{2\pi}\left(\bar{T} - T\right)$$

Sasha Zamolodchikov '04:

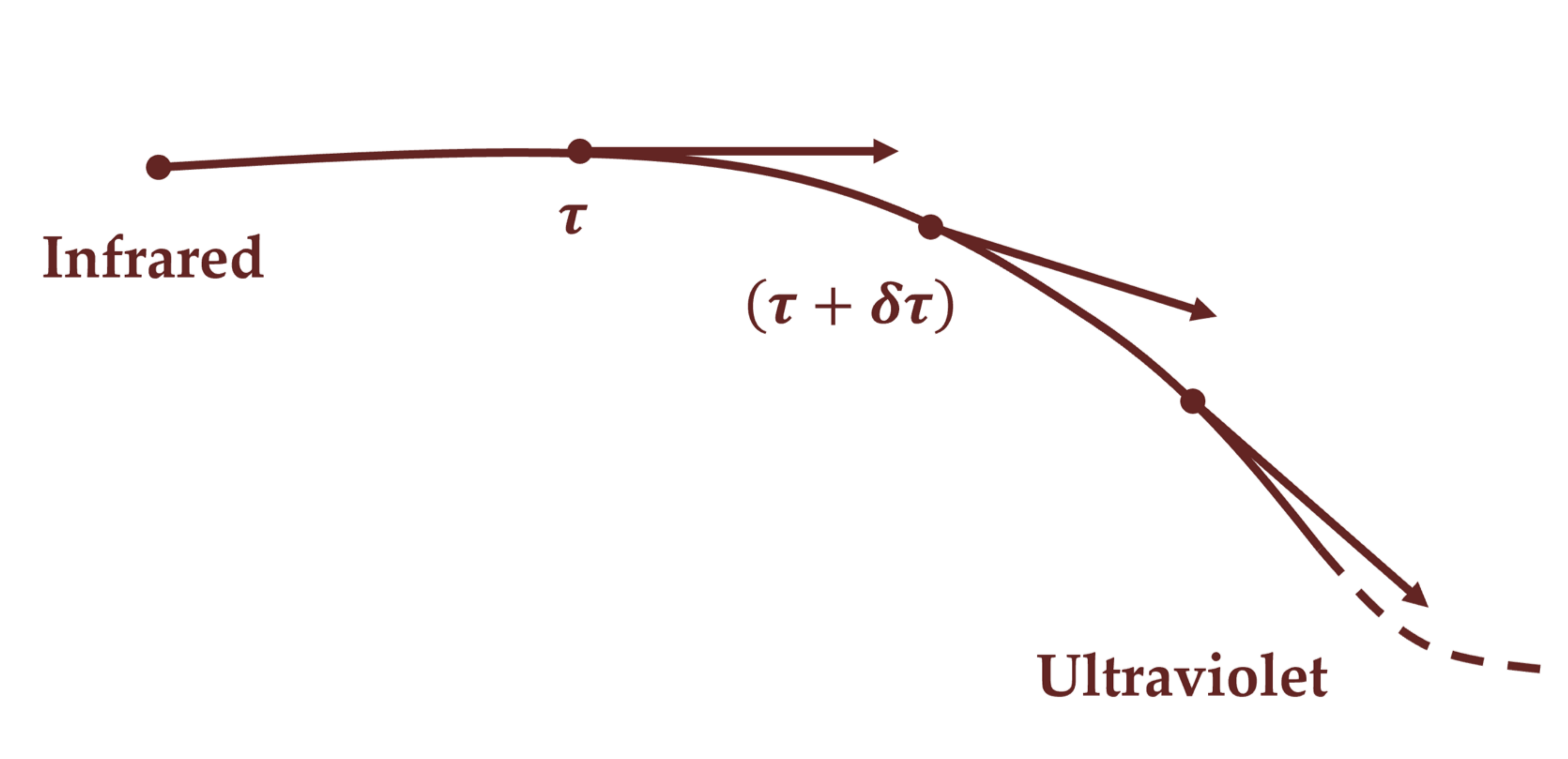
$$\mathrm{T\bar{T}}(z,\bar{z}) := \lim_{(z,\bar{z})\to(z',\bar{z}')} T(z,\bar{z})\bar{T}(z',\bar{z}') - \Theta(z,\bar{z})\Theta(z',\bar{z}') + \text{total derivatives}$$

Therefore, up to total derivatives and a classical level:

$$\mathcal{O}_{T\bar{T}} := -\det\left[T^{\mu}_{\nu}\right] = \frac{1}{2}\left(T^{\mu\nu}T_{\mu\nu} - T^{\mu}_{\mu}T^{\nu}_{\nu}\right) = -\frac{1}{2}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}T^{\mu\rho}T^{\nu\sigma}$$

The $T\bar{T}$ Lagrangian flow equation

$$\begin{cases} \partial_\tau \mathscr{L}(\tau) = \mathscr{O}^{(\tau)}_{T\bar{T}}, \\ \\ T_{\mu\nu}(\tau) = -\dfrac{2}{\sqrt{|g|}} \dfrac{\partial \mathscr{L}(\tau)}{\partial g^{\mu\nu}} \end{cases}$$



(Smirnov-Zamolodchikov ’16)

(In the following, we will initially focus on $T\bar{T}$ deformation in flat spacetime.)

Simple example: bosons with generic potential

$$\mathscr{L}^V(0) = \mathscr{L}(0) - V$$

with

$$\mathscr{L}(0) = \frac{1}{2} g^{\mu\nu} \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi}, \qquad V = V(\vec{\phi})$$

$$\Downarrow$$

$$\mathscr{L}^V(\tau) = -\frac{V}{1 - \tau V} + \frac{1}{2\bar{\tau}} \left(1 - \sqrt{1 - 4\bar{\tau}\, \mathscr{L}(0) - 4\bar{\tau}^2\, \mathscr{B}} \right)$$

with $\bar{\tau} = \tau \left(1 - \tau V \right)$ **and** $\mathscr{B} = |\, \partial \vec{\phi} \times \bar{\partial} \vec{\phi}\, |^2$

(Cavaglià-Negro-Szécsényi-RT ’16 , Bonelli-Doround-Zue ’18)

[See also earlier results earlier hints: Dubovsky-Flauger-Gorbenko ’12, Caselle-Fioravanti-Gliozzi-RT ’13]

The deformed Lagrangian can be obtained by solving the following PDE

$$\frac{\partial \mathscr{L}}{\partial \tau} = 2 g^{\mu\rho} g^{\nu\sigma} \frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} \frac{\partial \mathscr{L}}{\partial g^{\rho\sigma}} - 2 \left(g^{\mu\nu} \frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} \right)^2 + 2 \mathscr{L} g^{\mu\nu} \frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - \mathscr{L}^2 \,.$$

The latter equation can be solved either perturbatively or by using the method of characteristics.

- Free bosons $\longrightarrow$ Nambu-Goto theory in the static gauge.

- $YM_2 \longrightarrow$

$$\mathscr{L}^{YM_2}(\tau) = \frac{3}{4\tau}\left({}_3F_2\left(-\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}; \frac{1}{3}, \frac{2}{3}; \frac{256}{27}\,\tau\,\mathscr{L}^{YM_2}(0)\right) - 1\right) \, .$$

However, performing a Legendre transformation:

$$\mathscr{H}^{YM_2}(\tau) = \frac{\mathscr{H}^{YM_2}(0)}{1 - \tau\,\mathscr{H}^{YM_2}(0)}\, .$$

Formally identical equations also hold in the case of perturbations of quantum mechanical models.
(Gross-Kruthoff-Rolph-Shaghoulian).

- **Fermions can be included *(Bonelli-Doroud-Zue, Frolov, Lee-Yi-Yoon).***

- **The perturbation is compatible with supersymmetry *(Baggio-Sfondrini-Tartaglino Mazzucchelli-Walsh).***

Geometric interpretations

1) There exists a *random geometry interpretation* of the $T\bar{T}$ deformation *(Cardy '18)*

The infinitesimally deformed action near a point $t = -\tau$ is

$$S(t + \delta t) = S(t) + \frac{\delta t}{2} \int_{\mathscr{M}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} T^{\mu\rho} T^{\nu\sigma} d^2 x \, ,$$

where $\mathscr{M}$ is the domain on which the theory is defined.

Since $\det[T^{\mu}_{\nu}]$ is quadratic in components of the stress energy tensor, we can rewrite the infinitesimal deformation of the partition function by performing a Hubbard-Stratonovich transformation.

The relevant identity is

$$e^{\frac{\delta t}{2}\int_{\mathcal{M}} d^2x\,\epsilon_{\mu\nu}\epsilon_{\rho\sigma}T^{\mu\rho}T^{\nu\sigma}} \propto \int \mathcal{D}h\, e^{-\frac{1}{2\delta t}\int_{\mathcal{M}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}h_{\mu\rho}h_{\nu\sigma}\,d^2x+\int_{\mathcal{M}} d^2x\,h_{\mu\nu}T^{\mu\nu}}\,.$$

By the definition of stress energy tensor $T_{\mu\nu}$, the second term is equivalent to an infinitesimal change in the metric:

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}\,.$$

The infinitesimal parameter δt appears in the action as $1/\delta t$.

Since $\delta t \to 0$, the "gravity" sector is dominated by the saddle-point.

n principle, the functional integral should be taken over all possible geometries, including those with non-zero curvature.

However, by computing the variation of the Ricci curvature and imposing the conservation of the stress-energy tensor, one can show that it is sufficient to restrict to diffeomorphisms.

2) The $T\bar{T}$- deformation of a generic field theory is equivalent to coupling the undeformed field theory to d=2 *ghost-free massive gravity (Tolley '19).*

- $S_0[\phi, e^a_\mu]$ is an arbitrary undeformed action, where ϕ indicates a generic collection of matter fields.
- e^a_ν denotes an auxiliary dynamical zweibein, with associated metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$.
- e^a_ν is coupled to a second zweibein f^a_μ, associated to the metric tensor $h_{\mu\nu} = \eta_{ab} f^a_\mu f^b_\nu$.
- f^a_μ will eventually emerge as the metric of the manifold on which the $T\bar{T}$ deformed theory lives.

Then, the $T\bar{T}$ deformation can be generated from the action:

$$S_\tau[\phi, e^a_\mu, f^a_\mu] = S_0[\phi, e^a_\mu] + S_{\rm grav}[e^a_\mu, f^a_\mu]\,,$$

Where the topological gravity action is:

$$S_{\text{grav}}[e^a_\mu, f^a_\mu] = \frac{1}{2\tau}\int d^2x\epsilon^{\mu\nu}\epsilon_{ab}(e^a_\mu - f^a_\mu)(e^b_\nu - f^b_\nu)\,,$$

The deformed action is obtained by extremising the total action $S^{(\tau)}[\phi, e^a_\mu, f^a_\mu]$ with respect to the auxiliary zweibein e^a_μ.

Denoting the solution of the equation of motion by $e{*}^a_\mu$, then:

$$S_\tau[\phi, f^a_\mu] = S_\tau[\phi, e{*}^a_\mu, f^a_\mu]\,.$$

3) Field-dependent coordinate transformation.

(Conti et al. ’18, Coleman-Aguilera Damia-Freedman-Son, ’19)

The deformed theory is related to the original theory by a field-dependent coordinate transformation.

The deformed theory can be reformulated as

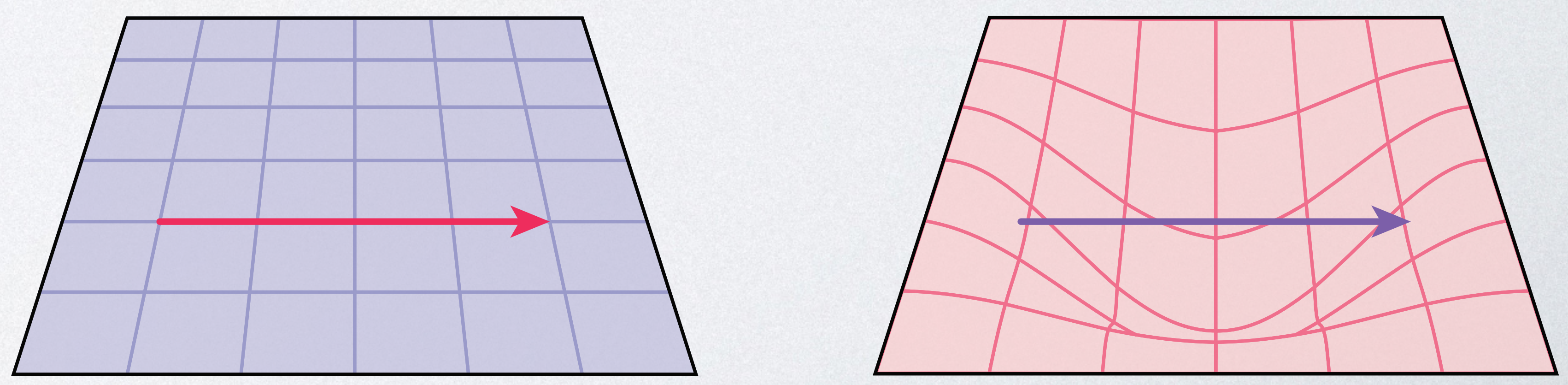
$$S_\tau = \int d^2w \mathscr{L}_\tau = \int \frac{d^2w}{\det J}(\mathscr{L}_0(\phi(x(w))) - \tau \mathcal{O}^{(\tau=0)}), \qquad (J_{ij} = \partial_{x^i} w^j\,, \,)$$

where x and w are coordinates for the original and the deformed theory, respectively.
They are related by a field-dependent coordinate transformation involving the stress–energy tensor:

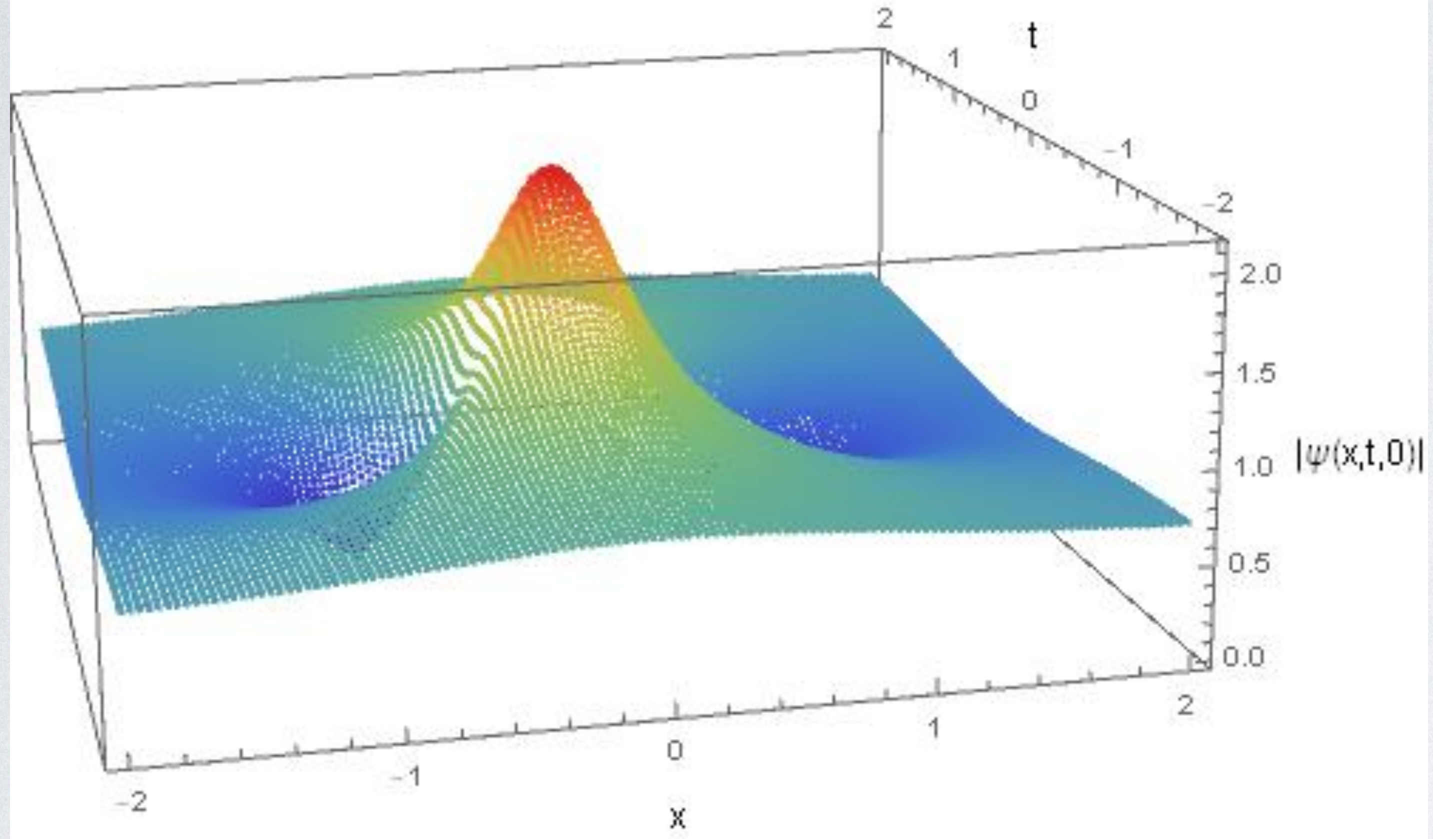
$$dw^\mu = dx^\mu - \tau \epsilon^{\mu\beta} \epsilon_{\alpha\nu} T^\alpha{}_\beta dx^\nu\,.$$

This also implies that the $T\bar{T}$-deformed solutions of the equations of motion may be constructed directly from the undeformed solutions through the corresponding transformation:

$$\phi(x,\tau) = \phi(w(x),0)\,.$$

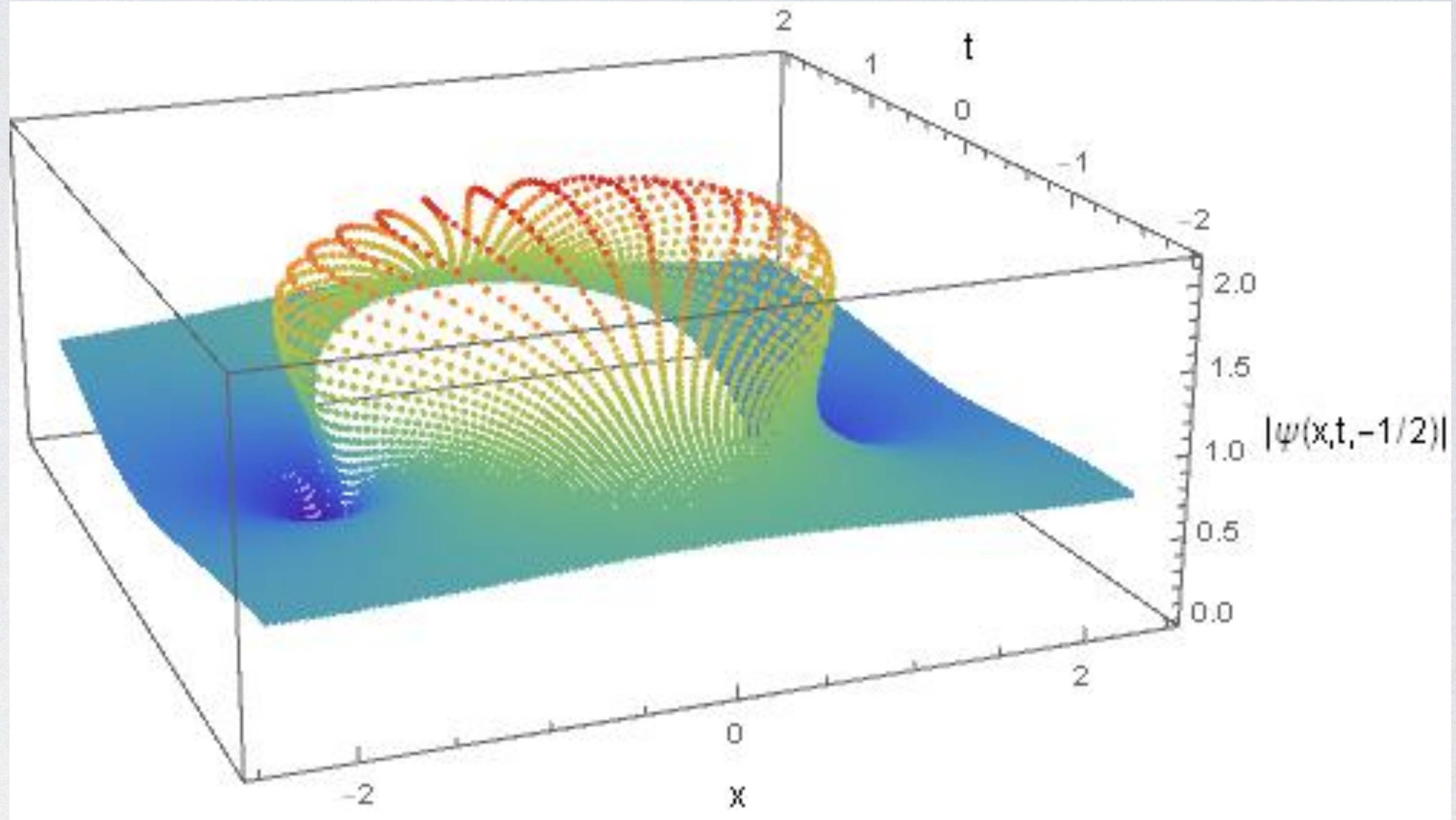


Deformed solutions (NLS - Peregrine's solutions)

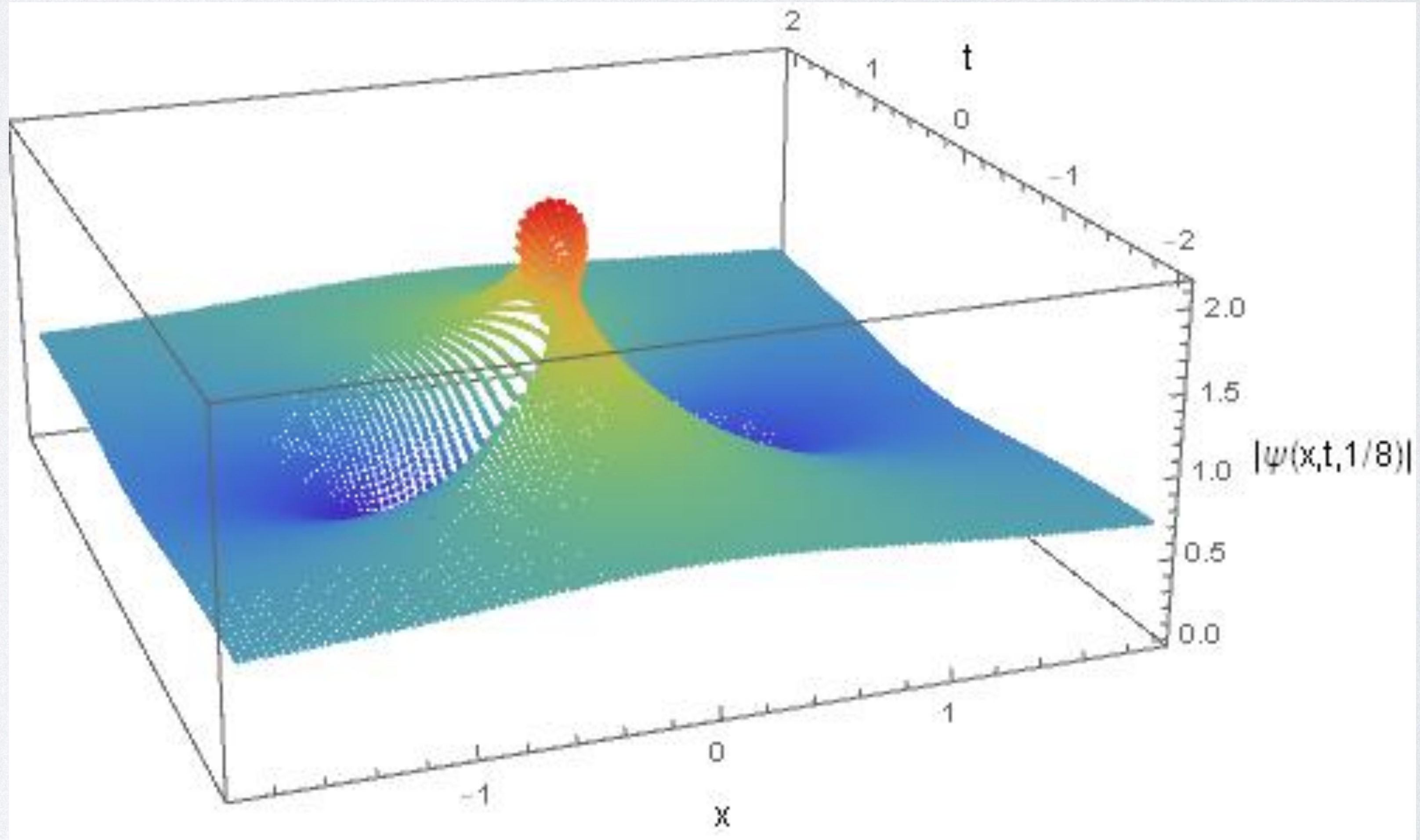


$\tau = 0$

$\tau > 0$



$\tau < 0$



(Conti et al., Esper-Frolov)

4) Any $T\bar{T}$-deformed field theory is dynamically equivalent to the corresponding undeformed theory coupled to flat Jackiw–Teitelboim gravity. ***(Dubovsky-Gorbenko-Mirbabayi '17)***

$$S_{\mathrm{M},\tau} \simeq S_{\mathrm{M}} + \int \mathrm{d}^2\mathbf{x}\sqrt{-g}\,(\varphi R - \Lambda_2) \qquad \tau \propto \Lambda_2^{-1}$$

5) $T\bar{T}$-deformed and the light-cone gauge ***(Frolov '19)***

- It gives a direct relation between the condition of the gauge invariance of the target space-time energy and momentum of a (non-critical) string theory quantised in a generalised uniform light-cone gauge which depends on the deformation parameter and homogeneous inviscid Burgers equation associated to the evolution of the finite-size spectrum under the $T\bar{T}$ perturbation.

The root-$T\bar{T}$ deformation

A novel classically marginal deformation in d=2, was recently introduced, and denoted root-$T\bar{T}$,

$$\partial_\gamma \mathscr{L}(\gamma) = \sqrt{\frac{1}{2}T^{\mu\nu}T_{\mu\nu} - \frac{1}{4}T^{\mu}{}_{\mu}T^{\nu}{}_{\nu}}\,.$$

It commutes with the $T\bar{T}$ deformation

$$\partial_\gamma \partial_\tau \mathscr{L}(\gamma,\tau) = \partial_\tau \partial_\gamma \mathscr{L}(\gamma,\tau)\,.$$

$$\begin{array}{ccc} \mathscr{L}(0,0) & \longrightarrow & \mathscr{L}(0,\tau) \\ \downarrow & & \downarrow \\ \mathscr{L}(\gamma,0) & \longrightarrow & \mathscr{L}(\gamma,\tau) \end{array}$$

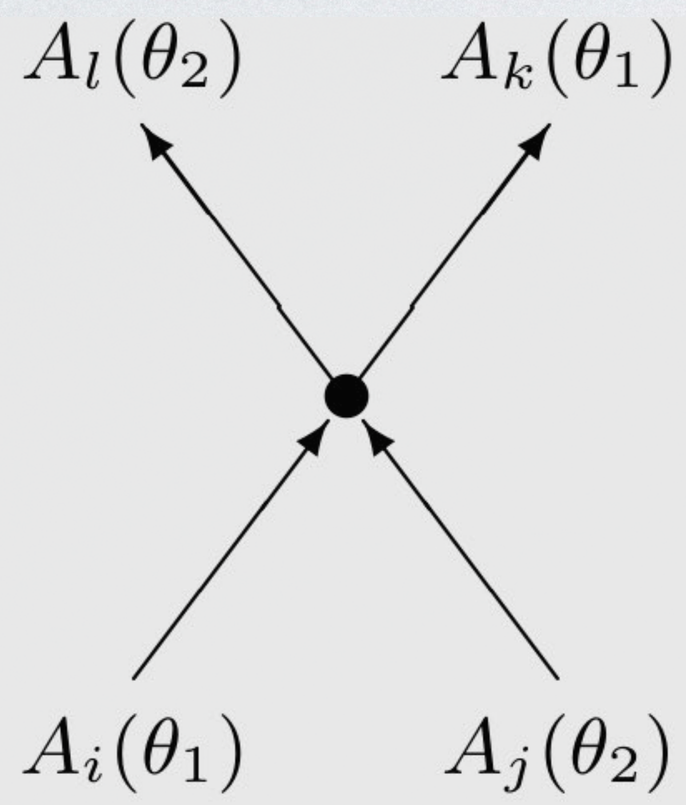
It corresponds to a change of the metric, but not to a global change of coordinates.

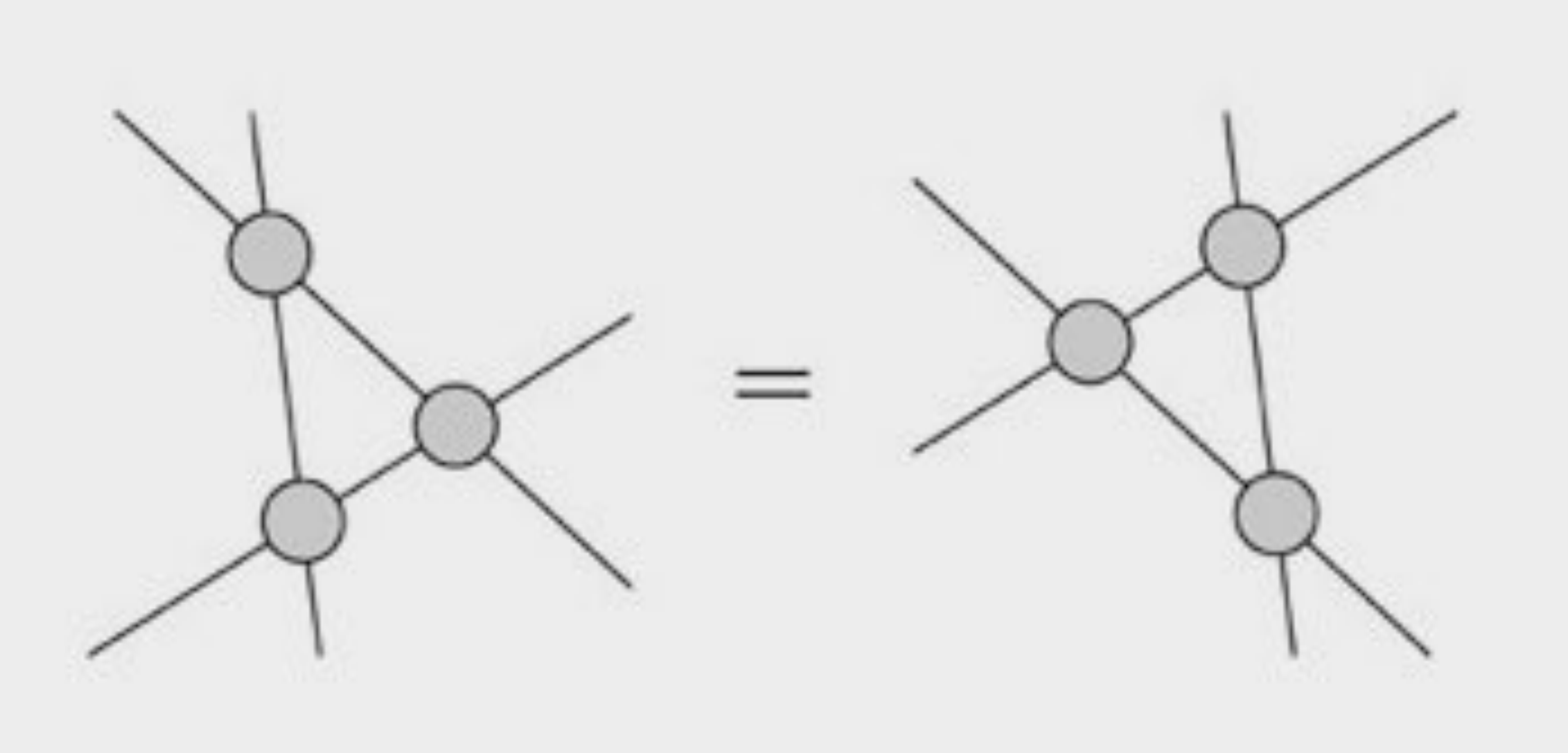
(Conti et al., Ferko-Sfondrini-Smith-Tartaglino Mazzucchelli, Babaei Aghbolagh-Babaei Velni-Mahdavian Yekta -Mohammadzadeh)

Exact Quantum Integrability: S-matrix and CDD ambiguity

Consider a relativistic integrable field theory with factorised scattering:

$$S_{ij}^{kl}(\theta_1 - \theta_2) \quad =$$



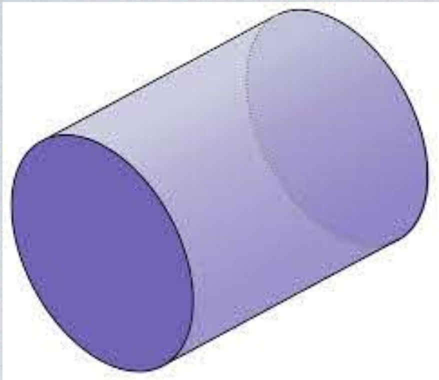


Castillejo-Dalitz-Dyson ambiguity: $S_{ij}^{kl}(\theta) \to S_{ij}^{kl}(\theta)\, e^{i\delta_{ij}^{(\tau)}(\theta)}$

The simplest possibility, consistent with the crossing and unitarity relations is:

$$\delta_{ij}^{(\tau)}(\theta) = \delta^{(\tau)}(m_i, m_j, \theta) = \tau\, m_i m_j \,\sinh(\theta)$$

(Aliosha Zamolodchikov, Mussardo-Simon, Dubovsky-Flauger-Gorbenko, Caselle et al.)



CDD-deformed Conformal Field Theories

The deformed energy, obtained introducing the CDD factor in the Thermodynamic Bethe Ansatz equations, is:

$$
\begin{aligned}
E(R,\tau) &= E^{(+)}(R,\tau) + E^{(-)}(R,\tau) \\
&= -\frac{R}{2\tau} + \sqrt{\frac{R^2}{4\tau^2} + \frac{2\pi}{\tau}\left(n_0 + \bar{n}_0 - \frac{c_{\text{eff}}}{12}\right) + \left(\frac{2\pi(n_0 - \bar{n}_0)}{R}\right)^2}
\end{aligned}
$$

$$c_{\text{eff}} = c - 24\,\Delta \quad \text{(primary)}$$

which matches the form of the (D=26, $c_{eff} = 24$) Nambu-Goto spectrum. However, this result holds for a generic CFT!

(Dubovsky-Flauger-Gorbenko '12, Caselle-Gliozzi-Fioravanti-RT '13)

Quantum $T\bar{T}$-deformations on infinite cylinder of circumference R

$$\partial_\tau \mathscr{H}(\tau) = \det\big(T_{\mu\nu}(\tau)\big) \quad \longrightarrow \quad \partial_\tau \langle n | \mathscr{H}(\tau) | n \rangle = \langle n | \det\big(T_{\mu\nu}(\tau)\big) | n \rangle \, .$$

Assuming the validity of Zamolodchikov's factorisation property also for the fully deformed theory, one can derive closed flow equations for the finite-volume spectrum.

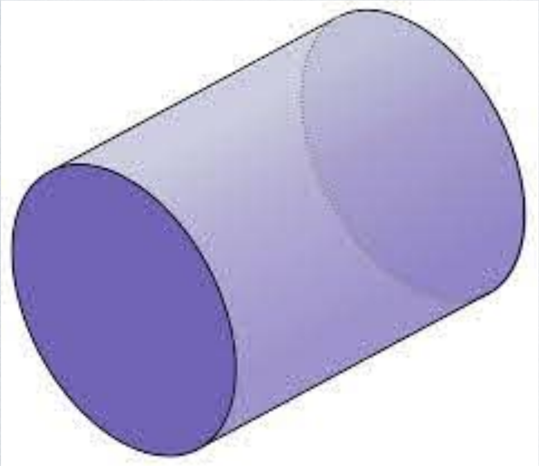
$$\langle n | \det\big(T_{\mu\nu}(\tau)\big) | n \rangle = \langle n | T_{11} | n \rangle \langle n | T_{22} | n \rangle - \langle n | T_{12} | n \rangle \langle n | T_{21} | n \rangle \, ,$$

with

$$E_n(R,\tau) = -R \langle n | T_{22} | n \rangle \ , \ \partial_R E_n(R,\tau) = -\langle n | T_{11} | n \rangle \ , \ P_n(R) = -\mathbb{i} R \langle n | T_{12} | n \rangle$$

and

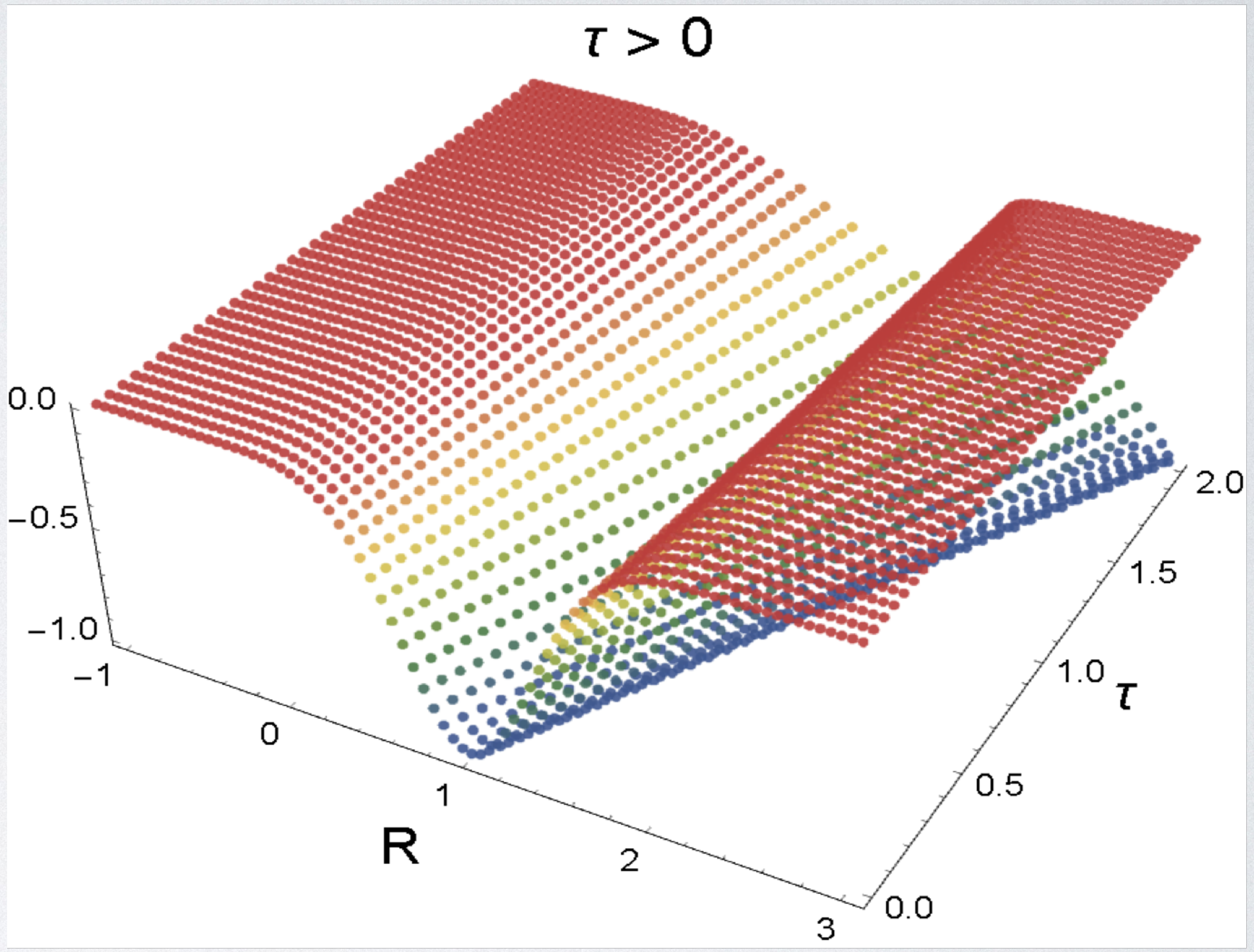
$$P(R,\tau) = P(R) = \frac{2\pi k}{R}, \quad k \in \mathbb{Z}.$$



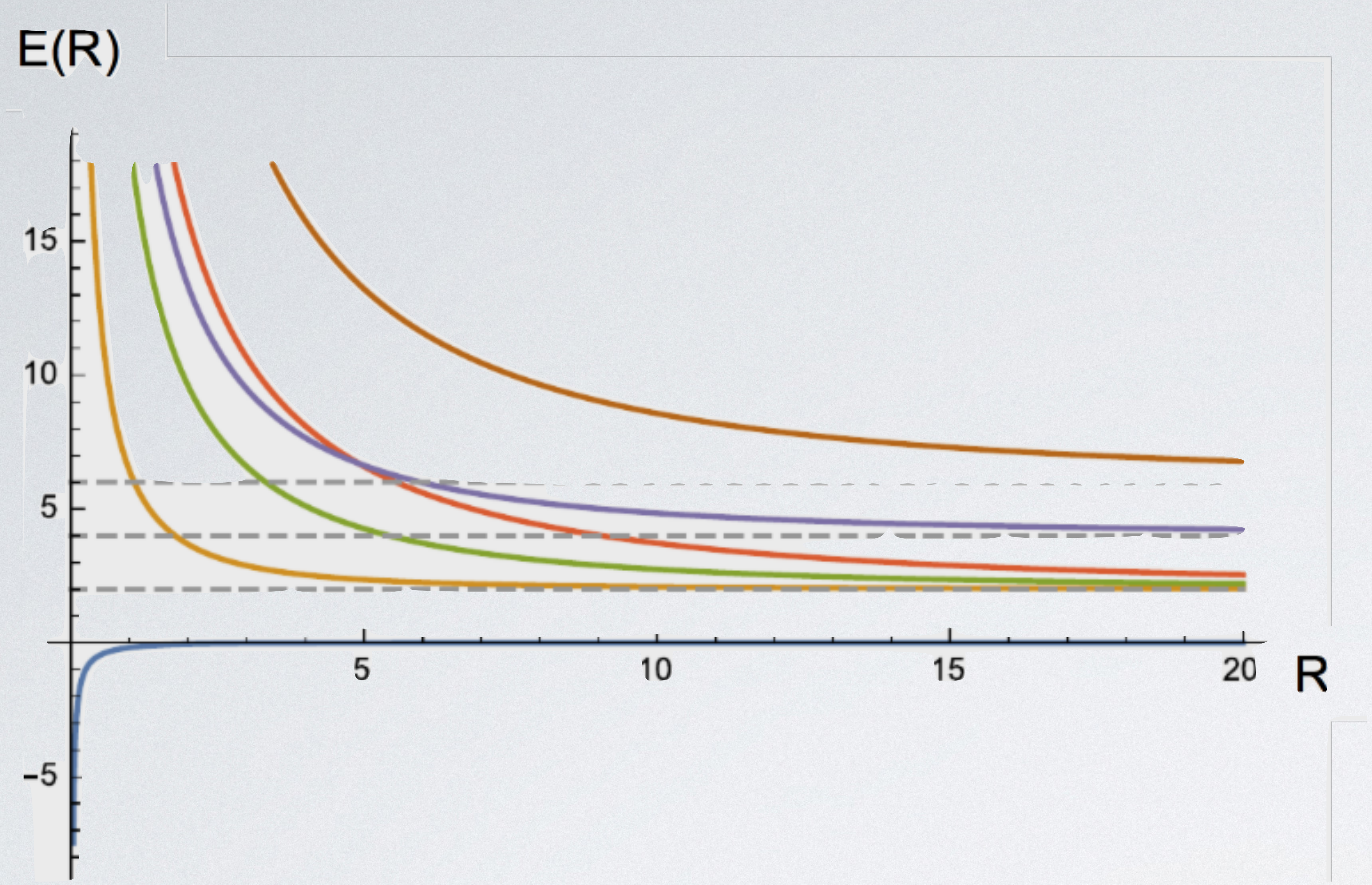
The inviscid Burgers equation for the quantum spectrum

$$\partial_\tau E_n(R,\tau) = E_n(R,\tau)\,\partial_R E_n(R,\tau) + \frac{P_n^2(R)}{R}$$

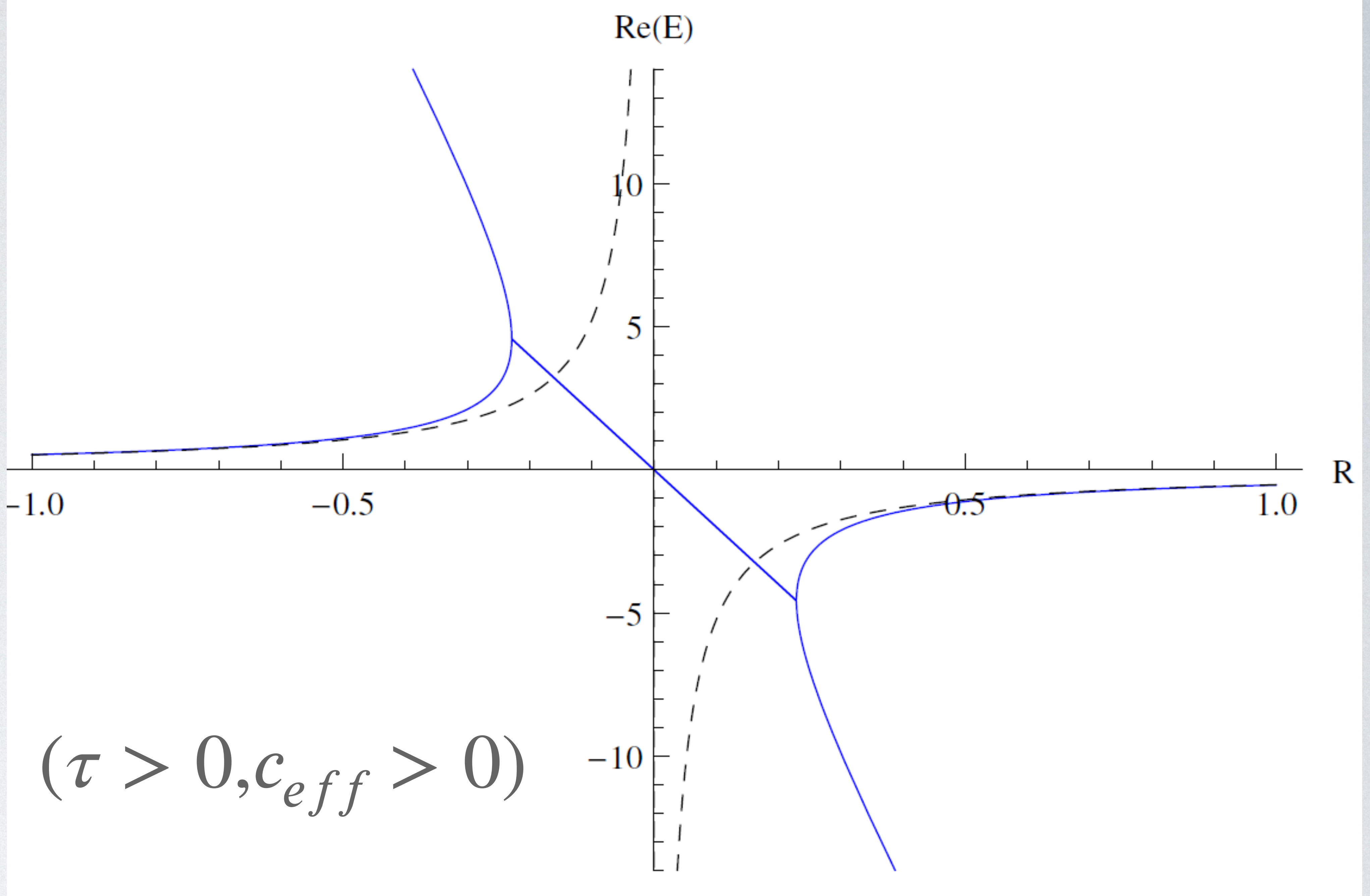
$$P_n = 0 \quad \longrightarrow \quad E_n(R,\tau) \;=\; E_n\big(R + \tau\, E_n(R,\tau), 0\big)$$

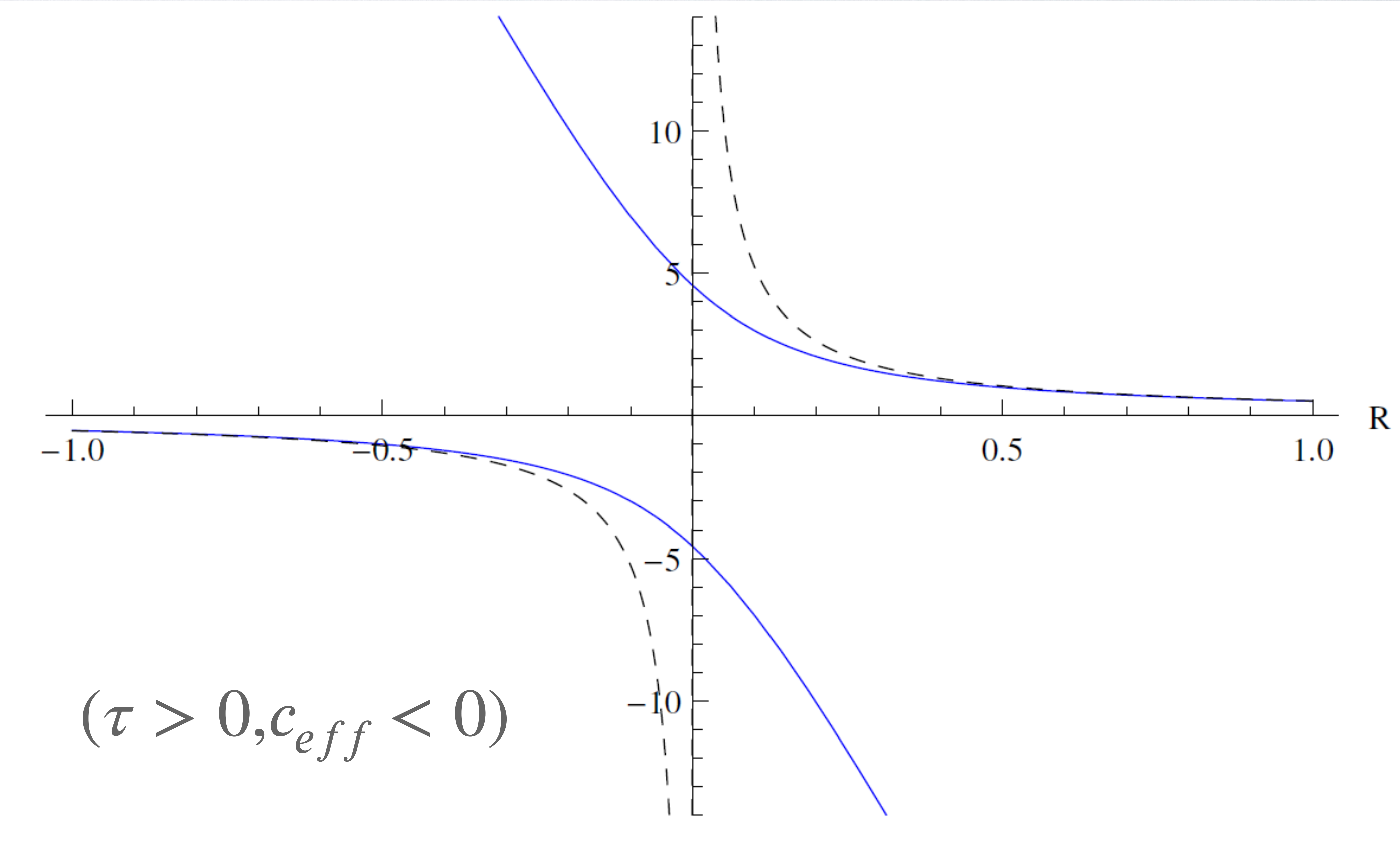


(The same result was obtained using the TBA/NLIE together with a set of relations that can essentially be regarded as the quantum version of the 'uniform light-cone' approach to the $T\bar{T}$-deformation introduced by Frolov.)



(Typical $\tau = 0$ finite-volume spectrum)

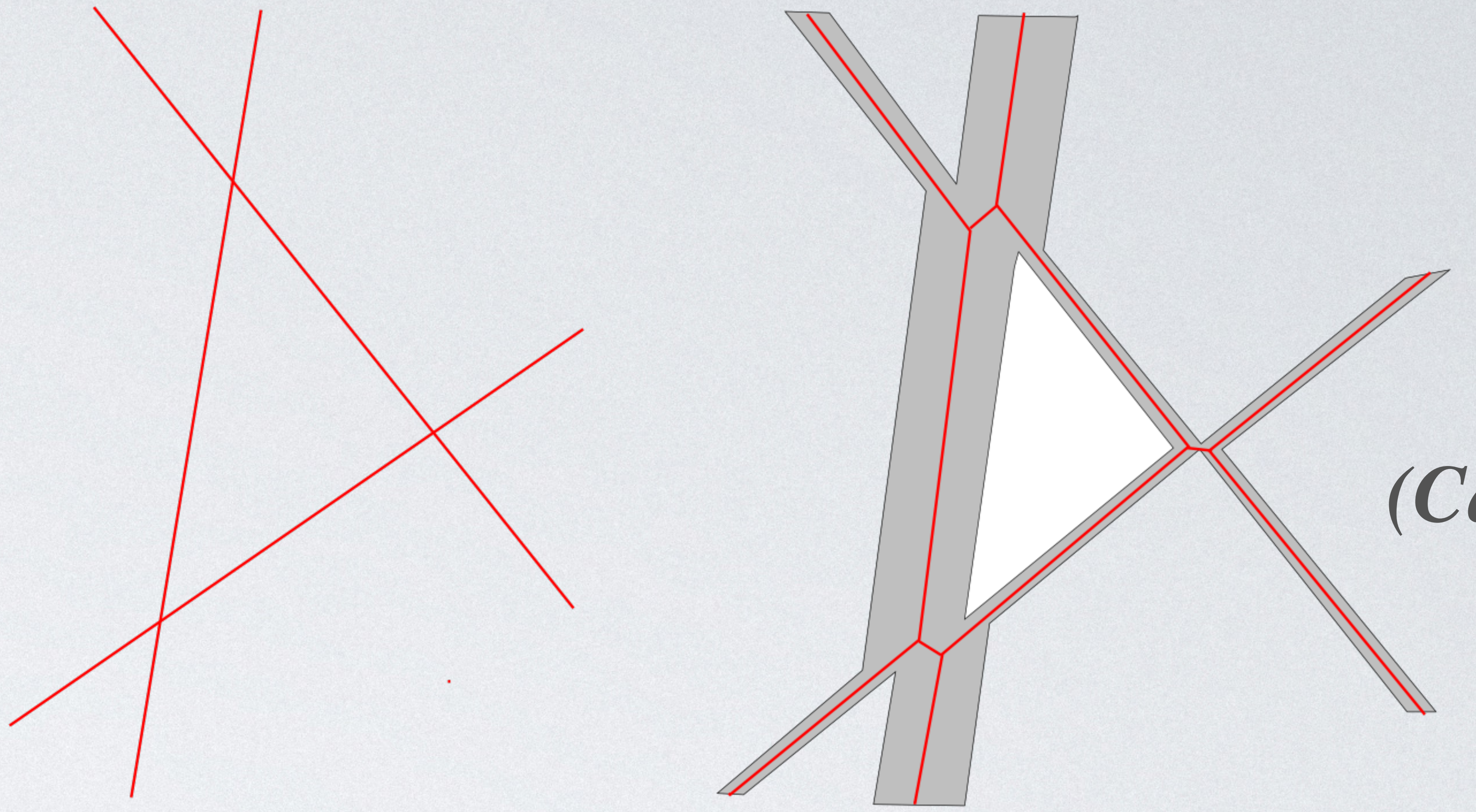




$$E(R,0) \sim -\pi \frac{c_{\rm eff}}{6\,R}, \quad R \sim 0,$$

A possible quantum interpretation:

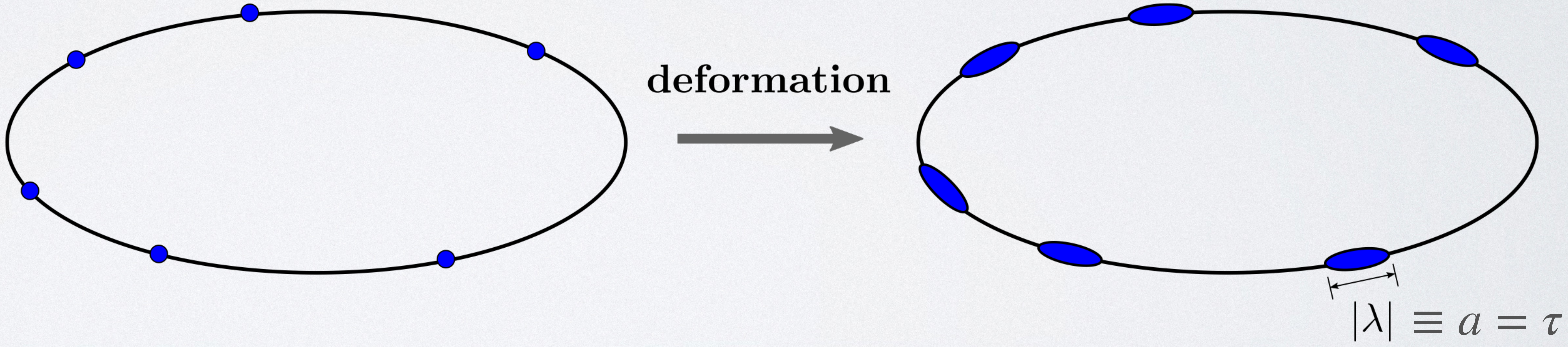
Point particles $\longrightarrow$ **Finite size particles**



(Cardy-Doyon)

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i<j}^{N} v(x_i - x_j)$$

$$v(x) = \begin{cases} \infty, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$



point-particle gas

hard rod gas

(Jiang)

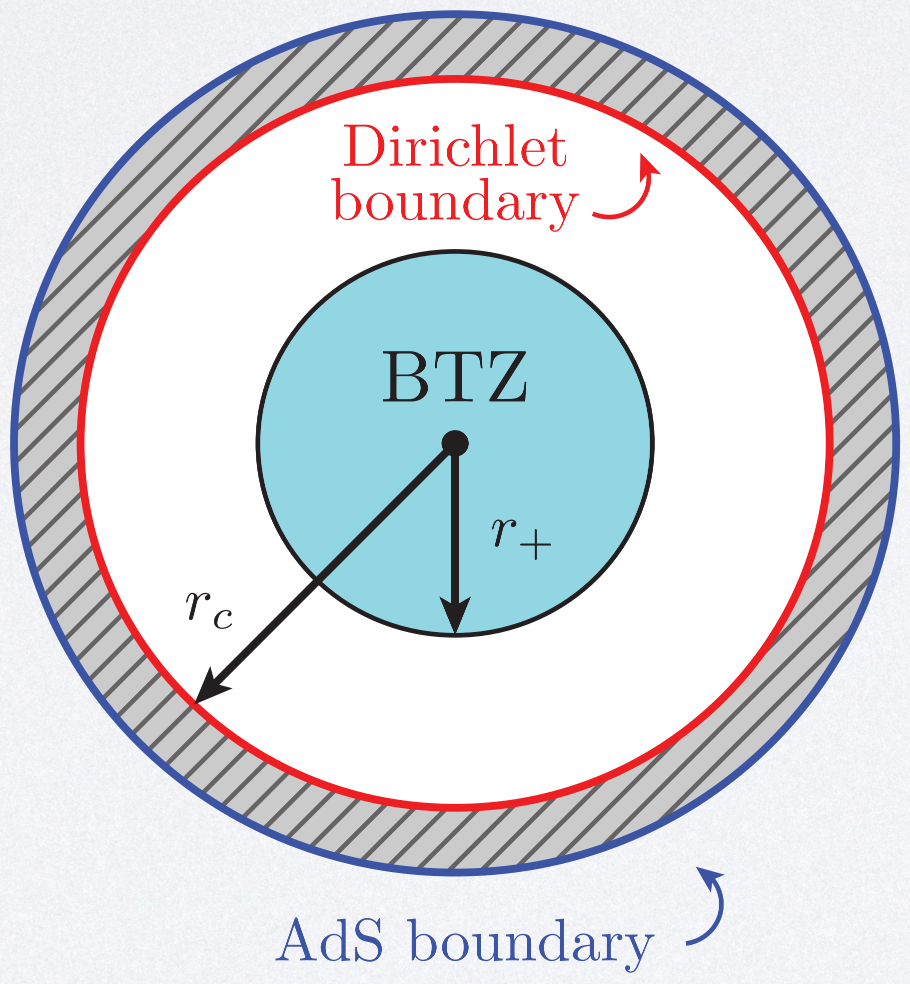
$T\bar{T}$ and AdS/CFT

T$\bar{\text{T}}$ deformations provide an exactly solvable example of irrelevant flows in 2D QFT.

In AdS/CFT, for $\tau < 0$, they correspond to introducing a finite radial cutoff with Dirichlet boundary conditions.

In black hole backgrounds, as the cutoff approaches the outer horizon, the spectrum becomes complex — signaling instabilities and loss of unitarity.

$\longrightarrow$ the complex energy levels are not part of the spectrum.

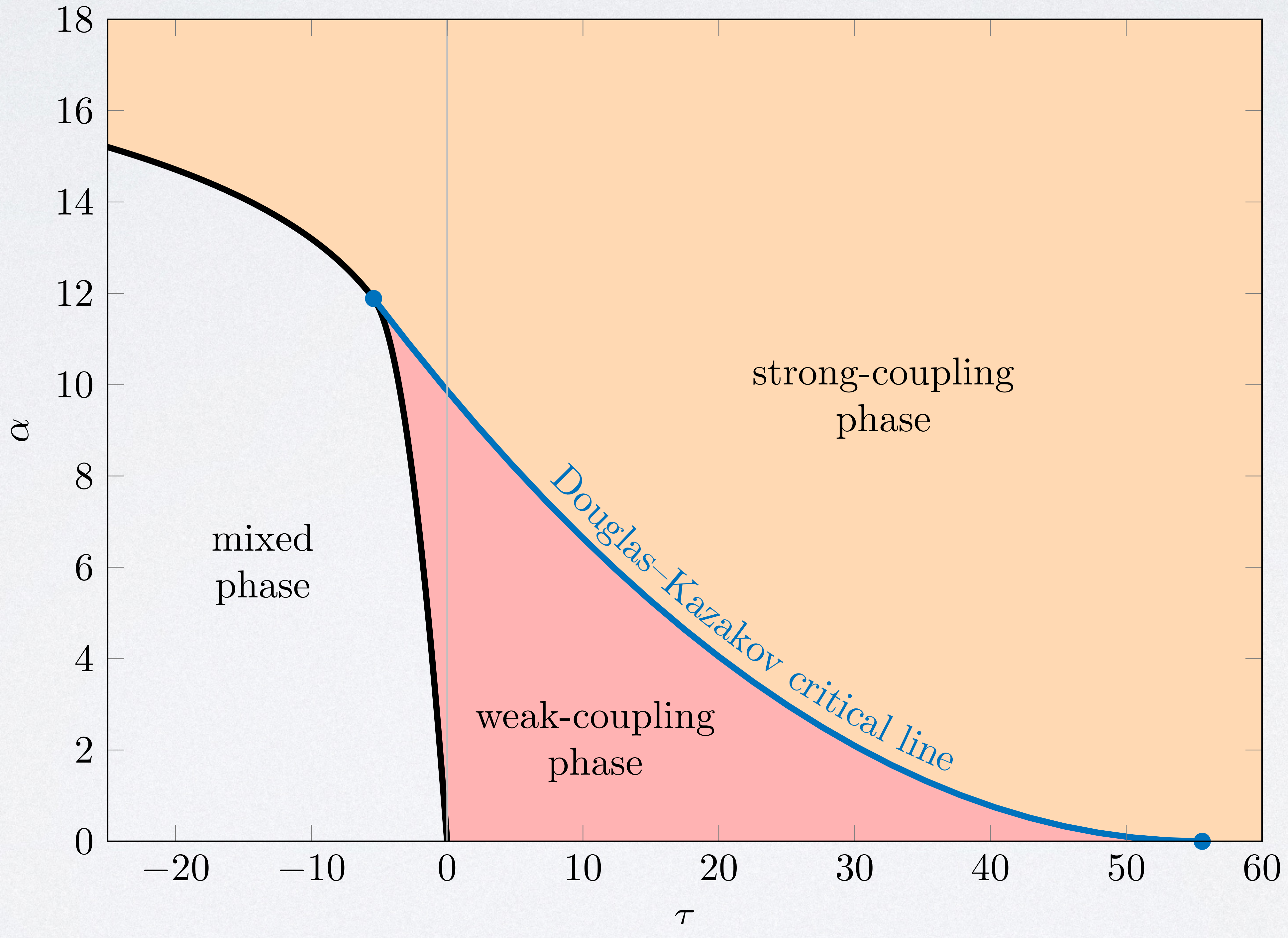


(McGough-Mezei-Verlinde '16, Guica-Monten '19)

YM_2 on the sphere $T\bar{T}$ - perturbed

In YM_2, for a theory on the sphere, it has been shown that the energy levels which develop singularities (in this case, pole singularities) disappear from the spectrum for $\tau > 0$.

The correct regularisation of the partition function for $\tau < 0$, involving instanton-like subtractions, was also identified.



(Griguolo–Panerai–Papalini–Seminara '22)

$T\bar{T}$ in d>2
(Stress-energy tensor deformations)

- They can be always interpreted as metric deformations.

- Connections with modified gravity models emerge, in particular with massive gravity.

- $T\bar{T}$-deformed theory can be coupled to Einstein gravity.

- There are connections with nonlinear electrodynamics and the so-called Modified Maxwell theory (ModMax).

(Taylor, Cardy, Conti, Romano, Morone, Negro, Ferko, Sethi, Bielli, Smith, Sorokin, Tartaglino-Mazzucchelli, Kuzenko, Lechner, Tsolakidis, Blair, Lahnsteiner, Obers, Yan, Babaei-Aghbolagh, He, Ouyang, Hou,…)

The Born-Infeld lagrangian is:

$$\mathscr{L}_{BI}(\tau) = \frac{-1 + \sqrt{1 - \tau F^{\mu\nu}F_{\mu\nu} + \frac{\tau^2}{4}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)^2}}{2\tau},$$

this Lagrangian describes the effective dynamics of the gauge field on the worldvolume of a D3-brane in string theory, and the parameter $\tau \propto 1/T$, where T is the tension of the brane.

We have

$$\frac{\partial \mathscr{L}_{BI}(\tau)}{\partial \tau} = \frac{1}{8}\left(T^{(\tau)\mu\nu}T^{(\tau)}_{\mu\nu} - \frac{1}{2}(T^{(\tau)\mu}{}_{\mu})^2\right).$$

Another interesting and recently discovered model of non-linear electrodynamics is the Modified Maxwell (ModMax) theory, described by the Lagrangian

$$\mathscr{L}_{ModMax}(\gamma) = \frac{1}{4}\left(-\cosh(\gamma)F^{\mu\nu}F_{\mu\nu} + \sinh(\gamma)\sqrt{\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)^2}\right).$$ **(Bandos, Lechner, Sorokin, Townsend ’20)**

Unique, nonlinear duality-invariant conformal extension of Maxwell’s theory!

Similarly, the ModMax Lagrangian satisfies the d=4 version of the so-called root-$T\bar{T}$ flow equation

$$\frac{\partial \mathscr{L}^{(\gamma)}}{\partial \gamma} = \frac{1}{2}\sqrt{\widehat{T}^{(\gamma)\mu\nu}\,\widehat{T}^{(\gamma)}_{\mu\nu}}\,,$$

(Babaei-Aghbolagh, Velni, Yekta, Mohammadzadeh)

where

$$\widehat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T^{\rho}_{\rho}\,.$$

Like ordinary Maxwell theory, both the Born–Infeld and ModMax theories also possess invariance under electric–magnetic duality rotations.

Quadratic stress-energy tensor deformations

Consider the family of deformations defined by the flow equation

$$\partial_\tau \mathscr{L}(\tau) = \mathcal{O}^{(\tau)}(d,r) = \frac{1}{2d}\left(T^{(\tau)\mu\nu}T^{(\tau)}_{\mu\nu} - r\,(T^{(\tau)\mu}{}_{\mu})^2\right).$$

Then, denoting the deformed metric by $h^{(\tau)}_{\mu\nu}$, such that $h^{(0)}_{\mu\nu} = g_{\mu\nu}$, one has

$$\frac{dh^{(\tau)}_{\mu\nu}}{d\tau} = \frac{2}{d}\left(T^{(\tau)}_{\mu\nu} - rT^{(\tau)\alpha}{}_{\alpha}h^{(\tau)}_{\mu\nu}\right).$$

This nonlinear equation can be again exactly-solved using the method of characteristics:

$$h^{(\tau)}_{\mu\nu} = g_{\mu\nu} - \frac{2\tau}{d}\left(-T^{(0)}_{\mu\nu} + rT^{(0)\alpha}{}_{\alpha}g_{\mu\nu}\right) + O(\tau^2).$$

- For the d=2 $T\bar{T}$ perturbation, this expansion truncates at second order in τ, reproducing the known result previously quoted.

- In d=4, iff the matrix T^{μ}_{ν} has only two independent eigenvalues each of multiplicity 2, the series truncates at first order.

Under the change of metric we have:

$$S_\tau[h_{\mu\nu}, \phi] = \left\{ S[g_{\mu\nu}, \phi] - \tau \int d^d x \sqrt{|g|}\ \mathcal{O}^{(\tau=0)}(d, r) \right\}\Bigg|_{g=g(h)},$$

which generalises to arbitrary dimensions the results obtained from the geometrical interpretations in d=2.

Thank you for your attention!