A variational formulation of the Hitchin system

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Motivation

General principal: [Costello '13, Costello-Witten-Yamazaki '18, '19, ...]

$\boxed{\text{d-dimensional Integrable Field Theory on } M}$ $\longleftrightarrow$ $\boxed{(d+2)\text{-dimensional topological/holomorphic gauge theory on } M \times C}$

Examples:

3d mixed BF on $\mathbb{R} \times C$	4d CS on $\mathbb{R}^2 \times C$	5d phCS 5d 2CS	6d hCS on $\mathbb{PT}^3 \cong \mathbb{R}^4 \times \mathbb{CP}^1$	...
This talk $\updownarrow$ [BV-Winstone]	$\updownarrow$ [Costello-Yamazaki]	$\updownarrow$	$\updownarrow$ [Bittleston-Skinner]	
Hitchin's system	2d IFT on $\mathbb{R}^2$	3d IFT	4d IFT on $\mathbb{R}^4$	
C of any genus g	$C = \mathbb{CP}^1$ or E_τ	$C = \mathbb{CP}^1$	$C = \mathbb{CP}^1$	

Completely integrable Hamiltonian systems

Let M be an n-dimensional manifold.

A Hamiltonian system on T^*M with Hamiltonian $H : T^*M \to \mathbb{R}$ is **completely integrable** (or **Liouville integrable**) if there exists

$$H_1, \dots, H_n : T^*M \longrightarrow \mathbb{R}$$

that are: 1) conserved: $\frac{d}{dt} H_i = \{H_i, H\} = 0 \qquad i = 1, \dots, n$

2) in involution: $\{H_i, H_j\} = 0 \qquad i,j = 1, \dots, n$

3) functionally independent: $dH_1 \wedge \dots \wedge dH_n \neq 0$ a.e. on T^*M.

Remark: Usually take $H_1 = H$. Convenient/natural to treat all H_i, $i = 1, \dots, n$ on an equal footing $\rightsquigarrow$ **integrable hierarchy**!

Lagrangian multiforms

Provide a variational criterion for integrability. [Lobb–Nijhoff '09]

<u>Individual</u> Hamiltonian flow (fixed $i \in \{1,\dots,n\}$)

$$\frac{dq^\mu}{dt^i} = \frac{\partial H_i}{\partial p_\mu} \,, \qquad \frac{dp_\mu}{dt^i} = -\frac{\partial H_i}{\partial q^\mu}$$

on T^*M is encoded variationally in the first order Lagrangian

$\mathcal{L}_i(p,q) := p_\mu \frac{dq^\mu}{dt^i} - H_i(p,q)$ for a map

$$(p,q) : \mathbb{R} \longrightarrow T^*M \,, \quad t^i \longmapsto (p_\mu(t^i), q^\mu(t^i)) \,.$$

Individual action: $\displaystyle S_i[p,q] = \int_{\mathbb{R}} \mathcal{L}_i(p,q)\, dt^i \,.$

NB: No sum!

Q: How to encode $\{H_i, H_j\} = 0$ variationally ?

Lagrangian 1-forms

Key idea: [Lobb-Nijhoff, Caudrelier, Singh, Sleigh, Stoppato, Suris, Vermeeren,...]

Introduce the Lagrangian 1-form on $T^*M \times \mathbb{R}^n$ [Caudrelier-Harland '25]

$$\mathscr{L}(p,q,t) := p_\mu \, dq^\mu - H_i(p,q)\, dt^i .$$

Now summed!

[summation convention.]

For any immersion from multitime space

$$\Sigma : \mathbb{R}^n \longrightarrow T^*M \times \mathbb{R}^n , \quad (u^j) \longmapsto \left(p_\mu(u), q^\mu(u), t^i(u)\right).$$

consider the family of actions

$$S_\Gamma[\Sigma] := \int_0^1 \Gamma^* \Sigma^* \mathscr{L} = \int_0^1 \left(p_\mu \frac{\partial q^\mu}{\partial u^j} - H_i(p,q) \frac{\partial t^i}{\partial u^j} \right) \frac{du^j}{ds} \, ds .$$

parametrised by curves $\Gamma : (0,1) \longrightarrow \mathbb{R}^n$, $s \longmapsto (u^j(s))$.

Lagrangian 1-forms

<u>Equations of motion:</u> Find Σ s.t. $\delta_\Sigma S_\Gamma[\Sigma] = 0$ for all Γ in $\mathbb{R}^n$.

For p_r: (1) $$\frac{\partial q^r}{\partial u^j} = \frac{\partial H_i}{\partial p_r}\frac{\partial t^i}{\partial u^j}$$

For q^r: (2) $$\frac{\partial p_r}{\partial u^j} = -\frac{\partial H_i}{\partial q^r}\frac{\partial t^i}{\partial u^j}$$

For t^i: (3) $$\frac{\partial H_i}{\partial p_r}\frac{\partial p_r}{\partial u^j} + \frac{\partial H_i}{\partial q^r}\frac{\partial q^r}{\partial u^j} = 0 \overset{(1)+(2)}{\Longrightarrow} \{H_i, H_j\}\frac{\partial t^i}{\partial u^j} = 0.$$

Also (1) + (2) $\Rightarrow$ $\left(\frac{\partial t^i}{\partial u^j}\right)$ is invertible since $d_u\Sigma = \left(\frac{\partial p_r}{\partial u^j}, \frac{\partial q^r}{\partial u^j}, \frac{\partial t^i}{\partial u^j}\right)$ is injective.

Therefore,

$$\boxed{\frac{\partial q^r}{\partial t^i} = \frac{\partial H_i}{\partial p_r}, \quad \frac{\partial p_r}{\partial t^i} = -\frac{\partial H_i}{\partial q^r} \quad \& \quad \{H_i, H_j\} = 0} \quad i,j = 1,\dots,n.$$

Gauged Lagrangian 1-forms

Let G be a Lie group acting on T^*M with moment map $\mu : T^*M \longrightarrow \mathfrak{g}^*$.

Suppose $\boxed{\{H_i, \mu_a\} = 0}$ for all $i = 1, \ldots, n$ and $a = 1, \ldots, \dim \mathfrak{g}$. [$\{X_a\}$ basis of $\mathfrak{g}$]

Implement Hamiltonian reduction to $\mu^{-1}(0)/G$ as follows:

(i) Enforce $\mu(p,q) = 0$ via Lagrange multiplier

$$S_\Gamma[\Sigma, \mathcal{A}] := \int_0^1 \Gamma^* \left(\Sigma^* \mathcal{L} + \langle \Sigma^* \mu, \mathcal{A} \rangle \right)$$

with $\mathfrak{g}$-valued 1-form field $\mathcal{A} = \mathcal{A}_j \, du^j = \mathcal{A}_j^a X_a \, du^j \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$.

(ii) Pass to quotient space by gauging G-symmetry:

$$\Sigma(u) \longmapsto g(u).\Sigma(u),$$

$$\mathcal{A} \longmapsto g \mathcal{A} g^{-1} - d_{\mathbb{R}^n} g \, g^{-1} \qquad \text{for any } g \in C^\infty(\mathbb{R}^n, G).$$

[$\mathcal{A}$ promoted to gauge field along $\mathbb{R}^n$!]

Gauged Lagrangian 1-forms

<u>Equations of motion:</u> Can be rewritten as

$$\tilde{A}_i^a := \frac{\partial u^j}{\partial t^i} A_j^a$$

For A^a: (0) $\quad \mu^a(p,q) = 0\,,$

For p_μ: (1) $\quad \dfrac{\partial q^\mu}{\partial t^i} = \dfrac{\partial}{\partial p_\mu}\Big(H_i(p,q) - \mu_a(p,q)\,\tilde{A}_i^a(t)\Big)$

For q^μ: (2) $\quad \dfrac{\partial p_\mu}{\partial t^i} = -\dfrac{\partial}{\partial q^\mu}\Big(H_i(p,q) - \mu_a(p,q)\,\tilde{A}_i^a(t)\Big)$

For t^i: (3) $\quad \{H_i, H_j\} = 0\,.$

Crucially, if action of G is free then

$$(1) + (2) \implies \boxed{\frac{\partial \tilde{A}_j^a}{\partial t^i} - \frac{\partial \tilde{A}_i^a}{\partial t^j} + f_{bc}{}^a\,\tilde{A}_i^b\,\tilde{A}_j^c = 0.}$$

$[X_b, X_c] = f_{bc}{}^a X_a\,.$

Hitchin's completely integrable system

Let C be a Riemann surface of genus $g \geq 2$ (for simplicity here).

The Hitchin system is a completely integrable system on

$$T^* \mathrm{Bun}_G(C)$$

[Isomorphism classes of stable holomorphic principal G-bundles.]

of dimension $2(g-1)\dim \mathfrak{g}$.

Hamiltonians given by Hitchin fibration

$$p : T^*_{[P]}\mathrm{Bun}_G(C) \cong H^0(C, \Lambda^{1,0}C \otimes \mathfrak{g}^*_P) \longrightarrow \bigoplus_{r=1}^{\mathrm{rk}\,\mathfrak{g}} \underbrace{H^0\left(C, (\Lambda^{1,0}C)^{\otimes(d_r+1)}\right)}_{\text{dimension } (g-1)(2d_r+1) =: m_r}$$

$$[p_r : \mathfrak{g}^* \to \mathbb{C}] \qquad B_{hol} \longmapsto (p_1(B_{hol}), \dots, p_{\mathrm{rk}\,\mathfrak{g}}(B_{hol}))$$

Remarkable fact:

$$\#\text{ Hamiltonians} = \sum_{r=1}^{\mathrm{rk}\,\mathfrak{g}} m_r = (g-1)\sum_{r=1}^{\mathrm{rk}\,\mathfrak{g}}(2d_r+1) = (g-1)\dim\mathfrak{g} =: n .$$

Holomorphic structures on principal G-bundles

Fix a smooth principal G-bundle $P \to C$.

Let $\mathcal{M}$ be affine space of $\mathfrak{g}$-valued (0,1)-connections A'' on P, i.e.

family $A''_{\bar{z}_I}(z_I, \bar{z}_I)\, d\bar{z}_I \in \Omega^{0,1}(U_I, \mathfrak{g})$ s.t.

$$A''_{\bar{z}_I} d\bar{z}_I = g_{IJ} A''_{\bar{z}_J} g_{IJ}^{-1} d\bar{z}_J - \bar{\partial} g_{IJ}\, g_{IJ}^{-1} \quad \text{on } U_I \cap U_J \neq \emptyset .$$

- Any $A'' \in \mathcal{M}$ defines a unique holomorphic structure on P:

$$0 \overset{!}{=} \tilde{A}''_{\bar{z}_I} d\bar{z}_I = h_I A''_{\bar{z}_I} h_I^{-1} d\bar{z}_I - \bar{\partial} h_I\, h_I^{-1} \quad \Rightarrow \quad \bar{\partial}\, \tilde{g}_{IJ} = 0 .$$

- Any $A'', \tilde{A}'' \in \mathcal{M}$ define isomorphic holomorphic structures iff

$$\tilde{A}'' = g A'' g^{-1} - \bar{\partial} g\, g^{-1} \quad \text{for some } g \in \mathcal{G} := \mathrm{Aut}\, P .$$

Hitchin phase space as symplectic quotient

Action $\mathcal{G} \circlearrowright \mathcal{M}$ lifts to Hamiltonian action $\mathcal{G} \circlearrowright T^*\mathcal{M}$ with **moment map**

$$\mu : T^*\mathcal{M} \longrightarrow \mathrm{Lie}(\mathcal{G})^*, \qquad \boxed{\mu_{(B,A'')}(X) = \frac{1}{2\pi i}\int_C \langle B, \bar\partial^{A''} X\rangle} .$$

Notation: • $A'' \in \mathcal{M}$

• $B \in T^*_{A''}\mathcal{M} \cong \Gamma(C, \Lambda^{1,0}C \otimes \mathfrak{g}_P^*)$, i.e. $B_{z_I}(z_I, \bar z_I)\, dz_I \in \Omega^{1,0}(U_I, \mathfrak{g}^*)$ s.t.

$$B_{z_I}\, dz_I = \mathrm{Ad}^*_{g_{IJ}} B_{z_J}\, dz_J \qquad \text{on} \qquad U_I \cap U_J \neq \emptyset .$$

• $X \in \mathrm{Lie}(\mathcal{G}) \cong \Gamma(C, \mathfrak{g}_P)$, i.e. $X_{\bar z_I}(z_I, \bar z_I) \in C^\infty(U_I, \mathfrak{g})$ s.t.

$$X_{z_I} = g_{IJ} X_{z_J}\, g_{IJ}^{-1} \qquad \text{on} \qquad U_I \cap U_J \neq \emptyset .$$

• $\bar\partial^{A''} X = \bar\partial X + [A'', X] \in \Gamma(C, \Lambda^{0,1}C \otimes \mathfrak{g}_P)$.

We then have $\qquad \boxed{T^*\mathrm{Bun}_G(C) \cong \mu^{-1}(0)/\mathcal{G}}$.

Lagrangian 1-form on $T^*\mathcal{M}$

Recall Hitchin fibration

$$p : T^*_{[P]} \mathrm{Bun}_G(C) \cong H^0(C, \Lambda^{1,0}C \otimes \mathfrak{g}^*_P) \longrightarrow \bigoplus_{r=1}^{\mathrm{rk}\,\mathfrak{g}} \underbrace{H^0\big(C, (\Lambda^{1,0}C)^{\otimes (d_r+1)}\big)}_{\text{dimension } (g-1)(2d_r+1) =: m_r}$$

$$B_{hol} \longmapsto \big(p_1(B_{hol}), \dots, p_{\mathrm{rk}\,\mathfrak{g}}(B_{hol})\big)$$

Now instead have

$$\tilde{p} : T^*_{A''}\mathcal{M} \cong \Gamma(C, \Lambda^{1,0}C \otimes \mathfrak{g}^*_P) \longrightarrow \bigoplus_{r=1}^{\mathrm{rk}\,\mathfrak{g}} \Gamma\big(C, (\Lambda^{1,0}C)^{\otimes (d_r+1)}\big)$$

smooth sections

$$B \longmapsto \big(p_1(B), \dots, p_{\mathrm{rk}\,\mathfrak{g}}(B)\big)$$

For each $r = 1, \dots, \mathrm{rk}\,\mathfrak{g}$ pick points $q_{(r,1)}, \dots, q_{(r,m_r)} \in C$.

$i \equiv (r,\ell)$

Define n Hamiltonians $H_i : T^*\mathcal{M} \to \mathbb{C}$ as $H_i(B) := p_r\big(B(q_{(r,\ell)})\big)$

$r = 1, \dots, \mathrm{rk}\,\mathfrak{g}$
$\ell = 1, \dots, m_r$

Then

$$\boxed{\mathcal{L} := \frac{1}{2\pi i} \int_C \langle B, \delta A'' \rangle - H_i(B)\, dt^i}$$

Gauged Lagrangian 1-form on $\mu^{-1}(0)/\mathcal{G}$

Given an immersion $\Sigma : \mathbb{R}^n \longrightarrow T^*\mathcal{M} \times \mathbb{R}^n$, $(u^j) \longmapsto \big(B(u), A''(u), t^i(u)\big)$

we have

$d_{\mathbb{R}^n} := du^j \frac{\partial}{\partial u^j}$

$$\Sigma^* \mathcal{L} = \frac{1}{2\pi i} \int_C \big\langle B(z,\bar z,u), d_{\mathbb{R}^n} A''(z,\bar z,u) \big\rangle - H_i\big(B(z,\bar z,u)\big)\, d_{\mathbb{R}^n} t^i .$$

Since Hamiltonians are $\mathcal{G}$-invariant $\big(H_i(gBg^{-1}) = H_i(B)\big)$,

can form gauged action

$\mathfrak{g}$-valued connection $\mathcal{A} = \mathcal{A}_j du^j \in \Omega^1(C \times \mathbb{R}^n, \mathfrak{g})$ along $\mathbb{R}^n$

$$S_\Gamma[\Sigma, \mathcal{A}] := \int_0^1 \Gamma^* \big(\Sigma^* \mathcal{L} + \langle \Sigma^* \mu, \mathcal{A} \rangle \big)$$

$\boxed{A := A'' + \mathcal{A}}$

connection on

$$\begin{tikzcd} \pi_C^* P \arrow[r, dashed] \arrow[d, dashed] & P \arrow[d] \\ C \times \mathbb{R}^n \arrow[r, "\pi_C"] & C \end{tikzcd}$$

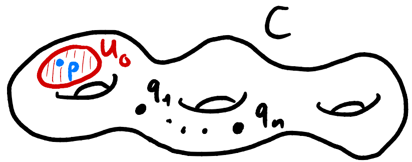
$$= \int_0^1 \Gamma^* \Big(\underbrace{\frac{1}{2\pi i} \int_C \langle B, d_{\mathbb{R}^n} A'' \rangle + \frac{1}{2\pi i} \int_C \langle B, \bar\partial^{A''} \mathcal{A} \rangle}_{} - H_i(B)\, d_{\mathbb{R}^n} t^i \Big)$$

$$= \int_0^1 \Gamma^* \Big(\frac{1}{2\pi i} \int_C \langle B, F_A \rangle - H_i(B)\, d_{\mathbb{R}^n} t^i \Big)$$

Multiform version of 3d mixed BF action!

Holomorphic description

Fix a $p \in C$ with neighbourhood $U_0 \ni p$, and let $U_1 := C \setminus \{p\}$. Then



(P, A'') is holomorphically trivial over both U_0 and U_1.

- Move to a new local trivialisation in which $A_0'' = A_1'' = 0$. Then $\pi_C^* P \to C \times \mathbb{R}^n$ is described by holomorphic (along C) transition function
$$\gamma : (U_0 \cap U_1) \times \mathbb{R}^n \longrightarrow G$$
- Solve constraint $0 = \mu_{(B,A'')}(X) = -\frac{1}{2\pi i}\int_C \langle \bar{\partial}^{A''} B, X \rangle \Rightarrow \bar{\partial} B = 0$, so Higgs field becomes holomorphic (along C) section
$$L := B_{hol} \in H^0\left(C \times \mathbb{R}^n, \pi_C^* \Lambda^{1,0} C \otimes \mathfrak{g}^*_{\pi_C^* P}\right), \qquad L_0 = \gamma L_1 \gamma^{-1}$$

Holomorphic description

Theorem: The action $S_\Gamma[\Sigma, \mathcal{A}]$ (3d mixed BF action) becomes

$$\boxed{S_{H,\Gamma}[L, \gamma, t] = \int_0^1 \Gamma^* \left(\frac{1}{2\pi i} \int_{c_p} \langle L_0, d_{\mathbb{R}^n} \gamma \gamma^{-1} \rangle - H_i(L)\, d_{\mathbb{R}^n} t^i \right).}$$

Equations of motion:

For L: $\exists\, M_{0i} \in \mathcal{O}_{hol}(U_0, \mathfrak{g})$, $M_{1i} \in \mathcal{O}_{hol}(U_1 \setminus \{q_i\}_{i=1}^n, \mathfrak{g})$ s.t.

$$M_{1i} = \gamma^{-1} M_{0i} \gamma - \gamma^{-1} \partial_{t^i} \gamma \quad , \quad \mathrm{res}_{q_i} M_{1i} = \nabla p_i (L(q_i))$$

For γ: $\quad \partial_{t^i} L_0 = [M_{0i}, L_0] \qquad \left(\Rightarrow \partial_{t^i} L_1 = [M_{1i}, L_1] \right)$

For t^i: $\quad \{H_i, H_j\} = 0$.

Stability $\Rightarrow \quad \partial_{t^i} M_{0j} - \partial_{t^j} M_{0i} + [M_{0i}, M_{0j}] = 0$.

$\rightsquigarrow$ Hitchin's system in Lax form.

Conclusion

- Obtained 1d multiform action for Hitchin system (C with $g \geqslant 2$)

$$T^*\mathcal{M} \xrightarrow{\text{gauging}} \mu^{-1}(0)/\mathcal{G} \xrightarrow[\text{description}]{\text{holomorphic}} T^*\mathrm{Bun}_G(C)$$

canonical action "$S = \int (p\,dq - H\,dt)$"

multiform 3d mixed BF

1d action for Hitchin's system

- Generalises to C with $g = 0, 1$ and ramification points.

<u>Examples:</u> Rational/Elliptic Gaudin, Elliptic spin Calogero-Moser.

Outlook

- Quantisation?
- Multiform versions of 4/5/6/... d CS actions?

  $\rightsquigarrow$ Lagrangian 2/3/4/...-form for 2/3/4/... d integrable hierarchies?
- Higher genus C in 4/5/6/... d context?