Holographic thermal coefficients, KMS condition and black holes

Ilija Burić
School of Mathematics and Hamilton Mathematics Institute
Trinity College Dublin

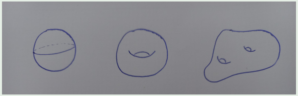
Eurostrings, Stockholm, August 27, 2025

based on 2505.10277 + 2508.08373 with I. Gusev and A. Parnachev

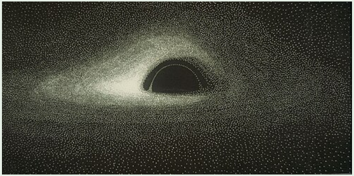
CFT at finite temperature

Reasons to study

- Experimental
- Heavy sectors of CFTs, numerics ...
- Broad picture: CFTs on different geometries



- Holography: black holes



Luminet '78

Thermal crossing equation

The setup

- Geometry: $S^1_\beta \times \mathbb{R}^{d-1}$
- Observable: scalar two-point function

$$g_\beta(\tau, \vec{x}) = \langle \phi(\tau, \vec{x}) \phi(0,0) \rangle$$

- Conformal block decomposition

$$g_\beta(\tau) = \tau^{-2\Delta_\phi} \sum_{\mathcal{O} \in \phi \times \phi} a_{\Delta_\mathcal{O}} \tau^{\Delta_\mathcal{O}}$$

- Kubo-Martin-Schwinger (KMS) condition [El-Showk,Papadodimas '11]

$$g_\beta(\tau) = g_\beta(\beta - \tau)$$

- Sum rules for thermal coefficients [Iliesiu et al '18], [Marchetto et al '23+...]

Holographic CFTs

Formulation of the problem

- Double traces and stress tensor composites

$$\phi \times \phi \sim 1 + [\phi\phi]_{k,\ell} + [T_{\mu\nu}]^n$$

- Scaling dimensions at large C_T

$$\Delta_{[\phi\phi]} = 2\Delta_\phi + 2m\,, \qquad \Delta_T = 4n$$

- Stress-tensor coefficients known from the bulk [Fitzpatrick,Huang '19]
- Aim for today : double-trace coefficients

Plan for the talk

Stress tensor thermal coefficients

Near-boundary expansion

- Klein-Gordon equation on the planar AdS-Schwarzschild

$$ds^2 = \left(r^2 - \frac{\mu}{r^2}\right) d\tau^2 + \frac{dr^2}{\left(r^2 - \frac{\mu}{r^2}\right)} + r^2 d\vec{x}^2$$

- Solutions in near-boundary expansion

$$\phi_T = \left(\frac{r}{w^2}\right)^{\Delta_\phi} \sum_{ijk} a^i_{jk} \frac{\rho^{2j} w^{2k}}{r^{4i}} \qquad \text{around} \qquad \partial = \{r \to \infty\}$$

- Solved order by order to obtain stress-tensor coefficients [Fitzpatrick,Huang '19]

$$\Lambda_n \equiv a^{(T)}_{4n} = a^n_{0,2n}$$

- Double-traces approached numerically [Parisini,Skenderis,Withers '23]

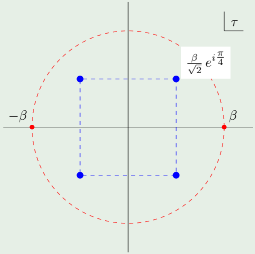
Bouncing singularities

Singularities of the stress-tensor sector

- Asymptotic behaviour of coefficients [Čeplak,Liu,Parnachev,Valach '24]

$$\Lambda_n \sim (-4)^n n^{2\Delta_\phi - 3}$$

- Singularities of g_T in the complex τ plane



- Not expected in the full correlator [Fidkowski,Hubeny,Kleban,Shenker '03]

KMS sum rules

Linear system

- Arise by taking derivatives at the crossing-symmetric point

$$\partial_\tau^{2k+1}\, g_\beta(\tau)\big|_{\tau=\frac{\beta}{2}} = 0$$

- Plugging in the block expansion [Marchetto,Miscioscia,Pomoni '23]

$$\sum_{m>k} \frac{a_{[\phi\phi]_{2\Delta_\phi+2m}}}{2^{2\Delta_\phi+2m}} \frac{\Gamma(2m+1)}{\Gamma(2(m-k))} = -\sum_{n=0}^{\infty} \frac{\Lambda_n}{2^{4n}} \frac{\Gamma(4n-2\Delta_\phi+1)}{\Gamma(4n-2\Delta_\phi-2k)}$$

- Viewed as equations for double-traces $a_{[\phi\phi]_{2\Delta_\phi+2m}}$

$$a_m \equiv \frac{(2m)!}{2^{2\Delta_\phi+2m}} a_{[\phi\phi]_{2\Delta_\phi+2m}}$$

Solutions to KMS

Structure of the equations [IB,Gusev,Parnachev 2505.10277]

- Upper triangular Toeplitz matrix

$$\begin{pmatrix} M_1 & M_2 & M_3 & \dots \\ 0 & M_1 & M_2 & \dots \\ 0 & 0 & M_1 & \dots \\ 0 & 0 & 0 & \dots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \dots \end{pmatrix}$$

- Formal solution

$$a_m = \sum_{k=0}^{\infty} (-1)^k \frac{\zeta(2k)}{\pi^{2k}} \left(2 - 4^{1-k}\right) F_{k+m-1}$$

- Divergent, can be Borel-resummed (or Mittag-Leffler-resummed)

$$a_m = \int_0^\infty dt\, e^{-t} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\zeta(2k)}{\pi^{2k}} \left(2 - 4^{1-k}\right) F_{k+m-1} t^{2k}$$

Asymptotic model

Checks and first applications [IB,Gusev,Parnachev 2505.10277]

- Recovers correct GFF double-trace coefficients
- Asymptotic model

$$\Lambda_n^{AM} \equiv \#(-4)^n n^{2\Delta_\phi - 3}$$

- KMS-invariant + holographic spectrum
- Free of singularities in the strip $0 < \mathrm{Re}(\tau) < \beta$
- Bounded as $\mathrm{Im}(\tau) \to \infty$
- Exactly solved for $\Delta_\phi \in \mathbb{N} + 1/2$, seems unique
- Borel resummation recovers the asymptotic model!

Padé-Borel method

Beyond asymptotic approximation [IB,Gusev,Parnachev 2508.08373]

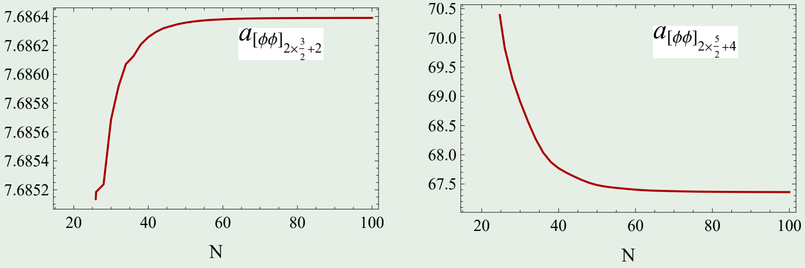
- Use exact Λ_n for $n = 1, \ldots, N$ and $\Lambda_n = 0$ for $n > N$
- Increase N and inspect convergence
- Borel resummation and Padé approximation needed

$$\begin{tikzcd}
a_m^{[N]}(z) = \sum_{n=0}^{N} a_m^{(n)} z^n \arrow[d, "\text{Borel}"'] & & a_m^{[N,N_G],\mathrm{PB}} = \sum_{i=1}^{N_G} \mathcal{P}^{[N]}(t_i)\omega_i \\
\mathcal{B}a_m^{[N]}(t^4) = \sum_{n=0}^{N} \frac{a_m^{(n)}}{(4n)!} t^{4n} \arrow[rr, "\text{Padé}"'] & & \mathcal{P}^{[N]}(t) = \frac{p_0 + p_1 t^4 + \ldots + p_{N/2} t^{2N}}{q_0 + q_1 t^4 + \ldots + q_{N/2} t^{2N}} \arrow[u, "\text{Gauss-Laguerre}"']
\end{tikzcd}$$

Table: Summary of the Padé-Borel method

Convergence of the Padé-Borel method

Convergence of double-trace OPE coefficients [IB,Gusev,Parnachev 2508.08373]



Main result: values of thermal coefficients

Padé-Borel resummed double-trace coefficients, $\Delta_\phi = 3/2$

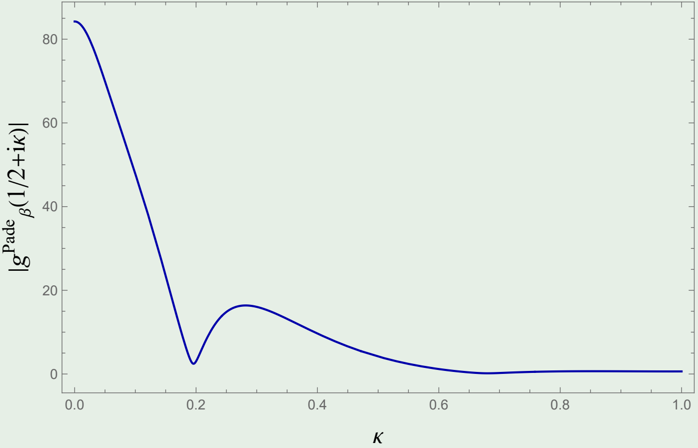
- Digits shown = stable under increase of N

operator $\mathcal{O}$	Padé-Borel resummed coefficient $a_{\mathcal{O}}$
$[\phi\phi]_{2\Delta_\phi+2}$	7.686391301178
$[\phi\phi]_{2\Delta_\phi+4}$	38.28355969692
$[\phi\phi]_{2\Delta_\phi+6}$	72.763467218
$[\phi\phi]_{2\Delta_\phi+8}$	56.85578514
$[\phi\phi]_{2\Delta_\phi+10}$	66.293380
$[\phi\phi]_{2\Delta_\phi+12}$	312.632
$[\phi\phi]_{2\Delta_\phi+14}$	499.994
$[\phi\phi]_{2\Delta_\phi+16}$	-212.0

- Uses $N = 300$ stress-tensor coefficients $\to$ recursion relations
- Similar results for $\Delta_\phi = 5/2$

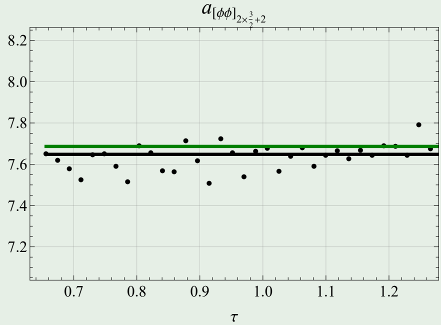
Check: boundedness

Two-point function in the complex plane



Comparison with the Klein-Gordon equation

Padé-Borel and numerical PDE estimates, $\Delta_\phi = 3/2$



- black dots = the values of $a_{[\phi\phi]_{2\Delta_\phi+2}}$
- green line = Padé-Borel resummed result

Comparison to the black hole 2

Double-trace coefficients using different methods

$\Delta_\phi = \frac{3}{2}$	Padé-Borel	PDE	GFF	AM
$[\phi\phi]_{2\Delta_\phi+2}$	7.686	7.7	12.44	7.549
$[\phi\phi]_{2\Delta_\phi+4}$	38.28	38	30.26	37.60
$[\phi\phi]_{2\Delta_\phi+6}$	72.76	72	56.11	71.54
$[\phi\phi]_{2\Delta_\phi+8}$	56.86	50	90.04	58.91

Comparison to the black hole 3

Double-trace coefficients using different methods

$\Delta_\phi = \frac{5}{2}$	Padé-Borel	PDE	GFF	AM
$[\phi\phi]_{2\Delta_\phi+2}$	37.67	40	30.25	-129.7
$[\phi\phi]_{2\Delta_\phi+4}$	67.36	80	140.3	-324.0
$[\phi\phi]_{2\Delta_\phi+6}$	851.8	800	420.2	1644
$[\phi\phi]_{2\Delta_\phi+8}$	2505	2000	990.1	4487

Summary and perspectives

Summary of results

- A method to solve KMS sum rules
- Requires Borel resummation for convergence
- Applied to holographic CFTs → double-trace thermal coefficients → two-point function at large N
- Consistency with KG on the AdS black hole

Future directions

- Generalisations: finite $\vec{x}$, finite R, other dimensions
- Lanczos coefficients, quasi-normal modes
- Other CFTs: $O(N)$ model at large N, ...
- Beyond large N, microscopics...

Thank you!