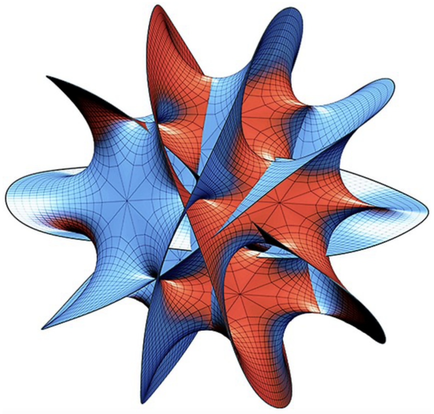

Eurostrings 2025

Nordita, Stockholm

Integration on higher-genus Riemann surfaces and string amplitudes

Oliver Schlotterer (Uppsala University
& Centre for Geometry and Physics)

based on joint work with E. D'Hoker, B. Enriquez, M. Hidding and F. Zerbini

Funded by the European Union

European Research Council
Established by the European Commission

August 28th 2025

MA∫CAMP

Thanks a lot for a wonderful conference !

Let's give the organizers an enthusiastic applause!

Pawel Caputa,	Niels Obers,
Michele Del Zotto,	Alessandro Sfondrini,
Magdalena Larfors,	Watse Sybesma,
Pietro Longhi,	Maxim Zabzine,
Anton Nedelin,	Konstantin Zarembo.

... with special thanks to Magdalena Larfors

for her fantastic job as the main organizer!

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Many many thanks for all the great support of

Jimmie Evenholt, Olga Lekka, Anastasios Mentesidis,

Elvira Pedone, Hans von Zur-Mühlen.

Outline

I. Motivation for integration on Riemann surfaces

$*$ computational motivation

$*$ physics motivation

II. Higher-genus polylogarithms

$*$ from single-valued connection

$*$ from meromorphic connection

$*$ link to string amplitudes

$*$ closure under integration

I. Motivation

Old problem from high-school days: $\int \mathrm{d}x$ is harder than $\frac{\mathrm{d}}{\mathrm{d}x}$

- for rational fct. $R(x)$ of $x \in \mathbb{C}$, also $\dfrac{\mathrm{d}R(x)}{\mathrm{d}x}$ is rational,

  but $\int \mathrm{d}x\, R(x)$ may be not: counterexample $\int_1^z \frac{\mathrm{d}x}{x} = \log(z)$

- many more "new functions" by imposing closure under integration

  including $\int \mathrm{d}x\, \left(-\frac{1}{x}\right) \log(1-x) \;=\; \mathrm{Li}_2(x)$,

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- many more "new functions" by imposing closure under integration

  including $\int \mathrm{d}x\, R(x) \log(1-x) \supset \mathrm{Li}_2(x)$, successively build

  function space of multiple polylogarithms (MPLs) via $G(\emptyset; z) = 1$ and

$$G(p_1,\ldots,p_w;z) = \int_0^z \frac{\mathrm{d}x}{x-p_1} G(p_2,\ldots,p_w;x)\,, \qquad p_i \in \mathbb{C}$$

[Poincaré, Lappo-Danilevsky, Goncharov, ...]

e.g. $G(p;z) = \log\left(1-\frac{z}{p}\right)$ and $G(\vec{0}^{n-1},1;z) = -\mathrm{Li}_n(z)$ for $n \geq 1$

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$$G(p_1, \ldots, p_w; z) = \int_0^z \underbrace{\frac{\mathrm{d}x}{x - p_1}}_{\text{"integration kernel"}} G(p_2, \ldots, p_w; x)\,, \qquad p_i \in \mathbb{C}$$

[Poincaré, Lappo-Danilevsky, Goncharov, ...]

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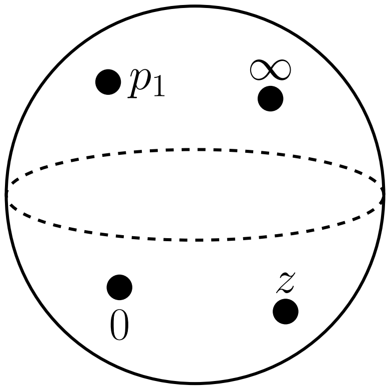
I. 1 Integration on the sphere



- MPLs defined by $G(\emptyset;z)=1$ and $(p_i\in\mathbb{C})$

$$G(p_1,\ldots,p_w;z) = \int_0^z \frac{\mathrm{d}x}{x-p_1} G(p_2,\ldots,p_w;x)$$

- MPLs $\Longrightarrow$ smallest function space $\supset$ {rational functions}

  that closes under integration over z or over any of $p_1,\cdots,p_w$

[Brown math/0606419]

- closure eventually relies on partial fraction relation

$$\frac{1}{(x-p_1)\,(x-p_2)} = \frac{1}{p_1-p_2}\left\{\frac{1}{x-p_1}-\frac{1}{x-p_2}\right\}$$

- shuffle-regularization of divergent cases $\begin{pmatrix} p_w=0 \\ p_1=z \end{pmatrix}$, e.g. $\begin{pmatrix} G(0;z)= & \log(z) \\ G(z;z)= & -\log(z) \end{pmatrix}$

[for review, see e.g. Panzer 1506.07243; Abreu, Britto, Duhr 2203.13014]

- MPLs tame $\int$ on Riemann sphere = arena of rational fct. $R(x)$ of $x\in\mathbb{C}$

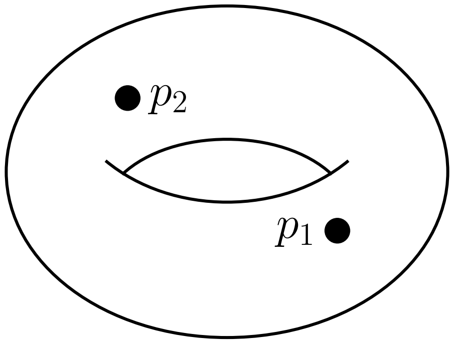
I. 2 Integration on the torus

Riemann sphere ($\leftrightarrow$ MPLs) is Riemann surface of genus $h=0$.

How to integrate on torus = Riemann surface of genus $h=1$??

- need closure under integration of rational fcts.

  $R(x,y)$ of $x,y \in \mathbb{C}$ subject to $y^2 \ = \ x^3 + b\,x + c$



- accomplished by elliptic MPLs: roughly speaking, iterated integrals

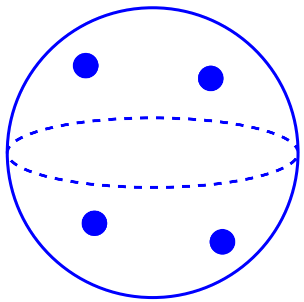
  of kernels $\frac{\mathrm{d}x}{y}$, $\frac{x\,\mathrm{d}x}{y}$ as well as $\frac{\mathrm{d}x}{y\,(x-p_i)}$ for marked points $p_1, p_2, \cdots$

- $\exists$ a variety of equivalent ways to span function space of elliptic MPLs

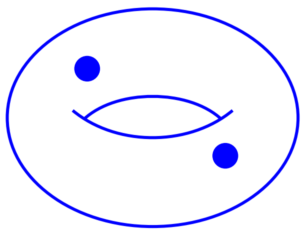
[e.g. Levin, Racinet 0703237; Brown, Levin 1110.6917; Broedel, Mafra, Matthes, OS 1412.5535; Broedel, Duhr, Dulat, Tancredi 1712.07089; Enriquez, Zerbini 2307.01833]

I. 3 Integration on more general geometries

Going beyond the well-explored cases of ...

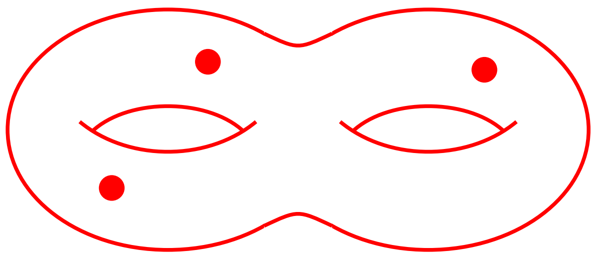


$\longleftrightarrow$ MPLs, kernels $\dfrac{\mathrm{d}x}{x-p_i}$



$\longleftrightarrow$ elliptic MPLs, kernels $\dfrac{\mathrm{d}x}{y}, \dfrac{x\,\mathrm{d}x}{y}, \dfrac{\mathrm{d}x}{y\,(x-p_i)}$

... this talk is dedicated to MPLs on Riemann surfaces of arbitrary genus h



also beyond hyperelliptic surfaces

with $y^2 = \prod_{j=1}^{2h+2}(x-u_j)$ description!

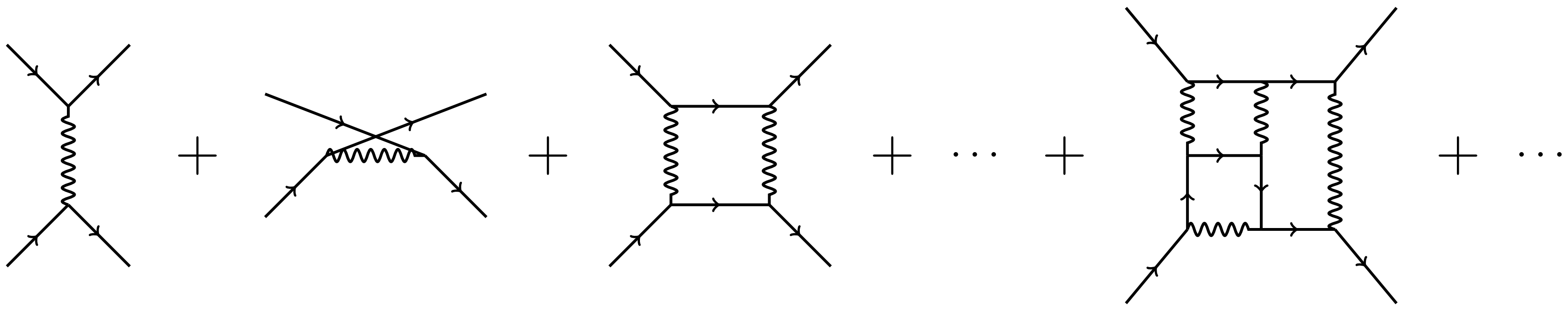
Important challenge beyond the scope of this talk: integration on

higher-dimensional varieties (K3 surfaces, Calabi-Yau n-folds, etc.)

I. 4 Physics motivation

Heavy demand for integration techniques from scattering amplitudes:

- Feynman integrals in scattering amplitudes of particle physics / gravity



$\longrightarrow$ e.g. higher-genus Riemann surfaces in Standard Model interactions

[e.g. Marzucca, McLeod, Page, Pögel, Weinzierl 2307.11497]

$\longrightarrow$ e.g. Calabi-Yau geometries for gravitational precision computations

[e.g. Driesse, Uhre Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch: Nature 641 (2025) no.8063, 603-607, 2411.11846]

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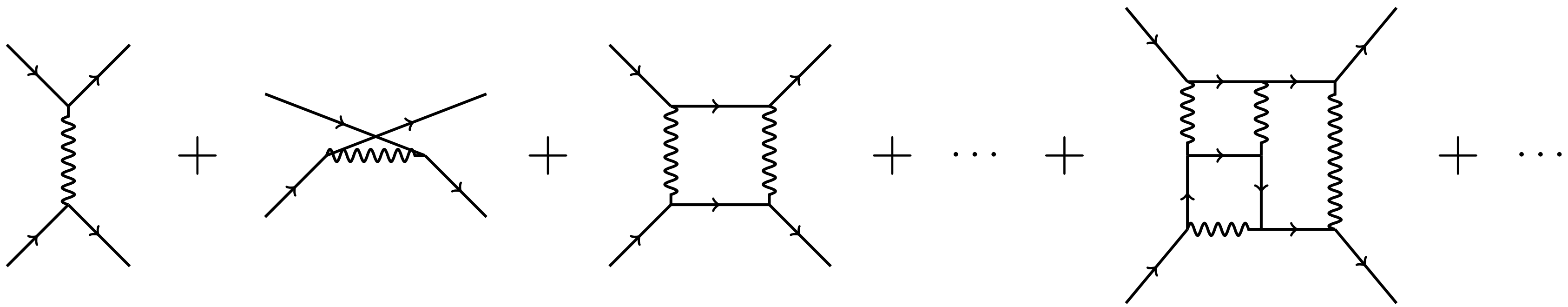
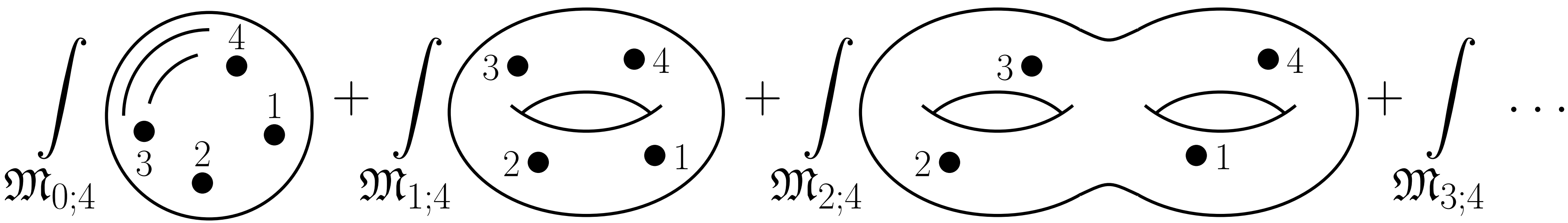


- moduli-space integrals over Riemann surfaces in string amplitudes



$\longrightarrow$ guiding the search for integration kernels on Riemann surfaces

[e.g. Broedel, Mafra, Matthes, OS 1412.5535; D'Hoker, Hidding, OS 2308.05044]

I. 4 Physics motivation

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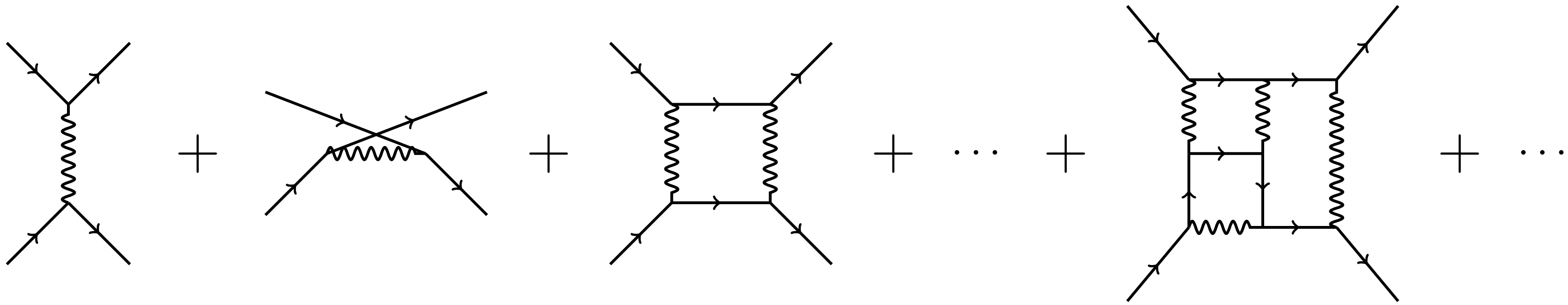
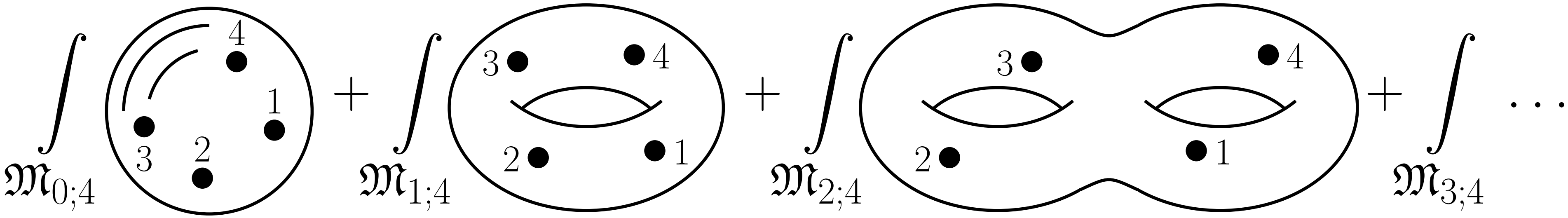


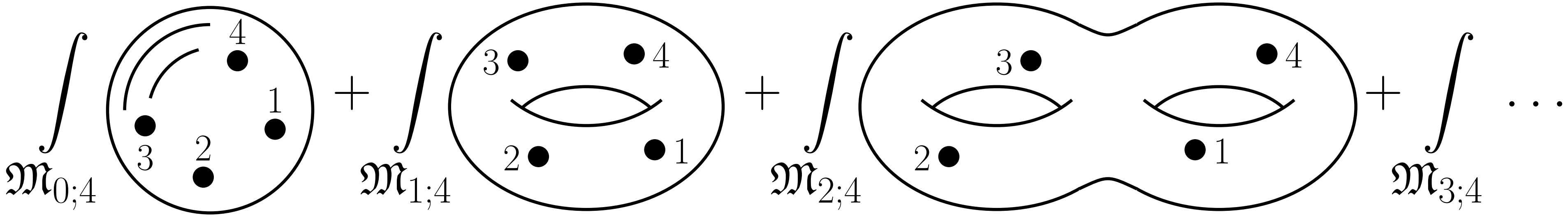
- moduli-space integrals over Riemann surfaces in string amplitudes



- your favorite other research problem / field besides high-energy physics

I. 5 String theory, polylogarithms and number theory

Perturbative string amplitudes $\Longrightarrow$ Riemann sufaces Σ_h at all $h \geq 0$



Polylogarithms on Σ_h & their integration kernels are valuable to:

- integrate over points in low-energy expansion (1 modulus at a time)

  $*$ multiple zeta values (MZVs) in α'-expansion of open/closed-string trees

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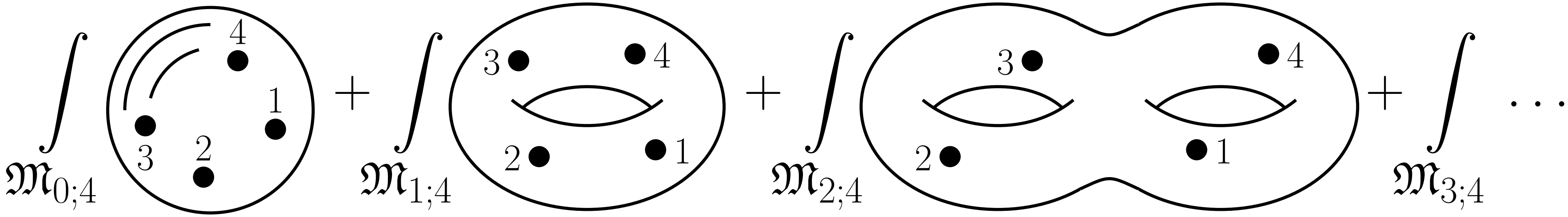

Polylogarithms on Σ_h & their integration kernels are valuable to:

- integrate over points in low-energy expansion (1 modulus at a time)

  $\ast$ multiple zeta values (MZVs) in α'-expansion of open/closed-string trees

  flat space: closed strings as "single-valued" open strings
  [Baune, Broedel, Brown, Dupont, OS, Schnetz, Stieberger, Taylor, Vanhove, Zerbini]

  recently extended to building blocks of string amplitudes in AdS
  [Alday, Fardelli, Giribet, Hansen, Silva; Alday, Nocchi, Strömholm 2504.19973, Baune 2505.23385, poster of Aurélie Strömholm-Sangaré]

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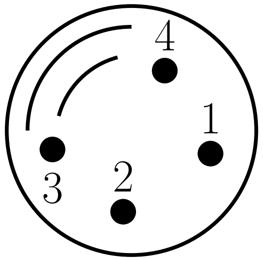
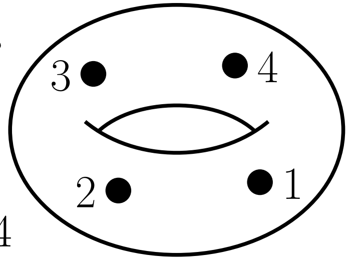
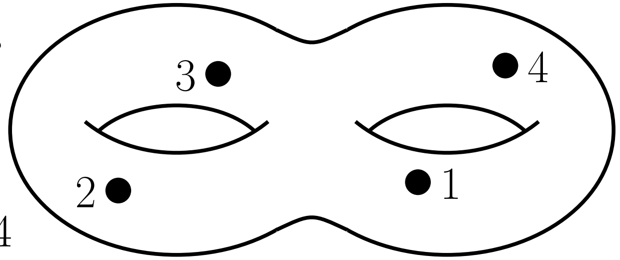
$$\int_{\mathfrak{M}_{0;4}} \quad + \int_{\mathfrak{M}_{1;4}} \quad + \int_{\mathfrak{M}_{2;4}} \quad + \int_{\mathfrak{M}_{3;4}} \ldots$$







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  $*$ multiple zeta values (MZVs) in α'-expansion of open/closed-string trees

  $*$ loop-level effective actions from elliptic MZVs/non-holo' modular graph forms / (sv) iterated Eisenstein integrals & higher-genus analogues

[refs. in 2024 textbook of D'Hoker-Kaidi, Snowmass White Paper 2203.09099 and SaGeX Review 2203.13021; poster of Yoann Sohnle]

I. 5 String theory, polylogarithms and number theory

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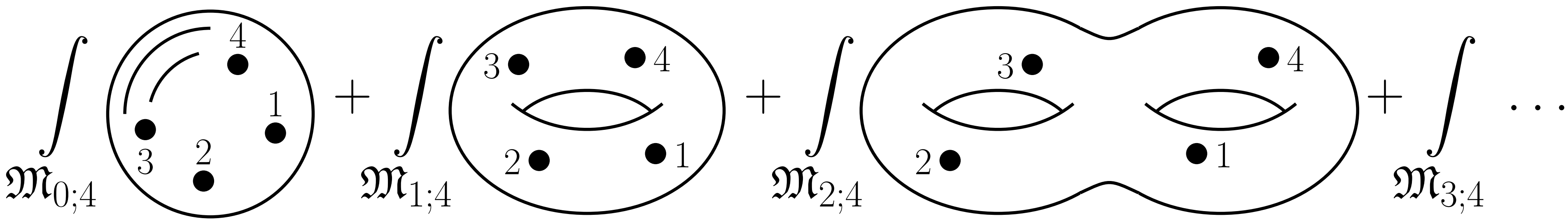
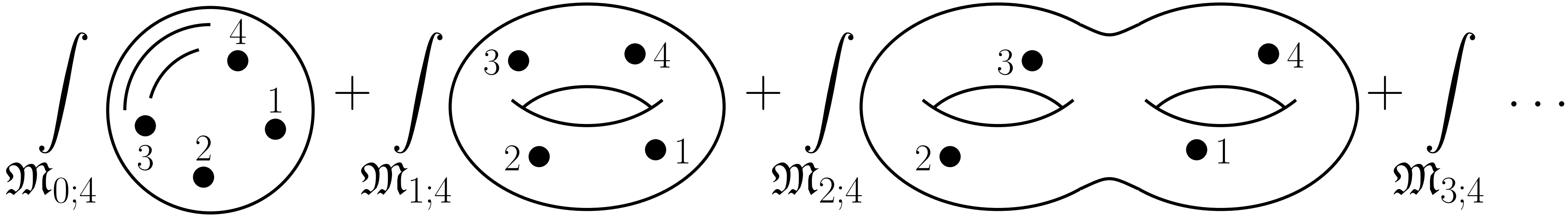


Polylogarithms on Σ_h & their integration kernels are valuable to:

- integrate over points in low-energy expansion (1 modulus at a time)
- organize / bootstrap $\mathfrak{M}_{h;n}$ integrand, i.e. delimit function space
  - $*$ int' kernels of elliptic MPLs universally capture 1-loop integrands
  - $*$ expect kernels for higher-genus polylogs to unlock integral rep's of superstring amplitudes beyond today's reach (e.g. 2-loop $\geq$ 6pt)

I. 5 String theory, polylogarithms and number theory

Perturbative string amplitudes $\Longrightarrow$ Riemann sufaces Σ_h at all $h \geq 0$



Polylogarithms on Σ_h & their integration kernels are valuable to:

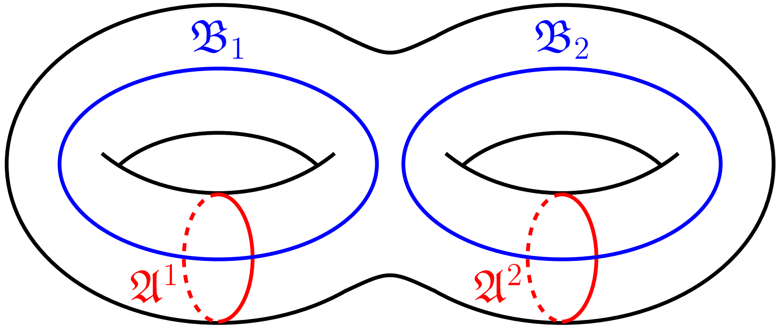
- integrate over points in low-energy expansion (1 modulus at a time)
- organize / bootstrap $\mathfrak{M}_{h;n}$ integrand, i.e. delimit function space
- symbiosis string theory questions / info $\leftrightarrow$ mathematical developments

… and concrete starting points to organize higher-genus polylogarithms!

II. Higher-genus polylogarithms

Integration kernels on Riemann surfaces Σ_h of arbitrary genus $h \geq 1$:

- holomorphic Abelian differentials $\omega_{I=1,2,\ldots,h}(z)$ on Σ_h



normalization $\oint_{\mathfrak{A}^I} \omega_J(z)\mathrm{d}z = \delta^I_J$

period matrix $\oint_{\mathfrak{B}_I} \omega_J(z)\mathrm{d}z = \Omega_{IJ}$

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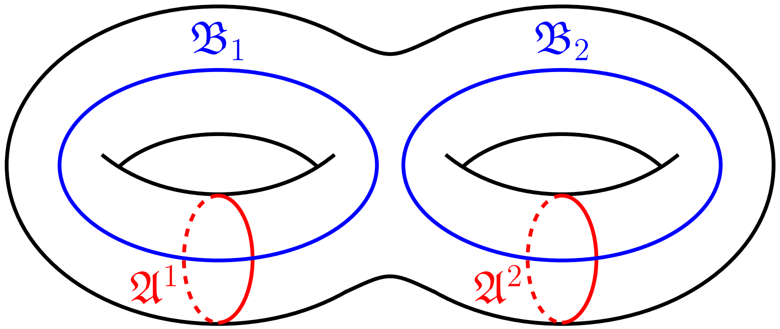

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- kernels with poles $\frac{\mathrm{d}z}{z-p_i}$ at points p_i by analogy with genus $h \leq 1$

- additional kernels: whatever it takes for closure under integration

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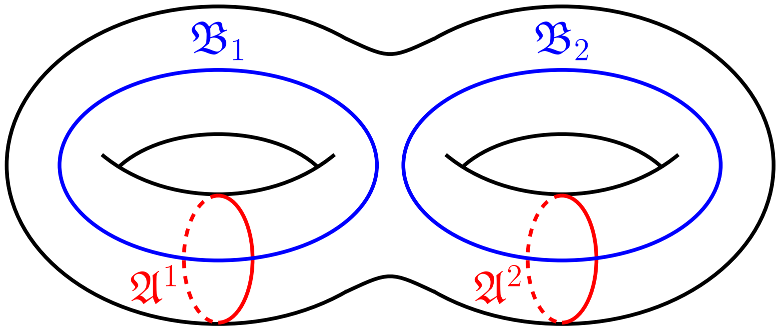

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- kernels with poles $\frac{\mathrm{d}z}{z-p_i}$ at points p_i by analogy with genus $h \leq 1$

- additional kernels: whatever it takes for closure under integration

$\longrightarrow$ gather entirety of higher-genus kernels in a flat connection $\mathcal{J}(z,p_i)$

$\mathrm{d}_z\mathcal{J}(z,p_i) = \mathcal{J}(z,p_i)\wedge\mathcal{J}(z,p_i) \ \Rightarrow\ $ homotopy invariant iterated integrals

II. Higher-genus polylogarithms

Integration kernels on Riemann surfaces Σ_h of arbitrary genus $h \geq 1$:

$\exists$ several alternatives for their flat connection having 2 out of 3 properties:

(i) meromorphicity

(ii) single-valuedness

(iii) at most simple poles in points

image from SleepingAngel.com

II. Higher-genus polylogarithms

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image from SleepingAngel.com

Intuitive reason why all 3 don't work:

already on torus $\Sigma_{h=1}$, cannot have

elliptic (meromorphic + single-valued)

functions with only simple pole (but $\exists$ elliptic Weierstraß $\wp(z) = \frac{1}{z^2} + \ldots$)

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image from SleepingAngel.com

This talk: 2 connections subject to (iii):

- single-valued / non-holo' $\mathcal{J}_{\rm DHS}(x,p)$
- meromorphic / multivalued $\mathcal{K}_{\rm E}(x,p)$

... which span same space of polylogs [D'Hoker, Enriquez, OS, Zerbini 2501.07640]

II. 1 Polylogarithms from single-valued connection

Singular kernels $\sim \frac{dx}{x-y}$ from (Arakelov) Green function $\mathcal{G}(x,y)$ on $\Sigma_h \times \Sigma_h$

$$\mathcal{G}(x,y) \ = \ -\ln \left| x-y \right|^2 + \text{sv completion}$$

Uniquely defined by (with canonical volume form $\kappa(x) = \frac{1}{h}\omega_I(x)\bar{\omega}^I(x)$)

- symmetry $\mathcal{G}(x,y) \ = \ \mathcal{G}(y,x)$

$$\bar{\omega}^I \ = \ [(\mathrm{Im}\,\Omega)^{-1}]^{IJ}\bar{\omega}_J$$

- Laplace equation $\partial_x \partial_{\bar{x}} \mathcal{G}(x,y) \ = \ \pi\kappa(x) - \pi\delta^2(x,y)$
- "integrates to zero", i.e. normalization $\int_{\Sigma_h} \kappa(x)\mathcal{G}(x,y) \ = \ 0$

Explicit realization in terms of "prime form" $E(x,y) = x-y+\mathcal{O}((x-y)^3)$

$\longrightarrow$ combination of odd Riemann $\theta[\nu](\int_y^x \omega|\Omega)$, see appendix

[Fay '73; Faltings '84; D'Hoker, Green, Pioline 1712.06135]

II. 1 Polylogarithms from single-valued connection

Embed simplest singular kernel $\partial_x \mathcal{G}(x,y) = \frac{\mathrm{d}x}{y-x} + \cdots$ into rk 2 tensor:

$$f^I{}_J(x,y) \;=\; \int_{\Sigma_h} \mathrm{d}^2 z \, \partial_x \mathcal{G}(x,z) \bar{\omega}^J(z) \omega_I(z) \;-\; \delta^I_J \, \partial_x \mathcal{G}(x,y)$$

$\longrightarrow$ need the $h \times h$ components for closure under integration, see later

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$$f^I{}_J(x,y) \; = \underbrace{\int_{\Sigma_h} \mathrm{d}^2 z \, \partial_x \mathcal{G}(x,z) \bar{\omega}^J(z) \omega_I(z)}_{} \; - \; \delta^I_J \, \partial_x \mathcal{G}(x,y)$$

$\nexists$ at genus 1 since $\mathcal{G}(x,z)$ integrates to zero against $\bar{\omega}^K(z)\omega_K(z)$

Higher-rank kernels $f^{I_1 \ldots I_n}{}_J$ with $n{+}1$ free indices from recursion

$$f^{I_1 \ldots I_n}{}_J(x,y) \; = \int_{\Sigma_h} \mathrm{d}^2 z \, \partial_x \mathcal{G}(x,z) \, \bar{\omega}^{I_1}(z) f^{I_2 \ldots I_n}{}_J(z,y)$$

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- kernels $f^{I_1 \ldots I_n}{}_J(x,y)$ at $n \geq 2$ are regular throughout $\Sigma_h \times \Sigma_h$,

  only $n=1$ case has simple pole $f^I{}_J(x,y) = \dfrac{\delta^I_J}{x-y} + \mathcal{O}\big((x-y)^0\big)$

- non-meromorphic: $\partial_{\bar{x}} f^{I_1 \cdots I_n}{}_J(x,y) = -\pi \bar{\omega}^{I_1}(x) f^{I_2 \cdots I_n}{}_J(x,y)$

[D'Hoker, Hidding, OS 2306.08644]

II. 1 Polylogarithms from single-valued connection

Assembly line for higher-genus polylogarithms **[D'Hoker, Hidding, OS 2306.08644]**

- combine f-tensors to flat connection

$$\mathcal{J}_{\mathrm{DHS}}(z,p) \;=\; -\pi \mathrm{d}\bar{z}\,\bar{\omega}^I(z)\,b_I + \mathrm{d}z\left(\omega_J(z) + \sum_{n=1}^{\infty} \mathrm{ad}_{b_{I_1}}\cdots \mathrm{ad}_{b_{I_n}} f^{I_1\cdots I_n}{}_J(z,p)\right) a^J$$

$*$ $(1,0)\oplus(0,1)$ form in $z\in\Sigma_h$, scalar in $p\in\Sigma_h$

$*$ valued in free Lie algebra with $2h$ generators $b_1,\cdots,b_h$ & $a^1,\cdots,a^h$

$*$ $\mathrm{d}_z\mathcal{J}_{\mathrm{DHS}} = \mathcal{J}_{\mathrm{DHS}}\wedge\mathcal{J}_{\mathrm{DHS}}$ by $\partial_{\bar{z}} f^{I_1\cdots I_n}{}_J(z,p) = -\pi\bar{\omega}^{I_1}(z) f^{I_2\cdots I_n}{}_J(z,p)$

II. 1 Polylogarithms from single-valued connection

Assembly line for higher-genus polylogarithms **[D'Hoker, Hidding, OS 2306.08644]**

- combine f-tensors to flat connection

$$\mathcal{J}_{\rm DHS}(z,p) \ = \ -\pi \mathrm{d}\bar z \, \bar\omega^I(z) \, b_I + \mathrm{d}z \left(\omega_J(z) + \sum_{n=1}^\infty \mathrm{ad}_{b_{I_1}} \cdots \mathrm{ad}_{b_{I_n}} f^{I_1 \cdots I_n}{}_J(z,p) \right) a^J$$

- expand homotopy-inv. path-ordered exp. in non-commutative var's a^J, b_I

$$\mathrm{Pexp}\left(\int_y^x \mathcal{J}_{\rm DHS}(z,p) \right) = 1 + \int_y^x \mathcal{J}_{\rm DHS}(z,p) + \int_y^x \mathcal{J}_{\rm DHS}(z_1,p) \int_y^{z_1} \mathcal{J}_{\rm DHS}(z_2,p) + \ldots$$

$$= \ 1 + a^J \, \Gamma_J(x,y) + b_I \, \Gamma^I(x,y) + a^J a^K \, \Gamma_{JK}(x,y) + b_I a^J \, \Gamma^I{}_J(x,y;p)$$

$$+ \, a^J b_K \, \Gamma_J{}^K(x,y;p) + b_I b_K \, \Gamma^{IK}(x,y;p) + \text{“}\geq 3 \text{ letters”}$$

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$$\mathcal{J}_{\rm DHS}(z,p) \;=\; -\pi {\rm d}\bar z\, \bar\omega^I(z)\, b_I + {\rm d}z \left(\omega_J(z) + \sum_{n=1}^\infty {\rm ad}_{b_{I_1}} \cdots {\rm ad}_{b_{I_n}} f^{I_1 \cdots I_n}{}_J(z,p) \right) a^J$$

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$${\rm Pexp}\left(\int_y^x \mathcal{J}_{\rm DHS}(z,p) \right) = 1 + \int_y^x \mathcal{J}_{\rm DHS}(z,p) + \int_y^x \mathcal{J}_{\rm DHS}(z_1,p) \int_y^{z_1} \mathcal{J}_{\rm DHS}(z_2,p) + \ldots$$

$$=\; 1 + a^J\, \Gamma_J(x,y) + b_I\, \Gamma^I(x,y) + a^J a^K\, \Gamma_{JK}(x,y) + b_I a^J\, \Gamma^I{}_J(x,y;p) + \cdots$$

- polylogarithm $\Gamma_{\ldots J \ldots}{}^{\ldots I \ldots}(x,y;p) :=$ coeff. of word $\ldots a^J \ldots b_I \ldots$, e.g.

$$\Gamma^I{}_J(x,y;p) = \int_y^x {\rm d}z\, f^I{}_J(z,p) - \pi \int_y^x {\rm d}\bar z_1\, \bar\omega^I(z_1) \int_y^{z_1} {\rm d}z_2\, \omega_J(z_2)$$

At genus $h=1$ reproduces elliptic polylogs of **[Brown, Levin 1110.6917]**

II. 1 Polylogarithms from single-valued connection

Assembly line for higher-genus polylogarithms **[D'Hoker, Hidding, OS 2306.08644]**

- combine f-tensors to flat connection

$$\mathcal{J}_{\rm DHS}(z,p) \ = \ -\pi {\rm d}\bar z\, \bar\omega^I(z)\, b_I + {\rm d}z \left(\omega_J(z) + \sum_{n=1}^\infty {\rm ad}_{b_{I_1}} \cdots {\rm ad}_{b_{I_n}} f^{I_1 \cdots I_n}{}_J(z,p) \right) a^J$$

- expand homotopy-inv. path-ordered exp. in non-commutative var's a^J, b_I

$$\text{Pexp}\left(\int_y^x \mathcal{J}_{\rm DHS}(z,p) \right) = 1 + \int_y^x \mathcal{J}_{\rm DHS}(z,p) + \int_y^x \mathcal{J}_{\rm DHS}(z_1,p) \int_y^{z_1} \mathcal{J}_{\rm DHS}(z_2,p) + \ldots$$

$$= \ 1 + a^J\, \Gamma_J(x,y) + b_I\, \Gamma^I(x,y) + a^J a^K\, \Gamma_{JK}(x,y) + b_I a^J\, \Gamma^I{}_J(x,y;p) + \cdots$$

- polylogarithm $\Gamma_{\ldots J \ldots}{}^{\ldots I \ldots}(x,y;p) :=$ coeff. of word $\ldots a^J \ldots b_I \ldots$

Both $f^{I_1 \cdots I_n}{}_J(x,y)$ & $\Gamma_{\ldots J \ldots}{}^{\ldots I \ldots}(x,y;p)$ transform as modular tensors under ${\rm Sp}(2h,\mathbb{Z}) \ni \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ taking $\Omega \rightarrow (A\Omega{+}B)(C\Omega{+}D)^{-1}$, e.g.

$$f^{I_1 \cdots I_n}{}_J(x,y) \ \rightarrow \ (C\Omega{+}D)^{I_1}{}_{K_1} \cdots (C\Omega{+}D)^{I_n}{}_{K_n}\, f^{K_1 \cdots K_n}{}_L(x,y) \left((C\Omega{+}D)^{-1} \right)^L{}_J$$

II. 2 Polylogarithms from meromorphic connection

Alternative to single-valued/non-holo connection $\mathcal{J}_{\rm DHS}$ with coeff's $f^{I_1\cdots I_n}{}_J$

$\longrightarrow$ meromorphic/multivalued Enriquez connection $\mathcal{K}_{\rm E}$ with coeff's $g^{I_1\cdots I_n}{}_J$

$$\mathcal{K}_{\rm E}(x,y) \ = \ {\rm d}x \left(\omega_J(x) + \sum_{n=1}^\infty {\rm ad}_{b_{I_1}} \cdots {\rm ad}_{b_{I_n}} \, g^{I_1\cdots I_n}{}_J(x,y) \right) a^J$$

defined through its functional properties in **[Enriquez 1112.0864]**

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defined through its functional properties in **[Enriquez 1112.0864]**

- no $\mathfrak{A}$-cycle monodromies $g^{I_1\ldots I_n}{}_J(\mathfrak{A}^K\cdot x,y) \ = \ g^{I_1\ldots I_n}{}_J(x,y)$

  but non-trivial $\mathfrak{B}$-cycle monodromies (familiar from genus $h=1$)

$$g^{I_1\ldots I_n}{}_J(\mathfrak{B}_L\cdot x,y) \ = \ \sum_{\ell=0}^{n} \frac{(-2\pi i)^\ell}{\ell!} \delta^{I_1}_L \ldots \delta^{I_\ell}_L \, g^{I_{\ell+1}\cdots I_n}{}_J(x,y)$$

- only $n=1$ case has simple pole $g^I{}_J(x,y) \ = \ \frac{\delta^I_J}{x-y} + \mathcal{O}\big((x-y)^0\big)$

  $\longrightarrow$ same poles as single-valued, but non-meromorphic $f^{I_1\cdots I_n}{}_J(x,y)$

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$$\mathcal{K}_{\rm E}(x,y) \ = \ {\rm d}x \left(\omega_J(x) + \sum_{n=1}^{\infty} {\rm ad}_{b_{I_1}} \cdots {\rm ad}_{b_{I_n}} \, g^{I_1\cdots I_n}{}_J(x,y) \right) a^J$$

defined through its functional properties in **[Enriquez 1112.0864]**

- $\exists$ Poincaré-series representations via Schottky uniformization **[Baune, Broedel, Im, Lisitsyn, Zerbini 2406.10051]**

- expressing $g^{I_1\ldots I_n}{}_J(x,y)$ in terms of $f^{I_1\ldots I_n}{}_J(x,y)$ & $\mathcal{J}_{\rm DHS}$ polylogs:

  gauge transformation & Lie-algebra automorphism of flat connections

$$\big({\rm d} - \mathcal{K}_{\rm E}(x,y;a,b)\big) \ = \ \mathcal{U}(x,y;b)^{-1} \big({\rm d} - \mathcal{J}_{\rm DHS}(x,y;\hat a,\hat b)\big) \, \mathcal{U}(x,y;b)$$

[D'Hoker, Enriquez, OS, Zerbini 2501.07640]

II. 2 Polylogarithms from meromorphic connection

Alternative to single-valued/non-holo connection $\mathcal{J}_{\rm DHS}$ with coeff's $f^{I_1\cdots I_n}{}_J$

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defined through its functional properties in [Enriquez 1112.0864]

- $\exists$ Poincaré-series representations via Schottky uniformization [Baune, Broedel, Im, Lisitsyn, Zerbini 2406.10051]

- expressing $g^{I_1\ldots I_n}{}_J(x,y)$ in terms of $f^{I_1\ldots I_n}{}_J(x,y)$ & $\mathcal{J}_{\rm DHS}$ polylogs:

  gauge transformation & Lie-algebra automorphism of flat connections

$$\big({\rm d} - \mathcal{K}_{\rm E}(x,y;a,b)\big) \ = \underbrace{\mathcal{U}(x,y;b)}_{\substack{\text{series in } \mathcal{J}_{\rm DHS} \\ \text{polylogs and } b_I}}{}^{-1} \big({\rm d} - \mathcal{J}_{\rm DHS}(x,y;\underbrace{\hat{a},\hat{b}}_{\substack{\text{series in } \mathcal{J}_{\rm DHS} \text{ polylogs} \\ \& \ {\rm ad}_{b_K} \text{ acting on } a^J,\, b_I}})\big)\, \mathcal{U}(x,y;b)$$

[D'Hoker, Enriquez, OS, Zerbini 2501.07640]

II. 2 Polylogarithms from meromorphic connection

Direct construction of $g^{I_1 \cdots I_n}{}_J(x,y)$ via $\mathfrak{A}$-cycle convolutions:

adapt surface integrals defining $f^{I_1 \cdots I_n}{}_J(x,y)$ according to

$$\int_{\Sigma_h} \mathrm{d}^2 z \, \bar\omega^I(z) \, \to \, \oint_{\mathfrak{A}^I} \mathrm{d}z \,, \qquad \partial_x \mathcal{G}(x,y) \, \to \, -\partial_x \ln E(x,y)$$

However, $\exists$ tail of additive lower-rank corrections with $\mathbb{Q}[2\pi i]$ coefficients

$$g^I{}_J(x,y) \, = \, -\oint_{\mathfrak{A}^I} \mathrm{d}z \, \omega_J(z) \, \partial_x \ln E(x,z) + \delta^I_J \, \partial_x \ln E(x,y) + i\pi \delta^I_J \, \omega_J(x)$$

$$f^I{}_J(x,y) \, = \, \int_{\Sigma_h} \mathrm{d}^2 z \, \bar\omega^I(z) \, \omega_J(z) \, \partial_x \mathcal{G}(x,z) - \delta^I_J \, \partial_x \mathcal{G}(x,y)$$

[D'Hoker, OS 2502.14769]

Valid for x,y inside fundamental domain within universal cover $\tilde\Sigma_h$ of Σ_h

$\longrightarrow$ see appendix for details

II. 2 Polylogarithms from meromorphic connection

Direct construction of $g^{I_1 \cdots I_n}{}_J(x,y)$ via $\mathfrak{A}$-cycle convolutions:

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$$g^I{}_J(x,y) = -\oint_{\mathfrak{A}^I} \mathrm{d}z \, \omega_J(z) \, \partial_x \ln E(x,z) + \delta^I_J \, \partial_x \ln E(x,y) + i\pi \delta^I_J \, \omega_J(x)$$

Recursion for $n \geq 2$ in parallel to surface-$\int$ representation of $f^{I_1 \ldots I_n}{}_J(x,y)$:

$$g^{I_1 \cdots I_n}{}_J(x,y) = -\oint_{\mathfrak{A}^{I_1}} \mathrm{d}z \, \partial_x \ln E(x,z) g^{I_2 \cdots I_n}{}_J(z,y) + \sum_{r=0}^{n-1} (2\pi i)^{n-r} \, (\text{rank } r)$$

$$f^{I_1 \ldots I_n}{}_J(x,y) = \int_{\Sigma_h} \mathrm{d}^2 z \, \bar\omega^{I_1}(z) \, \partial_x \mathcal{G}(x,z) \, f^{I_2 \ldots I_n}{}_J(z,y)$$

[D'Hoker, OS 2502.14769]

II. 2 Polylogarithms from meromorphic connection

Meromorphic polylogarithms constructed from Enriquez kernels

$$\tilde{\Gamma}\left(\begin{smallmatrix} \overrightarrow{I_1} & \overrightarrow{I_2} & \cdots & \overrightarrow{I_r} \\ J_1 & J_2 & \cdots & J_r \\ p_1 & p_2 & \cdots & p_r \end{smallmatrix}; x, y\right) = \int_y^x \mathrm{d}z \, g^{\overrightarrow{I_1}}{}_{J_1}(z, p_1) \, \tilde{\Gamma}\left(\begin{smallmatrix} \overrightarrow{I_2} & \cdots & \overrightarrow{I_r} \\ J_2 & \cdots & J_r \\ p_2 & \cdots & p_r \end{smallmatrix}; z, y\right)$$

with $\tilde{\Gamma}(\emptyset; x, y) = 1$ & $g^{\emptyset}{}_J(z, p) = \omega_J(z)$ and multi-indices $\overrightarrow{I_k} = I_k^1 \cdots I_k^s$.

[Baune, Broedel, Im, Lisitsyn, Zerbini 2406.10051; D'Hoker, OS 2407.11476; Baune, Broedel, Im, Lisitsyn, Möckli 2409.08208; Enriquez, Zerbini: work in progress]

- term-by-term homotopy invariant but non-tensorial under $\mathrm{Sp}(2h, \mathbb{Z})$

- first explorations of numerical evaluations in **[above 2406.10051]**,

  and of functional identities in **[above 2407.11476, 2409.08208]**

II. 2 Polylogarithms from meromorphic connection

Meromorphic polylogarithms constructed from Enriquez kernels

$$\tilde{\Gamma}\left(\begin{smallmatrix} \overrightarrow{I_1} & \overrightarrow{I_2} & \cdots & \overrightarrow{I_r} \\ J_1 & J_2 & \cdots & J_r \\ p_1 & p_2 & \cdots & p_r \end{smallmatrix} ; x,y\right) = \int_y^x \mathrm{d}z \, g^{\overrightarrow{I_1}}{}_{J_1}(z,p_1)\, \tilde{\Gamma}\left(\begin{smallmatrix} \overrightarrow{I_2} & \cdots & \overrightarrow{I_r} \\ J_2 & \cdots & J_r \\ p_2 & \cdots & p_r \end{smallmatrix} ; z,y\right)$$

with $\tilde{\Gamma}(\emptyset;x,y) = 1$ & $g^{\emptyset}{}_J(z,p) = \omega_J(z)$ and multi-indices $\overrightarrow{I_k} = I_k^1 \cdots I_k^s$.

[Baune, Broedel, Im, Lisitsyn, Zerbini 2406.10051; D'Hoker, OS 2407.11476; Baune, Broedel, Im, Lisitsyn, Möckli 2409.08208; Enriquez, Zerbini: work in progress]

- term-by-term homotopy invariant but non-tensorial under $\mathrm{Sp}(2h,\mathbb{Z})$
- first explorations of numerical evaluations in **[above 2406.10051]**,

  and of functional identities in **[above 2407.11476, 2409.08208]**
- $\mathcal{J}_{\rm DHS}$ polylogarithms expressible via $\tilde{\Gamma}$ multiplied by iterated int's of $\bar{\omega}^I$ **[D'Hoker, Enriquez, OS, Zerbini 2501.07640]**
- reduce to elliptic polylogs $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_r \\ p_1 & \cdots & p_r \end{smallmatrix} ; x\right)$ at genus $h=1$ and $y=0$

II. 3 Higher-genus kernels and string amplitudes

Constructed higher-genus kernels $f^{I_1 \cdots I_n}{}_J(x,y)$ from convolutions of

Green function $\mathcal{G}(z,y) \leftrightarrow \langle X(z)X(y)\rangle_{\Sigma_h}$ of **free** boson on Σ_h

central ingredient for moduli-space integrands of string amplitudes, e.g.

$$\mathcal{A}_{g;n} = \int_{\mathfrak{M}_{g;n}} \exp\left(\alpha' \sum_{1\leq i\leq j}^{n} k_i \cdot k_j \, \mathcal{G}(z_i,z_j)\right) \cdot \left(\begin{array}{c}\text{rest of CFT correlator}\\ \text{of vertex operators}\end{array}\right)$$

Contributions to "rest of CFT correlators" from polynomials in $\mathcal{G}$, f:

- $\partial_z \mathcal{G}(z,y)$ and $\partial_z \partial_y \mathcal{G}(z,y)$ and cc's from bosons $X(z) \in$ vertex operator

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Contributions to "rest of CFT correlators" from polynomials in $\mathcal{G}$, f:

- $\partial_z \mathcal{G}(z,y)$ and $\partial_z \partial_y \mathcal{G}(z,y)$ and cc's from bosons $X(z) \in$ vertex operator
- worldsheet fermions $\psi^\mu(z)$ of RNS superstrings: all z_i-dependence of current correlators $\langle {:}\psi^{\mu_1}(z_1) \psi^{\lambda_1}(z_1){:} \ldots {:}\psi^{\mu_n}(z_n) \psi^{\lambda_n}(z_n){:} \rangle_{\Sigma_h}$ expressible via modular tensors $f^{I_1 \cdots I_r}{}_J(x,y)$ or Enriquez kernels $g^{I_1 \cdots I_r}{}_J(x,y)$

[D'Hoker, Hidding, OS 2308.05044; D'Hoker, OS 2505.07947]

II. 3 Higher-genus kernels and string amplitudes

Constructed higher-genus kernels $f^{I_1 \cdots I_n}{}_J(x,y)$ from convolutions of

$$\text{Green function } \mathcal{G}(z,y) \;\leftrightarrow\; \langle X(z)X(y)\rangle_{\Sigma_h} \text{ of } \underline{\textbf{free}} \text{ boson on } \Sigma_h$$

central ingredient for moduli-space integrands of string amplitudes, e.g.

$$\mathcal{A}_{g;n} = \int_{\mathfrak{M}_{g;n}} \exp\left(\alpha' \sum_{1\leq i\leq j}^{n} k_i \cdot k_j \, \mathcal{G}(z_i,z_j)\right) \cdot \left(\begin{array}{c} \prod_i \text{ of } f^{I_1\cdots I_r}{}_J(z_i,z_j), \\ \text{their } \partial_{z_j}\text{'s and cc's} \end{array} \right)$$

- plan to bootstrap $\mathfrak{M}_{g;n}$ integrand of $g \geq 2$ amplitudes using ansaetze built from $f^{I_1\cdots I_r}{}_J(z_i,z_j)$ and imposing consistency conditions
- z_i integrals in low-energy expansion will become algorithmic using higher-genus polylogarithms and their single-valued versions
- to do: connect with $\int$ over modular parameters **[talk of Lorenz Eberhardt, [Baccianti, Chandra, Eberhardt, Hartmann, Manschot, Mizera, Wang 2022 - 2025]**

II. 4 Closure under integration

Products of $\tilde{\Gamma}\left(\begin{smallmatrix}\vec{I}_1 & \cdots & \vec{I}_r \\ J_1 & \cdots & J_r \\ p_1 & \cdots & p_r\end{smallmatrix}; z, y\right)$ with $g^{\vec{K}}{}_{L}(z, p_i)$ & $g^{\vec{M}}{}_{N}(p_i, z)$ close under integration over z and p_i by algebraic identities of integration kernels:

$$\tilde{\Gamma}\left(\begin{smallmatrix}\vec{I}_1 & \vec{I}_2 & \cdots & \vec{I}_r \\ J_1 & J_2 & \cdots & J_r \\ p_1 & p_2 & \cdots & p_r\end{smallmatrix}; x, y\right) = \int_y^x \mathrm{d}z \, g^{\vec{I}_1}{}_{J_1}(z, p_1) \, \tilde{\Gamma}\left(\begin{smallmatrix}\vec{I}_2 & \cdots & \vec{I}_r \\ J_2 & \cdots & J_r \\ p_2 & \cdots & p_r\end{smallmatrix}; z, y\right)$$

II. 4 Closure under integration

Products of $\tilde{\Gamma}\left(\begin{smallmatrix} \vec{I}_1 & \cdots & \vec{I}_r \\ J_1 & \cdots & J_r \\ p_1 & \cdots & p_r \end{smallmatrix}; z, y\right)$ with $g^{\vec{K}}{}_L(z, p_i)$ & $g^{\vec{M}}{}_N(p_i, z)$ close under integration over z and p_i by algebraic identities of integration kernels:

- warmup: how to integrate $\omega_K(z) g^K{}_J(p, z) \tilde{\Gamma}(\cdots ; z, y)$ over z?

"interchange lemma" $\omega_K(z) g^K{}_J(p, z) = \underbrace{-\,\omega_K(p) g^K{}_J(z, p)}_{\text{fewer } z\text{'s than on LHS}}$

- similarly, need to convert $\partial_p \to \partial_z$

"swapping identities" $\partial_p g^{I_1 I_2 \cdots I_s K}{}_J(z, p) = (-1)^s\, \partial_z g^{I_s \cdots I_2 I_1 K}{}_J(p, z)$

[D'Hoker, OS 2407.11476]

II. 4 Closure under integration

Products of $\tilde{\Gamma}\left(\begin{smallmatrix} \vec{I}_1 & \cdots & \vec{I}_r \\ J_1 & \cdots & J_r \\ p_1 & \cdots & p_r \end{smallmatrix}; z, y\right)$ with $g^{\vec{K}}{}_L(z, p_i)$ & $g^{\vec{M}}{}_N(p_i, z)$ close under integration over z and p_i by algebraic identities of integration kernels:

- "interchange lemma" $\omega_K(z) g^K{}_J(p, z) = -\omega_K(p) g^K{}_J(z, p)$
- "swapping identities" $\partial_p g^{I_1 I_2 \cdots I_s K}{}_J(z, p) = (-1)^s \, \partial_z g^{I_s \cdots I_2 I_1 K}{}_J(p, z)$
- more generally, Fay identities eliminate repeated appearance of z, e.g.

$$g^I{}_J(y, z) g^J{}_K(z, x) = g^I{}_J(y, x) g^J{}_K(z, x) - g^I{}_J(z, y) g^J{}_K(y, x)$$

$$- \omega_J(y) g^{IJ}{}_K(z, y) - \omega_J(y) g^{JI}{}_K(z, x) - \omega_J(z) g^{JI}{}_K(y, x)$$

no repeated z on RHS!

[D'Hoker, OS 2407.11476]

II. 4 Closure under integration

Products of modular $\Gamma_{\cdots J \cdots}{}^{\cdots I \cdots}(x, y; p)$ with kernels from $\mathcal{J}_{\rm DHS}$ close under integration over z and p by algebraic identities of integration kernels:

- "interchange lemma" $\omega_K(z) f^K{}_J(p, z) \ = \ -\omega_K(p) f^K{}_J(z, p)$
- "swapping identities" $\partial_p f^{I_1 I_2 \cdots I_s K}{}_J(z, p) \ = \ (-1)^s \, \partial_z f^{I_s \cdots I_2 I_1 K}{}_J(p, z)$
- more generally, Fay identities eliminate repeated appearance of z, e.g.

$$f^I{}_J(y, z) f^J{}_K(z, x) \ = \ f^I{}_J(y, x) f^J{}_K(z, x) - f^I{}_J(z, y) f^J{}_K(y, x)$$

$$- \omega_J(y) f^{IJ}{}_K(z, y) - \omega_J(y) f^{JI}{}_K(z, x) - \omega_J(z) f^{JI}{}_K(y, x)$$

- identical interchange/swapping/Fay identities for $g^{\vec{I}}{}_J(x, y) \leftrightarrow f^{\vec{I}}{}_J(x, y)$

$\Longrightarrow$ closely related integration algorithms for combinations of f's & g's

[D'Hoker, OS 2407.11476]

II. 4 Closure under integration

Tensorial Fay identities of bilinears in $f^{\vec{I}}{}_J(x,y)$ at arbitrary rank

$$
\begin{aligned}
f^{I_1\ldots I_r}{}_J(z,x) f^{P_1\ldots P_s J}{}_K(y,z) &= f^{I_1\ldots I_r}{}_J(z,x) f^{P_1\ldots P_s J}{}_K(y,x) \\
&+ \sum_{m=0}^{s} (-1)^{m-s-1} \sum_{\ell=0}^{r} f^{(P_s\cdots P_{m+1} \sqcup I_1 \cdots I_\ell)}{}_J(z,y) f^{P_1\cdots P_m J I_{\ell+1}\cdots I_r}{}_K(y,x) \\
&+ \sum_{m=0}^{s} (-1)^{m-s-1} f^{P_1\cdots P_m}{}_J(y,x) \big[f^{(P_s\cdots P_{m+1} J \sqcup I_1 \cdots I_{r-1}) I_r}{}_K(z,x) \\
&\qquad + f^{(P_s\cdots P_{m+1} \sqcup I_1 \ldots I_r) J}{}_K(z,y) \big]
\end{aligned}
$$

no repeated z on RHS!

- proven by (inductively) showing LHS - RHS is holomorphic in x, y, z and integrates to zero against $\bar{\omega}^L(z)\bar{\omega}^M(y)$ **[D'Hoker, OS 2407.11476]**
- analogous proof for $f^{\vec{I}}{}_J(x,y) \to g^{\vec{I}}{}_J(x,y)$ based on monodromy, reformulation via generating series and proof of completeness in **[Baune, Broedel, Im, Lisitsyn, Möckli 2409.08208]**

Summary

- 2 equivalent constructions of polylogs on higher-genus Riemann surfaces

  via single-valued *or* meromorphic kernels with at most simple poles

  $\longrightarrow$ by closure under integration, expect wide range of applications

- single-valued $f^{I_1\cdots I_n}{}_J$ from surface integrals of Arakelov Green function,

  $\longrightarrow$ arise naturally in worldsheet CFT for string amplitudes

  $\longrightarrow$ bootstrap approaches and algorithms for $\int \mathrm{d}z_i$ at small α'

Thank you for your attention !

Appendix: Arakelov Green function $\mathcal{G}$ from theta functions

- genus-h theta fct. with $\zeta \in \mathbb{C}^h$, odd characteristics $\kappa = [\kappa', \kappa''] \in \{0, \frac{1}{2}\}^{2h}$

$$\theta[\kappa](\zeta|\Omega) = \sum_{n \in \mathbb{Z}^h} e^{\pi i (n+\kappa')^t \Omega (n+\kappa') + 2\pi i (n+\kappa')^t (\zeta+\kappa'')}$$

- prime form: $E(x,y) = (x-y) + \mathcal{O}\big((x-y)^3\big)$ without extra zeros

$$E(x,y) = \frac{\theta[\kappa]\big(\int_y^x \omega_I\big)}{h_\kappa(x) h_\kappa(y)} \, ,$$

with half-differentials $h_\kappa(x)^2 = \sum_{I=1}^h \omega_I(x) \frac{\partial}{\partial \zeta_I} \theta[\kappa](0)$

$\longrightarrow$ independent on odd characteristics κ with $\theta[\kappa](-\zeta) = -\theta[\kappa](\zeta)$

- string Green function:

$$G(x,y) = -\log \big| E(x,y) \big|^2 + \overbrace{2\pi \, \mathrm{Im} \int_y^x \omega_I \, [(\mathrm{Im}\,\Omega)^{-1}]^{IJ} \, \mathrm{Im} \int_y^x \omega_J}^{\text{compensate } x \to \mathfrak{B}_I \cdot x \text{ monodromy of } \ln |E(x,y)|^2}$$

Appendix: Arakelov Green function $\mathcal{G}$ from theta functions

- string Green function:

$$G(x,y) = -\log\left|E(x,y)\right|^2 + 2\pi \overbrace{\mathrm{Im}\int_y^x \omega_I \,[(\mathrm{Im}\,\Omega)^{-1}]^{IJ}\,\mathrm{Im}\int_y^x \omega_J}^{\text{compensate } x \to \mathfrak{B}_I \cdot x \text{ monodromy of } \ln|E(x,y)|^2}$$

- can find straightforward shifts $\mathcal{G}(x,y) = G(x,y) + \ldots$ such that Arakelov Green function integrates to zero against $\kappa(x) = \frac{1}{h}\bar{\omega}^I(x)\omega_I(x)$:

$$\mathcal{G}(x,y) = G(x,y) - \int_{\Sigma_h} \kappa(z) G(z,y) - \int_{\Sigma} \kappa(z) G(z,x) + \int_{(\Sigma_h)^2} \kappa(z)\kappa(w) G(z,w)$$

- By the construction of $G(x,y)$ from $E(x,y)$ and hence $\theta[\kappa](\zeta|\Omega)$,

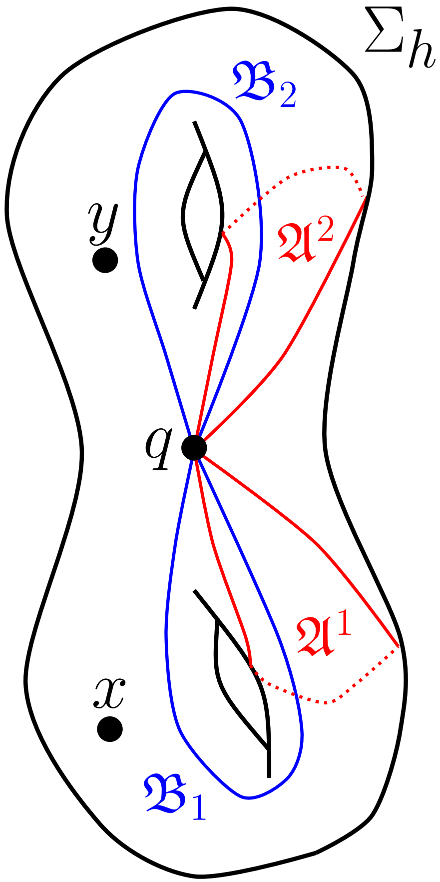
  Arakelov Green function $\longrightarrow$ theta functions **and** their integrals over Σ.

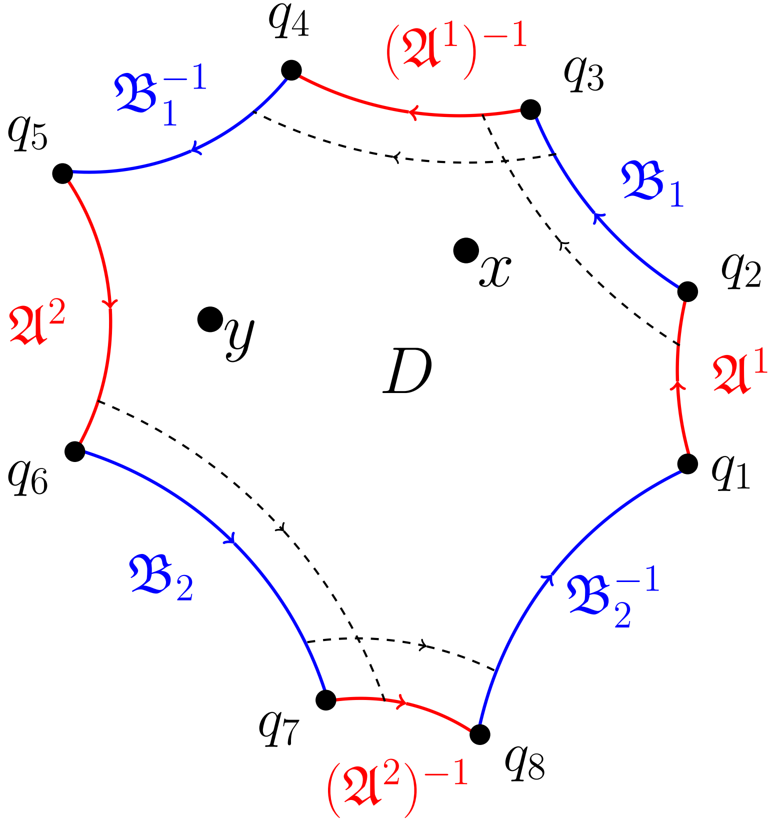
- Work in progress: expansion around degenerations suitable for numerics

Appendix: Fundamental domain D for Enriquez kernels

$\mathfrak{A}$-integral representation of Enriquez kernels $g^{I_1\cdots I_n}{}_J(x,y)$ applies to

$x,y \in$ (fundamental domain D) inside universal cover $\tilde{\Sigma}_h$ of surface Σ_h





Reduce x,y outside D to $x,y \in D$ via known monodromies of $g^{I_1\cdots I_n}{}_J(x,y)$