



BOUNDARY ENERGY-MOMENTUM TENSORS FOR ASYMPTOTICALLY FLAT SPACETIMES

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THE CENTER OF GRAVITY

The hunt for "Structure X"



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A HOLOGRAPHIC DESCRIPTION

FOR $\lambda=0$, IF THERE

REALLY IS SUCH A THING,

MUST INVOLVE NOT C.F.T.

BUT SOMETHING ELSE-

CALL IT "STRUCTURE X"

AS WE DON'T KNOW WHAT

IT IS. WHATEVER IT IS,

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0 + + + + +

A TOO-NAIVE VERSION

OF STRUCTURE X WOULD

BE A FELD THEORY

ON NULL INFINITY WITH

ITS PECULIAR DIFFERENTIAL

GEOMETRY.

Motivation



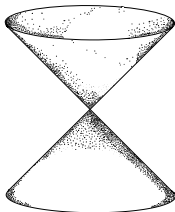
- In AdS/CFT, the boundary energy-momentum tensor (EMT) plays a crucial rôle
- ⇒ Defined via procedure known as holographic renormalisation [Balasubramanian, Kraus, '99; de Haro, Solodhukin, Skenderis, '00]

Goal: Do the same for asymptotically flat spacetimes

- The boundary geometry on $\mathcal{I}^+$ in asymptotically flat spacetime is conformally Carrollian [Duval, Gibbons, Horvarthy, '14]
- ⇒ Must understand near-boundary on-shell description of gravity with arbitrary carrollian data at $\mathcal{I}^+$

A taste of Carrollian symmetry & geometry

- Physically $c \rightarrow 0$; hence “ultra-local”



$c = 1$



$c \ll 1$



$c = 0$



- Carroll boosts act as

$$t \rightarrow t' = t + b_i x^i, \quad \vec{x} \rightarrow \vec{x}' = \vec{x}$$

Metric $g_{\mu\nu}$ replaced by $(v^\mu, h_{\mu\nu})$ satisfying $v^\mu h_{\mu\nu} = 0$

- A *conformal* Carrollian structure transforms as

$$\delta v^\mu = -\Lambda_D v^\mu, \quad \delta h_{\mu\nu} = 2\Lambda_D h_{\mu\nu}, \quad \delta \tau_\mu = \Lambda_D + \lambda_\mu, \quad \delta h^{\mu\nu} = -2\Lambda_D + 2^{(\mu} \lambda^{\nu)}$$

Plan

- ① AF spacetimes and Carroll-covariant Bondi–Sachs gauge
- ② Constraints, variations, and Ward identities
- ③ Holographic renormalisation



Carrollian geometry at $\mathcal{I}^+$ from Penrose construction

Let $(\mathcal{M}, g)$ be an AF $(d+2)$ -dimensional Lorentzian manifold:

- Write

$$g = -2UV + E^a E^a \quad (a = 1, \dots, d)$$

U, V null; E^a spacelike

- Treat $\mathcal{I}^+$ as a Penrose b'dary $\Rightarrow$ split $x^M = (r, x^\mu)$ [Penrose, '63]
- Partially fix local $\text{SO}(d+1, 1)$ & diffeos to get

$$\begin{aligned} g_{rr} &= 0, \quad g_{r\mu} = -V_\mu, \quad g_{\mu\nu} = -SV_\mu V_\nu + \Pi_{\mu\nu}, \\ g^{rr} &= S, \quad g^{r\mu} = U^\mu, \quad g^{\mu\nu} = \Pi^{\mu\nu} \quad (S = \mathcal{O}(r)) \end{aligned}$$

- Penrose construction fixes boundary conditions:

$$V_\mu|_{r \rightarrow \infty} = \tau_\mu, \quad r^{-2}\Pi_{\mu\nu}|_{r \rightarrow \infty} = h_{\mu\nu}, \quad U^\mu|_{r \rightarrow \infty} = v^\mu, \quad r^2\Pi^{\mu\nu}|_{r \rightarrow \infty} = h^{\mu\nu}$$

Carroll-covariant Bondi–Sachs gauge

This gauge corresponds to

$$g_{rr} = 0, \quad \Gamma_{rr}^\rho = 0, \quad \Gamma_{\mu r}^\mu = dr^{-1}$$

⇒ This fixes $V_\mu = e^\beta \tau_\mu$ ($\beta = \mathcal{O}(r^{-1})$) and $h^{\mu\nu} \Pi_{\mu\nu}$

👉 In “standard” Bondi gauge, metric fully gauge fixed to

$$\tau = du \quad \text{and} \quad h = ds_{\mathbb{CS}^d}^2 \quad (\mathcal{I}^+ \cong \mathbb{R} \times \mathbb{CS}^d)$$

- We, however, want *arbitrary* boundary data (τ, h) , so we can vary with respect to it



When $d \geq 2$, Einstein's equations constrain the boundary geometry

$$K_{\mu\nu} := -\frac{1}{2} \mathcal{L}_\nu h_{\mu\nu} = \frac{1}{d} K h_{\mu\nu}$$

but this “issue” can be dealt with

Solving Einstein's equations asymptotically

Einstein's equations allow to solve for β , S , $h_{\mu}^{\rho} v^{\sigma} \Pi_{\rho\sigma}$ ($h_{\nu}^{\mu} = e_a^{\mu} e_{\nu}^a$), and $\mathfrak{L}_{\nu} \left(h_{\langle\mu}^{\rho} h_{\nu\rangle}^{\sigma} \Pi_{\rho\sigma} \right)$

$$\Pi_{\mu\nu} = r^2 h_{\mu\nu} + r (C_{\mu\nu} - 2\tau_{(\mu} a_{\nu)}) + \mathcal{O}(1), \quad S = \frac{2}{d} Kr + \mathcal{O}(1), \quad \beta = \mathcal{O}(r^{-2})$$

with $C_{\mu\nu}$ the shear & $a_{\mu} = \mathfrak{L}_{\nu} \tau_{\mu}$

- Residual transformations $\xi^{\mu} = \chi^{\mu} + r^{-1} h^{\mu\nu} \lambda_{\nu} + \mathcal{O}(r^{-2})$ and $\xi^r = r\Lambda_D + \mathcal{O}(1)$ turn into Carroll transformations

$$\begin{aligned} \delta\tau_{\mu} &= \mathfrak{L}_{\chi} \tau_{\mu} + \Lambda_D \tau_{\mu} + \lambda_{\mu}, \\ \delta h_{\mu\nu} &= \mathfrak{L}_{\chi} h_{\mu\nu} + 2\Lambda_D h_{\mu\nu}, \\ \delta C_{\mu\nu} &= \mathfrak{L}_{\chi} C_{\mu\nu} + 2h_{\langle\mu}^{\rho} h_{\nu\rangle}^{\sigma} (\mathcal{D}_{\rho} \lambda_{\sigma} + a_{\rho} \lambda_{\sigma}) \end{aligned}$$



- When $d = 1$, the r -expansion truncates

On-shell near-boundary metric for $d = 2$

We denote by $X^{(n)}$ the coefficient of r^{-n} in the r -expansion of X

- The on-shell metric expands near $\mathcal{I}^+$ as

$$ds^2 = -2dr(\tau_\mu + \mathcal{O}(r^{-2}))dx^\mu + \left(r^2 h_{\mu\nu} + r g_{\mu\nu}^{(-1)} + g_{\mu\nu}^{(0)} + \mathcal{O}(r^{-1})\right) dx^\mu dx^\nu$$

where

$$\begin{aligned} g_{\mu\nu}^{(-1)} &= -K\tau_\mu\tau_\nu - 2\tau_{(\mu}a_{\nu)} + C_{\mu\nu}, \\ g_{\mu\nu}^{(0)} &= -(S^{(0)} - a^2)\tau_\mu\tau_\nu - 2\tau_{(\mu}P_{\nu)}^{(0)} + h_\mu^\rho h_\nu^\sigma \Pi_{\rho\sigma}^{(0)} \end{aligned}$$

- The free data not determined by Einstein's equations consists of $S^{(1)}$ and $h_\mu^\rho v^\sigma \Pi_{\rho\sigma}^{(1)} \Rightarrow$ generalised Bondi mass and angular momentum aspect

Constrained boundary variations and Ward identities

Consider the variation of a boundary action functional $S[\tau, h, C]$

$$\delta S[\tau, h, C] = \int d^{d+1}x e \left(T^\mu \delta \tau_\mu + \frac{1}{2} T^{\mu\nu} \delta h_{\mu\nu} + \frac{1}{2} S^{\mu\nu} \delta C_{\mu\nu} \right) \quad \text{⚠}$$

Together, the responses form an EMT-news complex

👉 Ward identities are obtained by varying the boundary action functional under the local symmetries of the boundary structure:

$$\chi^\mu : \quad 0 = -e^{-1} \partial_\mu (e [T^\mu{}_\nu + S^{\mu\rho} C_{\rho\nu}]) + T^\mu \partial_\nu \tau_\mu + \frac{1}{2} T^{\mu\rho} \partial_\nu h_{\mu\rho} + \frac{1}{2} S^{\mu\rho} \partial_\nu C_{\mu\rho} ,$$

$$\Lambda_D : \quad 0 = T^\mu \tau_\mu + T^{\mu\nu} h_{\mu\nu} + \frac{1}{2} S^{\mu\nu} C_{\mu\nu} ,$$

$$\lambda_\mu : \quad 0 = h^\mu{}_\rho T^\rho - (\mathcal{D}_\rho - a_\rho) S^{\rho\mu}$$

(In the above, $T^\mu{}_\nu = T^\mu \tau_\nu + T^{\mu\rho} h_{\rho\nu}$ is the EMT)

A well-posed variational principle

- Demand well-posed* variational principle for large r ; regularise on cut-off hypersurface $r = \Lambda$

$$\delta S_{\text{EH}} = \cdots + \int_{r=\Lambda} d^3 E J^r \quad (E := \sqrt{-g} = e^d e^\beta)$$

⇒ Need extrinsic GHY-type boundary term, but cut-off surface *indefinite* [Parattu, Chakraborty, Padmanabhan, '16]

$$S_{\text{ext}} = 2 \int_{r=\Lambda} d^{d+1} x \sqrt{-g} (\delta_P^M + V^M N_P) \nabla_M N^P, \quad N_M = \partial_M r \text{ and } V^M = -\delta_r^M$$

$$\Rightarrow \delta (S_{\text{EH}} + S_{\text{ext}}) = \cdots + 2 \int_{r=\Lambda} d^3 x E \left(\overbrace{\tilde{T}^\mu \delta V_\mu + \frac{1}{2} \tilde{T}^{\mu\nu} \delta \Pi_{\mu\nu}}^{=\mathcal{O}(r^{-1})} + \overbrace{\frac{1}{2} dr^{-1} E^{-1} \delta (ES)}^{\mathcal{O}(1)} \right)$$

👉 The leading term *contains variations of the free data* $S^{(1)} \Rightarrow$ must be cancelled by adding

$$S_{\text{norm}} = -2 \int_{r=\Lambda} d^{d+1} x E r^{-1} S$$

Holographic renormalisation for AF spacetimes

- $\delta (S_{\text{EH}} + S_{\text{ext}} + S_{\text{norm}})$ *still* diverges as $r \rightarrow \infty$, but cured by

$$S_{\text{int}} = - \int_{r=\Lambda} d^3x E r R[C]$$

- In addition to this, we add finite counterterms that make the on-shell action Weyl invariant (*improvements*)

$\Rightarrow$ This leads to

$$\delta (S_{\text{EH}} + S_{\text{ext}} + S_{\text{norm}} + S_{\text{int}}) \Big|_{r \rightarrow \infty} = \int d^3x e \left(T^\mu \delta \tau_\mu + \frac{1}{2} T^{\mu\nu} \delta h_{\mu\nu} + \frac{1}{2} S^{\mu\nu} \delta C_{\mu\nu} \right)$$

which happens because $\mathcal{T}^{\mu\nu} = \mathcal{O}(r^{-3})$

 The explicitly computed currents satisfy (cf. Ward identities)

$$\mathcal{A}_B^\mu = h_\rho^\mu T^\rho - (\mathcal{D}_\rho - a_\rho) S^{\rho\mu}$$

$$0 = T^\mu \tau_\mu + T^{\mu\nu} h_{\mu\nu} + \frac{1}{2} S^{\mu\nu} C_{\mu\nu}$$

Carroll-covariant Bondi loss equations

The diffeo Ward identity turns into the Bondi loss equations

$$0 = - \left(\mathcal{L}_v - \frac{3}{2} K \right) (\tau_\mu T^\mu) - \frac{1}{4} N^{\rho\sigma} N_{\rho\sigma} + (\mathcal{D}_\mu + a_\mu) (T^\rho h_\rho^\mu) ,$$

$$0 = - (\mathcal{L}_v - K) P_\kappa + h_{\kappa\sigma} \mathcal{D}_\mu \tilde{T}^{\mu\sigma} + \frac{1}{2} h_\kappa^\mu (\partial_\mu + 3a_\mu) \left(T^{\rho\sigma} h_{\rho\sigma} + \frac{1}{2} N^{\rho\sigma} C_{\rho\sigma} \right) \\ + \frac{1}{4} h_{\kappa\sigma} \mathcal{D}_\mu (N^{\mu\lambda} C_\lambda^\sigma - N^{\sigma\lambda} C_\lambda^\mu) + T^\sigma h_\sigma^\mu F_{\mu\kappa} - \frac{1}{4} N^{\mu\sigma} h_\kappa^\nu (\mathcal{D}_\nu + a_\nu) C_{\mu\sigma}$$

where $\tilde{T}^{\mu\nu} := h_{\langle\rho}^\mu h_{\sigma\rangle}^\nu T^{\rho\sigma}$ and $P_\mu = T^{\rho\sigma} \tau_\rho h_{\sigma\mu}$

- Covariant Weyl-invariant news tensor defined by

$$N_{\mu\nu} = -\mathcal{L}_v C_{\mu\nu} - \frac{1}{2} K C_{\mu\nu}$$

- Related to improved response via $S^{\mu\nu} = \frac{1}{2} h^{\mu\rho} h^{\nu\sigma} N_{\rho\sigma}$

Future directions

- Can we write down a Carrollian boundary CFT dual to gravity in AF spacetime?

- Connections to string theory?

[Costello, Paquette, Sharma, '22 & '23]

- Explicit relation to celestial holography?

à la [Donnay, Fiorucci, Herfray, Ruzziconi, '22]



THANK YOU FOR YOUR ATTENTION