Coulomb Branch of $\mathcal{N} = 4$ SYM and integrability

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[Ivanovskiy, Komatsu, M., Terziev, Zaigraev, Zarembo]

- SYM theory has many nice features: integrability and holography etc.
- Reasonably well understood at the conformal point.
- We are going to study $\mathcal{N} = 4,\ U(N)$ SYM theory in a symmetry breaking vacuum.
- Example: defect CFT [de Leeuw, Kristjansen, Zarembo],[de Leeuw, Kristjansen, Linardopoulos],[de Leeuw, Kristjansen, Linardopoulos]

Gauge theory set-up

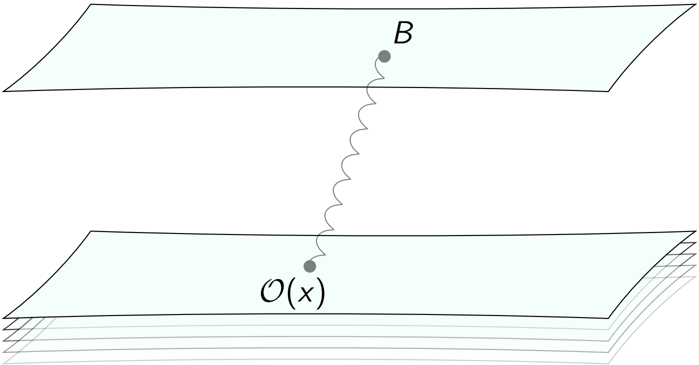
Give a vev to the scalar fields, $i = 1, \dots, 6$:

$$\Phi_i^{\mathrm{cl}} = \begin{pmatrix} v_i & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in U(N), \qquad \text{choose} \quad v_i = v\,\delta_{i,1}$$

Symmetry breaking pattern

- Conformal symmetry is broken to Poincare
- Gauge symmetry is higgsed $\mathrm{U(N)} \to \mathrm{U(N-1)} \times \mathrm{U(1)}$
- Massive fields from fields with $1j$ indices, for $j \neq 1$.

The holographic set up



Position in AdS $z_0 \sim \frac{1}{v}$, W-boson mass $m_W = v$

Motivation

- Interesting to test holography and integrability beyond conformal symmetry.
- Theory is closer to realistic ones. Natural IR regularization of the S-matrix [Alday,Henn,Plefka,Schuster] and spectrum of W-boson bound states [Caron-Huot,Henn] .
- Condensates (one-point functions) are important: QCD sum rules [Shifman,Vainshtein,Zakharov] , resurgence [Liua,Mariño]

One-point functions

- Chiral primaries – tree level exact and the results coincides with a holographic computation

$$\mathcal{C}_L(x, y) = \mathrm{Tr}\left((y \cdot \Phi)^L\right)(x), \quad y \cdot y = 0$$

- Other operators receive loop-corrections. Konishi operator:

$$K(x) = Z_K \sum_{i=1}^{6} \mathrm{Tr}\, \Phi_i \Phi_i(x)$$

Renormalization of operators

- In theories with SSB - renormalized operators have finite one-point function.
- Scheme dependence.

Renormalized Konishi one-point function

$$\langle K \rangle_v = \frac{\left\langle K^b(0) \right\rangle_v}{|x|^\Delta \left\langle K^b(x) K^b(0) \right\rangle^{\frac{1}{2}}} \, ,$$

- The result is scheme independent
- Denominator computed at conformal point
- Result:

$$\langle K \rangle_v = \frac{4\pi^2 v^2}{\sqrt{3}\,\lambda} \left[1 + \frac{3\lambda}{4\pi^2} \left(\gamma + \ln \frac{v}{2} \right) \right] .$$

- To one loop we have:

$$\langle K \rangle_v = c_K v^{\Delta_K}.$$

$$c_K = \frac{4\pi^2}{\sqrt{3}\,\lambda} + \sqrt{3}\,(\gamma - \ln 2)\,.$$

- γ contribution is scheme independent - reproduced by integrability.

First hint of integrability

- To one loop one-point function can be represented as an onverlap with a boundary state

$$\langle \mathcal{O} \rangle_v \sim \langle B | \Psi \rangle ,$$

$$B_{i_1 \dots i_L} = n_{i_1} \dots n_{i_L}, \qquad n_i \equiv \frac{v_i}{v}.$$

- The boundary state is integrable [de Leeuw, Gombor, Kristjansen, Linardopoulos, Pozsgay]

Structure of the two-point functions

- UV: OPE on the Coulomb branch
- IR: The dilaton Ward identity

OPE vs one-point functions

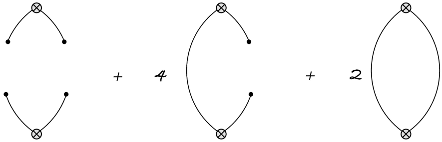
- BPS operator:

$$\mathcal{O} = \mathrm{Tr}\, Z^2(x)\,, \quad Z = \Phi_1 + i\Phi_2$$

- OPE:

$$\underbrace{\mathcal{O}^\dagger(x)\mathcal{O}(0) \approx \frac{\mathbb{1}}{x^4} + \frac{\overbrace{\tilde{\mathcal{O}}(0)}^{\mathrm{Tr}\left(3\bar{Z}Z - \sum_i \Phi_i\Phi_i\right)}}{x^2}}_{\text{not renormalized}} + \underbrace{\frac{K(0)}{x^{4-2\Delta_K}}}_{\text{renormalized}} +$$

- Diagrams:



- UV limit of the massive propagator:

$$D_v(x) \overset{x\to 0}{\simeq} \frac{1}{4\pi^2 x^2} + \left(\ln\frac{vx}{2} + \gamma - \frac{1}{2}\right)\frac{v^2}{8\pi^2}$$

- Structure constants at one loop:

$$C_{\mathcal{O}^\dagger\mathcal{O}\widetilde{\mathcal{O}}} = \frac{4}{\sqrt{6}\,N}\,, \qquad C_{\mathcal{O}^\dagger\mathcal{O}K} = \frac{2}{\sqrt{3}\,N}\left(1 - \frac{3\lambda}{8\pi^2} + \ldots\right).$$

- Consistency implies that:

$$\frac{2}{\sqrt{3}}\left[1 + (\Delta_K - 2)\ln x - \frac{3\lambda}{8\pi^2}\right]\langle K\rangle =$$
$$= 2v^2\left(\frac{4\pi^2}{3\lambda} + \ln\frac{vx}{2} + \gamma - \frac{1}{2}\right)$$

Convergence of OPE

- We find that one-loop corrections do not spoil the convergence of the OPE.
- Coulomb branch bootstrap:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle_{\rm vac} \overset{|x|\to\infty}{\sim} (\texttt{const}) + (\texttt{power}) + e^{-m_0|x|}$$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle_{\rm vac} = \sum_k c_{ijk}\langle \mathcal{O}_k\rangle_{\rm vac}(vx)^{\Delta_k-\Delta_i-\Delta_j}$$

- Estimates of growth of OPE data:

$$C(\Delta) \overset{\Delta\to\infty}{\sim} \frac{e^{-\pi i\Delta}\left(m_0/v\right)^{\Delta}}{\Gamma(\Delta)}$$

[Cuomo, Rastelli, Sharon]

Large distance behavior.

- Dilaton Ward identity dictates the large distance behavior

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_v \stackrel{|x|\to\infty}{\to} \langle O_1 \rangle_v \langle O_2 \rangle_v + \\ + \frac{\left(\Delta_1 \Delta_2 + \sum_{I=1}^{5} \hat{R}_I^{(1)} \hat{R}_I^{(2)} \right) \langle \mathcal{O}_1 \rangle_v \langle \mathcal{O}_2 \rangle_v}{f_\pi^2 v^2 |x|^2}$$

- At one loop the dilaton decay constant is not renormalized

$${}_v\langle 0 | T_{\mu\nu}(p) | \pi \rangle_v = \frac{f_\pi}{3} v p_\mu p_\nu$$

Integrability of SYM

- At $\lambda \ll 1$ - spin chain.
- Can be extended to any λ via BA, QCS
- At $\lambda \gg 1$ - string sigma model

Integrability on the Coulomb branch

- At loop level - overlap with a integrable boundary state.
- Integrability extends to all λ [Coronado, Komatsu, Zarembo] - closed form determinant formula.
- String σ-model with the bulk D3 brane b.c. is integrable [Demjaha, Zarembo]
- Integrable properties of amplitudes [Alday, Henn,Plefka,Schuster],[Loebbert,Miczajka, Müller,Münkler]

Outlook

- Coulomb branch is integrable
- OPE vs condensates
- Large charge