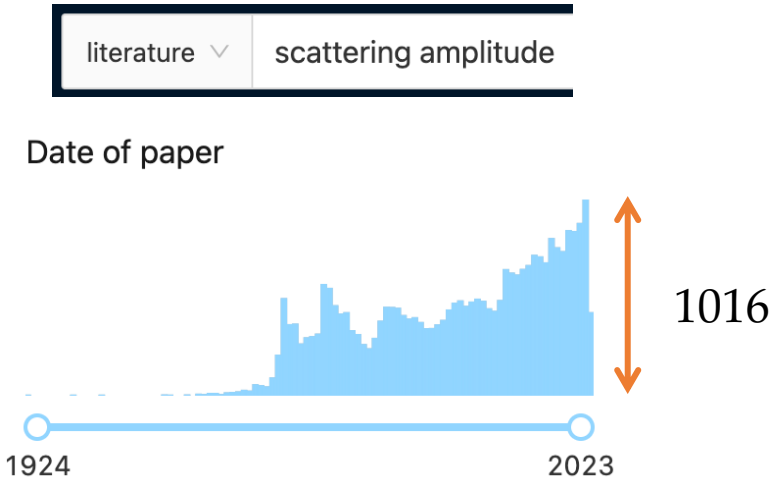
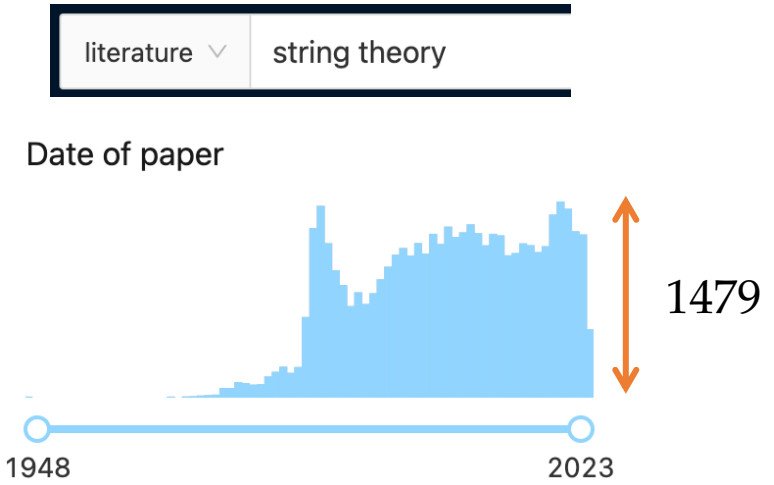
Formal aspects of scattering amplitudes:

Lessons and Challenges in 2023

Sebastian Mizera (IAS)

The field of scattering amplitudes continues to be enormously rich and fast-growing

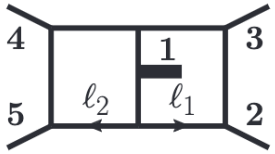




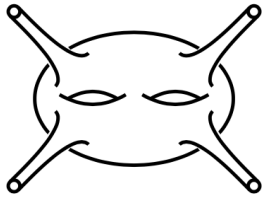
Can’t possibly do it justice in 60 mins!

Today, I'll cover some of the highlights in the following *formal* areas:

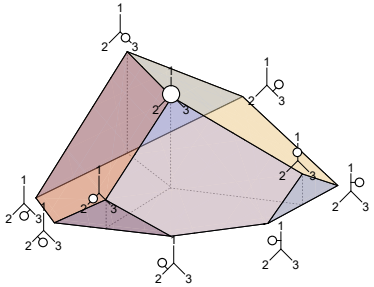
1) Precision frontier



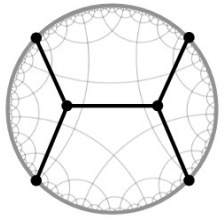
i) String amplitudes



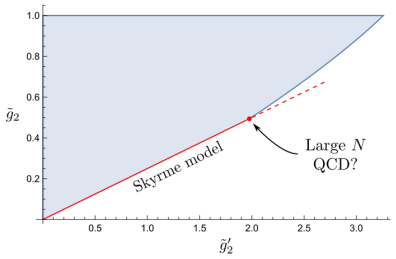
I) Polytopes and tropical geometry



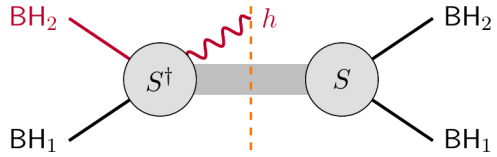
a) Amplitudology in curved space



A) S-matrix bootstrap



α) Crossing symmetry



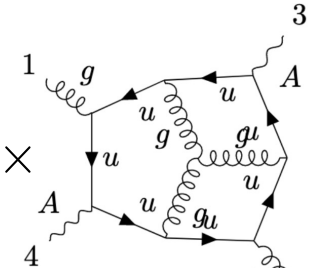
1) PRECISION FRONTIER

Life cycle of a scattering amplitude in 2023

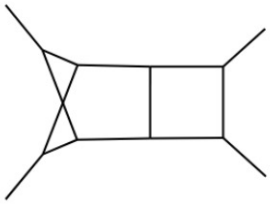
(simplified version)

[Talks on Monday]

$\mathcal{O}(10^3) \times$



$\mathcal{O}(10^6) \times$



- Draw all Feynman diagrams
- Reduce colors/tensor structures to scalar Feynman integrals
- Express in terms of master integrals $\vec{I}, \quad |\vec{I}| = \mathcal{O}(10^2)$
- Find canonical differential equations $\mathrm{d}\vec{I} = \epsilon\, \mathbf{\Omega}\, \vec{I}$
- Get the analytic form for all $\vec{I}$ ⟵ Skip?!
- Use to evaluate observables numerically

[Recent books by Badger, Henn, Plefka, Zoia 2306.05976 and Weinzierl 2201.03593]

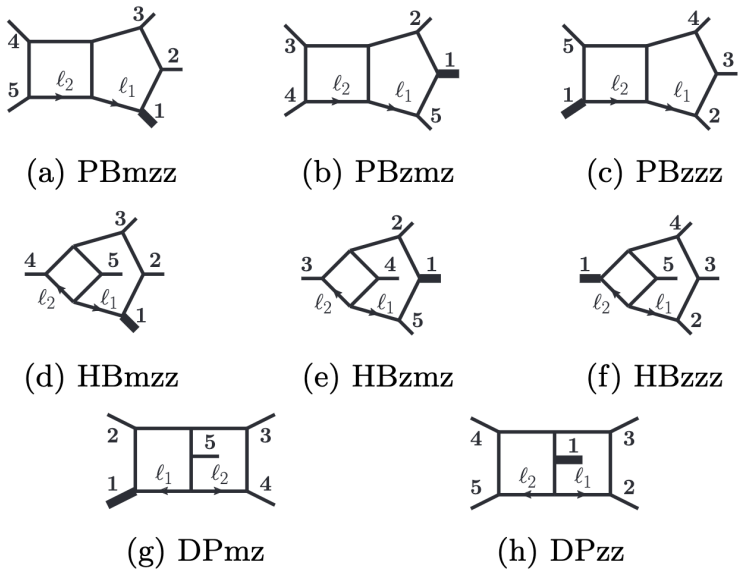
Challenge: Applying similar computational pipeline in cosmology?

[Review talk by Stefanyszyn]

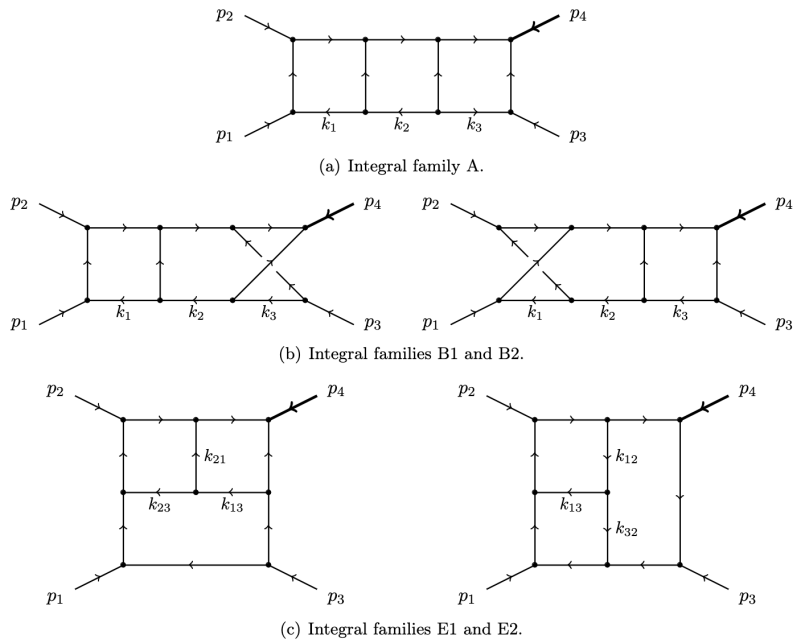
[Talk by Lee]

Example cutting-edge computations: polylogarithmic

(Higgs+jet production at LHC)



[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431]



[Henn, Lim, Bobadilla 2302.12776]

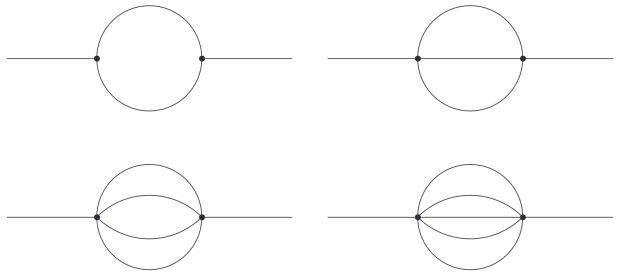
Lesson: No longer clear having an analytic expression is useful!

Challenge: Rethink the way we *represent* scattering amplitudes

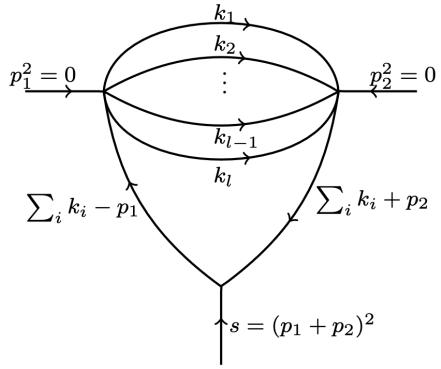
Example cutting-edge computations: elliptic and beyond

(more formal progress, symbology)

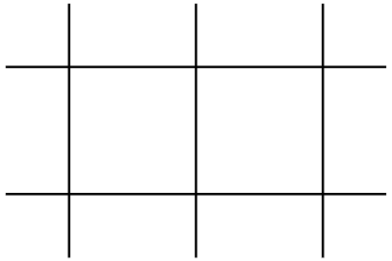
Rule of thumb:
Cut with 3 massive particles
$\Rightarrow$ not a polylog



[Pogel, Wang, Weinzierl 2212.08908]



[Duhr, Klemm, Nega, Tancredi 2212.09550]



[Wilhelm, Zhang 2206.08378]

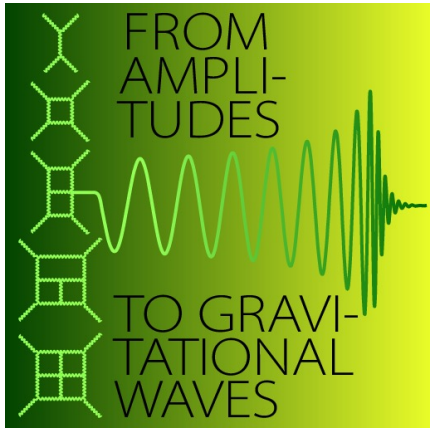
Many other ideas in multi-loop computations:

- Pentagon functions
- Prescriptive unitarity
- Intersection theory
- …

Recent highlights in software for Feynman integrals

- **PySecDec 1.6:** up to 5x speedup in numerical evaluation [Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk 2305.19768]
- **AMFlow:** fast numerics with auxiliary mass flow [Liu, Ma 2201.11669]
- **FeynTrop:** evaluate integrals up to 17-loop (finite only) [Borinsky, Munch, Tellander 2302.08955]
- **INITIAL:** canonical differential equations for elliptics [Dlapa, Henn, Wagner 2211.16357]
- **NeatIBP:** small-size IBP relations [Wu, Boehm, Ma, Xu, Zhang 2305.08783]
- **PLD:** predicting singularities of Feynman integrals [Fevola, SM, Telen]

There's been enormous progress on



[Talks by Cangemi, Carrillo Gonzalez, Cristofoli, Heissenberg, Kälin, van de Meent, Mogull, Pound, Roiban, Ruf, Skvortsov, Travaglini]

Lessons/challenges: [Review talk by Buonanno]

We'll come back to gravitational waveforms in the last part of the talk

Infrared divergences

- **QED:** Bloch-Nordsieck, Chung-Faddeev-Kulish dressing, eikonal exponentiation, Wilson lines, …
- **Perturbative gravity:** Chung-Faddeev-Kulish-like dressing, eikonal exponentiation, Wilson lines, …
- **QCD:** Non-forward factorization, otherwise ¯_(ツ)_/¯ [Sterman 2207.06507]

[Review by Agarwal, Magnea, Signorile-Signorile, Tripathi 2112.07099]

Lesson: No theorem guaranteeing inclusive cross-sections are finite

[Frye, Hannesdottir, Paul, Schwartz, Yan 1810.10022]

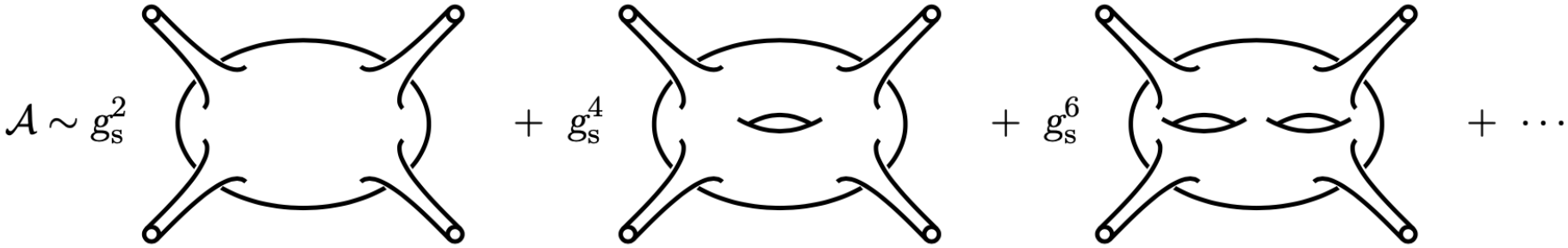
Some recent progress:

- Trouble with non-perturbative gravity [Prabhu, Satishchandran, Wald 2203.14334]
- Soft-collinear effective theory for gravity [Beneke, Hager, Szafron 2210.09336]
- Local unitarity: compute finite cross-sections [Capatti, Hirschi, Ruijl 2203.11038]
- Connections to celestial amplitudes [Review talk by Raclariu]

[Talks by Taylor, Yelleshpur Srikant]

i) STRING AMPLITUDES

String perturbation theory



[Recent reviews: Mafra, Schlotterer 2210.14241, Snowmass 2203.09099]

Formal expression: $\mathcal{A}_{g,n} \sim \int_{\mathcal{M}_{g,n}} \langle \mathcal{V}_1(z_1)\mathcal{V}_2(z_2)\cdots\mathcal{V}_n(z_n)\rangle \, \mathrm{d}\mu_{g,n}$

Genus = #loops #legs

Four avenues of recent progress

- Tree-level scattering
- Computing the correlation function & relations to double copy
- Number theory in the α' expansion
- Computing the amplitude

Recent progress on tree-level scattering

- **Coon amplitude:** accumulation point in the spectrum [Jepsen 2303.02149] [Geiser, Lindwasser 2210.14920] [Bhardwaj, De, Spradlin, Volovich 2212.00764] [Cheung, Remmen 2302.12263] [Li, Sun 2307.13117]

  Challenge: Do any of such Veneziano-like expressions come from a consistent model?

- **Bounds on the rank of gauge group:** N=32 seems to be special [Bachu, Hillman 2212.03871]

- **Chaotic scattering off excited strings?** [Bianchi, Firrotta, Sonnenschein, Weissman 2303.17233] [Talk by Bianchi]

Correlation functions/loop integrands

- **State of the art:** One-loop for any n [Mafra, Schlotterer, …]
  Two-loop for $n \leqslant 5$ [D'Hoker, Mafra, Pioline, Schlotterer 2008.08687]
  Three-loop for $n \leqslant 4$ (conjecturally) [Geyer, Monteiro, Stark-Muchao 2106.03968]

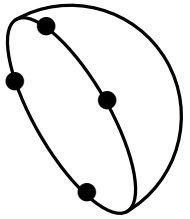
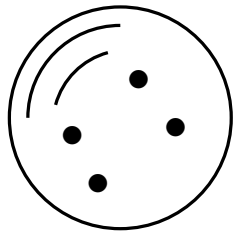
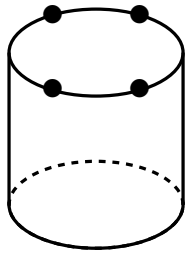
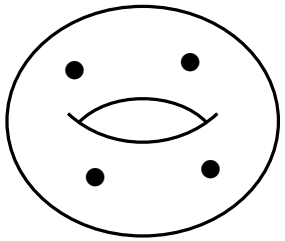
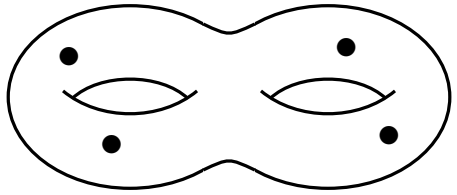
- **Color-kinematics dual representation of one-loop integrands**
  [Edison, He, Johansson, Schlotterer, Teng, Zhang 2211.00638]

- **KLT-like relations for the correlation function at one loop** [Stieberger 2212.06816]

- **New ideas in representing the correlator in bosonic string theory (surfacehedron)**
  [Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas]

[Slide from yesterday's Strings discussion by Schlotterer]

Mathematical input on α'-expansions of string amplitudes

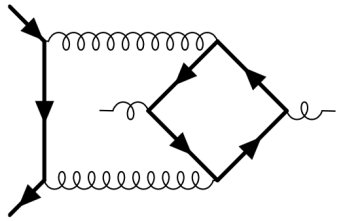
	open strings		closed strings	
tree level	motivic MZVs and their "f-alphabet" [Brown 1102.1310] [Stieberger-OS '12] Drinfeld associator [Terasoma '02] [Broedel-OS-Stieberger-Terasoma '13]	$\int$  multiple zeta values (MZVs)	single-valued integration & periods [Brown, Schnetz '13] [Brown-Dupont '18] stringy applications [OS, Schnetz, Stieberger, Taylor, Vanhove Zerbini '12 - '18]	$\int$  only "single-valued" subclass of MZVs
one loop	introducing eMZVs & their properties [Enriquez '13, Zerbini '18] stringy applications [Broedel, Mafra, Matthes, Richter, OS, Zerbini '14 - '18]	$\int$  elliptic MZVs (eMZVs)	non-holo' modular from involving MZVs [D'Hoker, Green, Gürdogan, Vanhove '15/16] equiv. Eisenstein int's [Brown '17] matched with MGFs [2209.06772]	$\int$  modular graph functions / forms (MGFs)
higher loop	higher-genus ϑ-functions [Fay, Mumford, etc. '70s, '80s] import to string amplitudes [Alvarez-Gaume et al. '80s] single-valued "Arakelov" Green function [Faltings '84] non-holomorphic modular tensors [Kawazumi '16, '17, '22] 2-loop α'-expansion [D'Hoker, Green, Pioline, Russo, OS '13-20] higher-genus polylogs: function space [Enriquez, Zerbini '21, 22] ... and an explicit construction [D'Hoker, Hidding, OS '23]			 higher-genus modular graph forms or tensors

α'-expansion and number theory

- **Polylogs for higher-genus Riemann surfaces**

[D'Hoker, Hidding, Schlotterer 2306.08644]
[Enriquez, Zerbini 2212.03119]
[Talk by Hidding]

Challenge: Applications to Feynman integrals?



- **Closed strings as single-valued version of open?**

[Snowmass 2203.09099]
[Alday, Hansen 2306.12786]
[Baune, Broedel 2306.16257]
[Talk by Hansen]

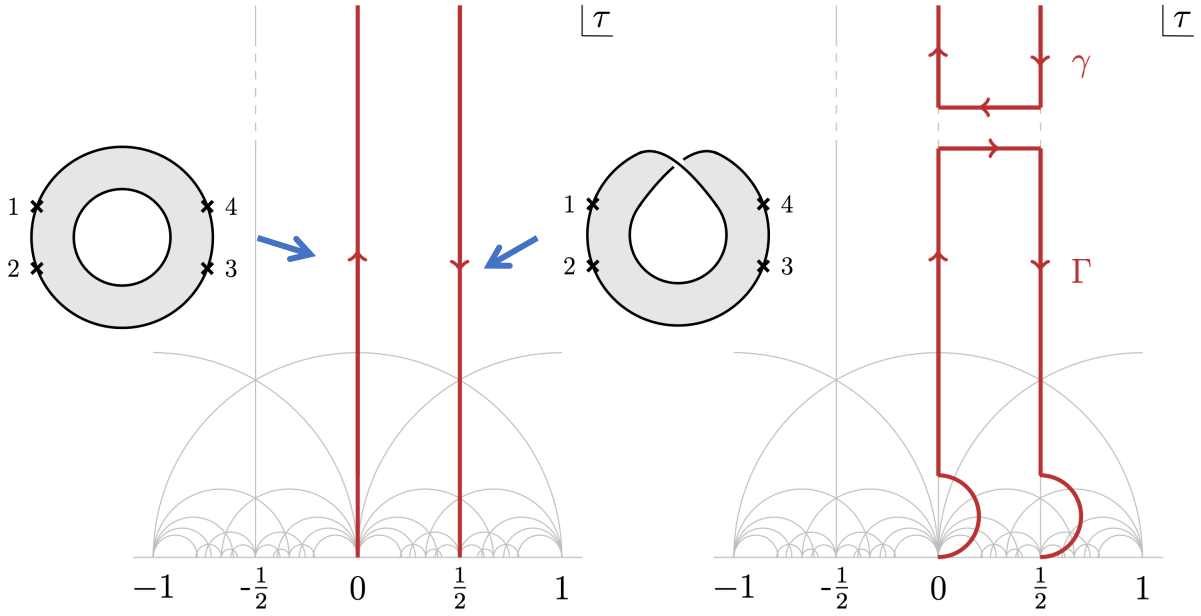
- **Exact integrated correlators in N=4 super Yang-Mills**

[Talk by Wen]

Computing string amplitudes

Formal integrals $\Rightarrow$ numbers

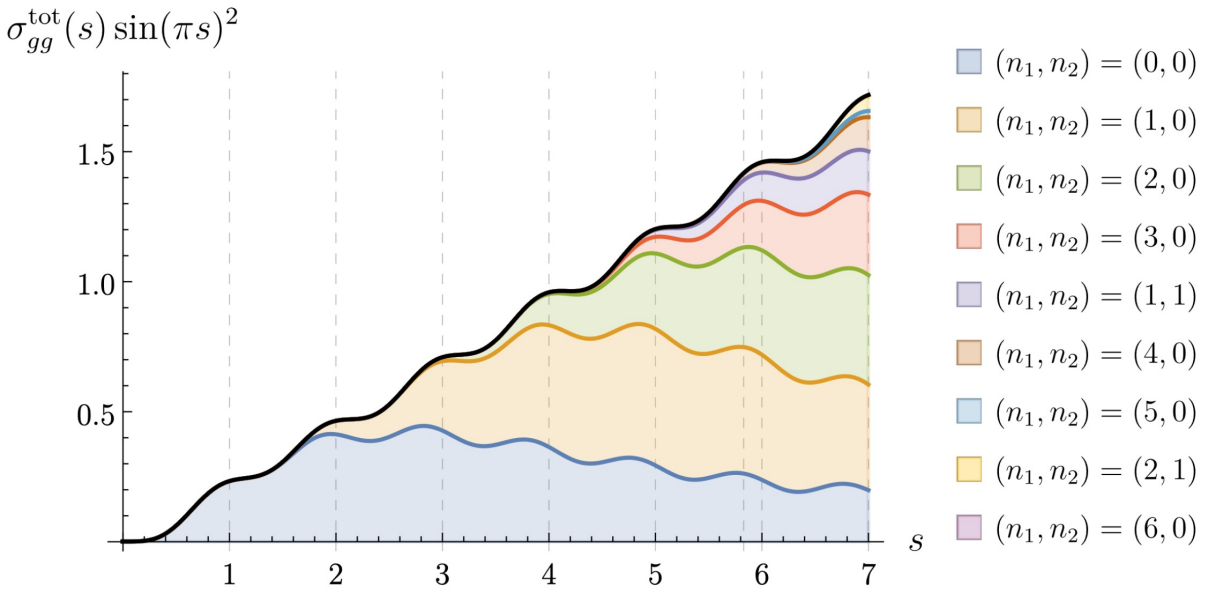
New understanding of the integration contour consistent with **causality** and **unitarity** at genus one



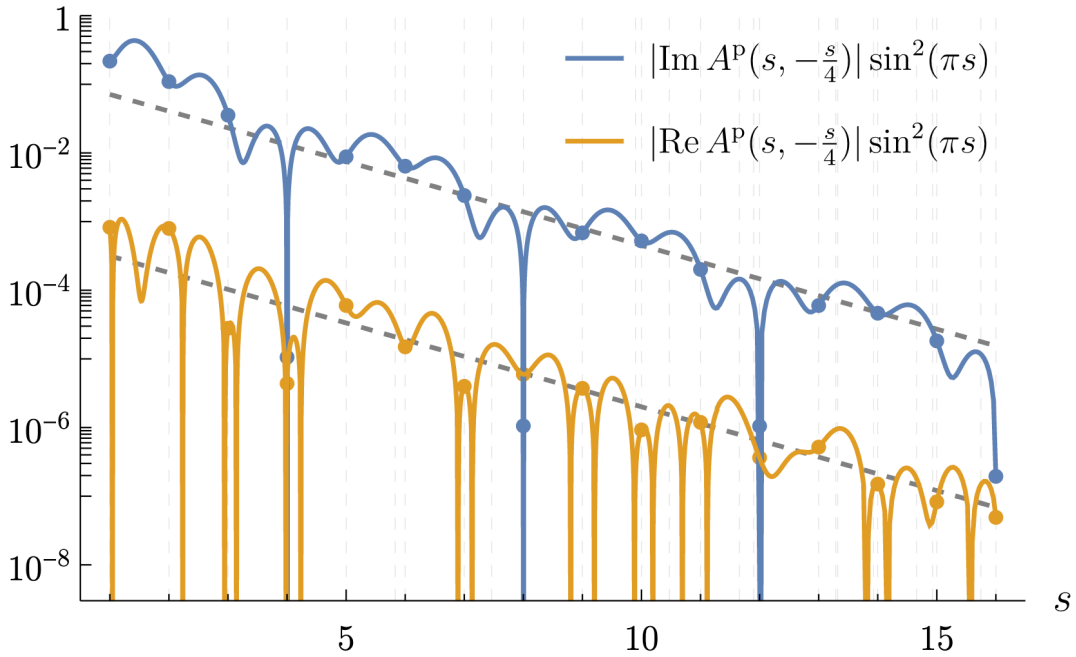
Textbook

Correct

String cross-section



Fixed-angle scattering



[Eberhardt, SM 2208.12233, 2302.12733]

Lesson: We can finally compute string amplitudes in Lorentzian kinematics

Challenge: Directly verify/disprove old conjectures

I) CONNECTIONS TO MATHEMATICS

Much-needed mathematical input

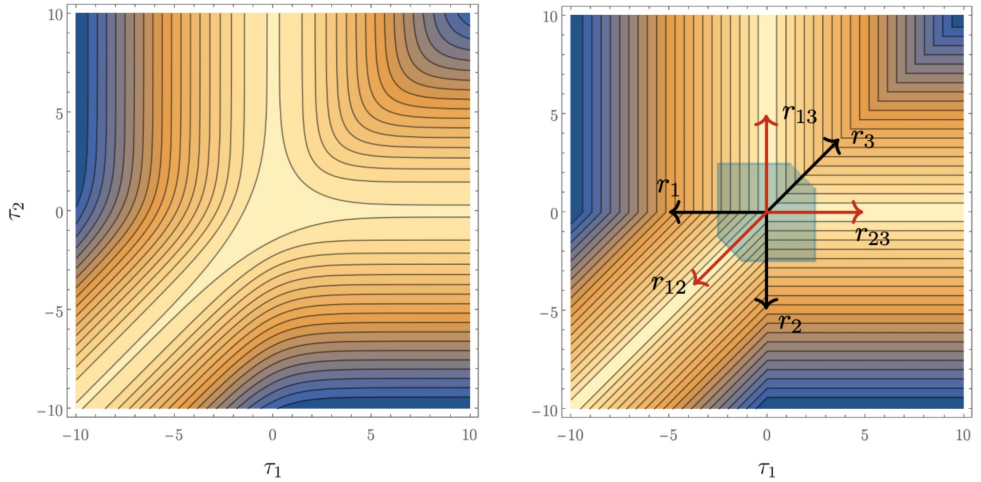
- **Expanding integrals by regions** (tropical and polyhedral geometry)
- **Integration by parts** (syzygy, twisted cohomology)
- **Convergence** (polytopes)
- **Efficient numerical evaluation** (tropical geometry)
- **Computing differential equations** (D-modules, GKZ systems)
- **Planar N=4 SYM** (projective geometry, cluster algebras)
- **Function space of amplitudes** (algebraic topology, motives, Calabi-Yau)
- …

[Lecture notes by Matsubara-Heo, SM, Telen 2306.13578]

Lesson: If you're doing heavy computations, it pays off to know cutting-edge mathematics

Tropical and polyhedral geometry

(is all about understanding limits)



Original function Tropical approximation

- **Measure for Monte-Carlo computations**
- **Soft and collinear divergences**
- **Method of regions**
- **N=4 SYM and cluster algebras**

[Borinsky 2008.12310]

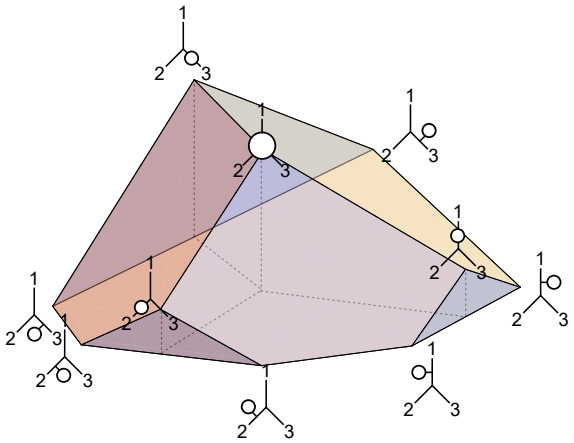
[Arkani-Hamed, Hillman, SM 2202.12296]

[pySecDec 2108.10807]

[Gardi, Herzog, Jones, Ma, Schlenk 2211.14845]

[Drummond et al., Henke et al.]

[insert noun]-hedron



- **Surfacehedron:** organizing principle for loop integrands

[Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas]

- **Non-perturbative negative geometries:** amplitude-like objects at strong coupling

[Arkani-Hamed, Henn, Trnka 2112.06956]

- **ABJM amplituhedron**

[He, Huang, Kuo 2306.00951]

[Lukowski, Stalknecht 2306.07312]

- **Related: New letters in the N=4 SYM heptagon symbol alphabet?**

[Lippstreu, Spradlin, Yelleshpur, Volovich 2305.17069]

a) AMPLITUDOLOGY IN CURVED SPACE

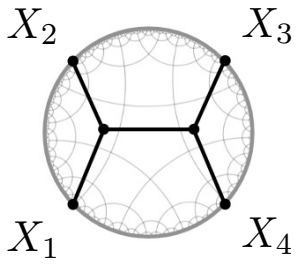
Vast topic with many exciting results

[Review talk by Stefanyszyn] [Snowmass 2203.08121]

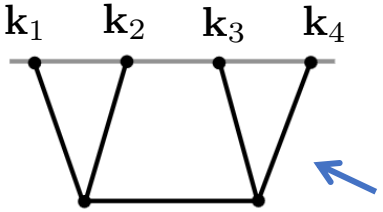
[Talks by Cohen, Lee, Nagy, Sleight]

Different representations of (A)dS amplitudes

- **Coordinate space**



- **Momentum space** (TOPT-like)



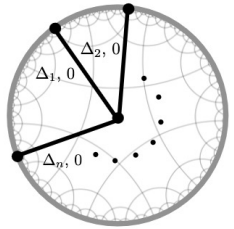
Cosmological bootstrap

- **Mellin amplitudes**

$$A(x_i) = \frac{\mathcal{N}}{(2\pi i)^{n(n-3)/2}} \int d\delta_{ij}\, M(\delta_{ij}) \prod_{i<j}^{n} \Gamma(\delta_{ij}) \left(x_{ij}^2\right)^{-\delta_{ij}}$$

- **Differential representation**

$$\sum \prod \frac{1}{D_i \cdot D_j} \left\{ \right.$$



$$\left. \right\}$$

Double copy in curved space?

- **Theoretical data:** Cosmological bootstrap gave the tree-level graviton 4-pt function

[Bonifacio, Goodhew, Joyce, Pajer, Stefanyszyn, Nagy 2212.07370]

[Talk by Stefanyszyn]

- **Double-copy-like relation** to the gluon 3, 4-pt functions

[Armstrong, Goodhew, Lipstein, Mei 2304.07206]

[Lee, Wang 2212.11282]

- **Previous progress on color-kinematics duality in AdS:**

BCJ in differential representation

[Herderschee, Roiban, Teng 2201.05067] [Li 2212.13195]

[Cheung, Parra-Martinez, Sivaramakrishnan 2201.05067]

Double-copy in Mellin space

[Zhou 2106.07651]

- **Self-dual kinematic algebras in AdS?**

[Lipstein, Nagy 2304.07141] [Talk by Nagy]

Challenge: Use your favorite formalism to predict a gravity correlator that wasn't known before

Other amplitudes-inspired highlights

- **Tree-level Virasoro-Shapiro amplitude in AdS (Mellin space)**

$$A^{(k)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}_{\text{tot}}(S,T,z)\,,$$

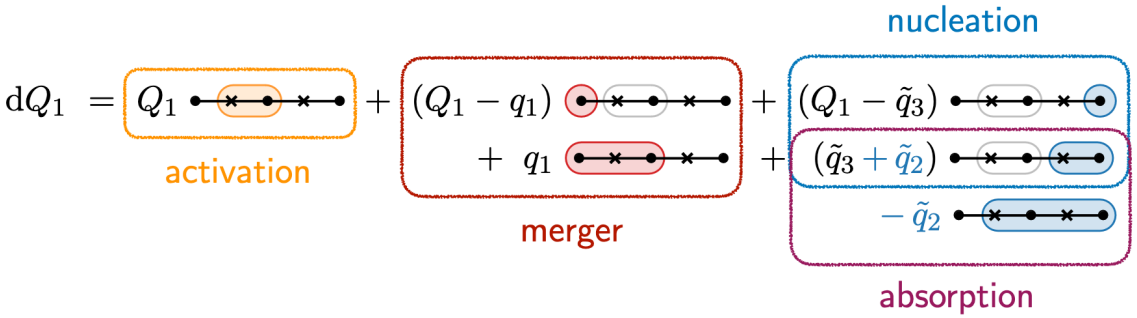
single-valued polylogs

[Alday, Hansen, Silva 2209.06223, 2305.03593]

[Alday, Hansen 2306.12786]

[Talk by Hansen]

- **Mathematics of cosmological wavefunctions (momentum space)**



[Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel]

[Talk by Lee]

A) S-MATRIX BOOTSTRAP

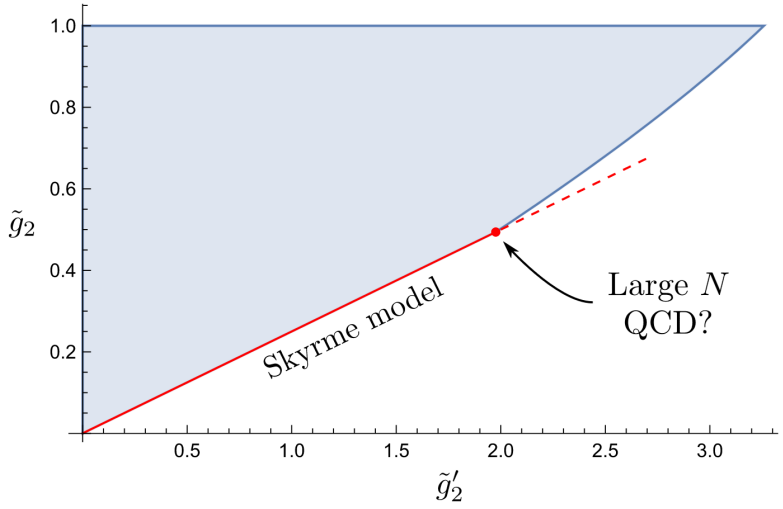
Old idea: use causality, locality, unitarity, crossing symmetry etc. to constrain the S-matrix

[Lecture notes, SM 2306.05395]

New philosophy inspired by the success of the CFT bootstrap:
Instead of try to determine the S-matrix uniquely, carve out the space of theories and look for special corners

Example: Bootstrapping large-N confining gauge theories

Four-derivative couplings



[Albert, Rastelli 2307.01246]

Not to be confused with other bootstrap approaches we've seen a lot of progress in:

- **N=4 SYM bootstrap: antipodal duality** [Dixon, Gürdoğan, Liu, McLeod, Wilhelm 2212.02410]
- **Cosmological bootstrap** [Talks by Stefanyszyn, Lee]
- **Bootstrapping individual Feynman integrals** [Morales, Spiering, Wilhelm, Yang, Zhang 2212.09762]
- **Bootstrapping Mellin amplitudes** [Alday, Gonçalves, Nocchi, Zhou 2307.06884]
- **Bootstrapping double copy** [Chen, Elvang, Herderschee 2302.04895] [Brown, Kampf, Oktem, Paranjape, Trnka 2305.05688]

Lesson: Being inventive about bootstrap axioms helps

Challenge: Apply tools from large language models to discovering new patterns

Where's string theory in the space of QFT's?

[Guerrieri, Murali, Penedones, Vieira 2212.00151]

Parametrize 4-graviton SUGRA amplitude through

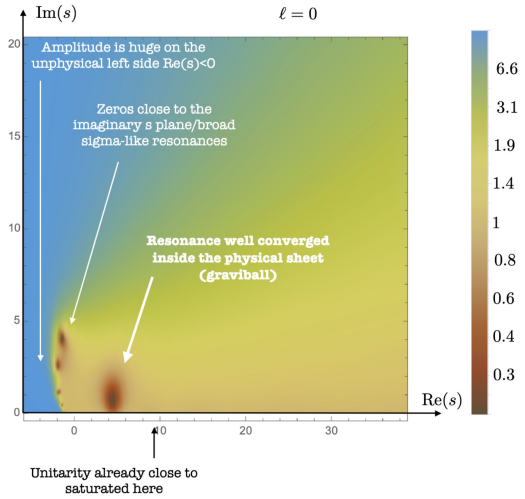
$$\frac{T(s,t,u)}{8\pi G_N} = s^4\left(\frac{1}{stu} + \alpha\,\ell_P^6 + \dots\right)$$

want to bound

bootstrapped minimal value

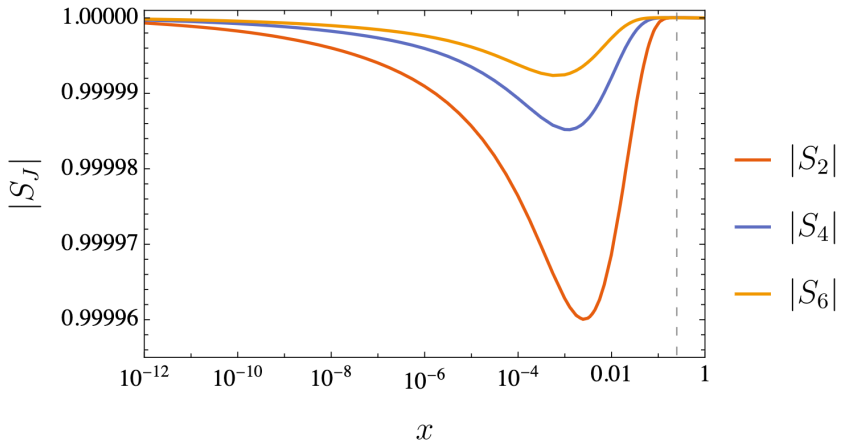
Dimension	Bootstrap	String/M–Theory
9	0.223 ± 0.002	0.241752
10	0.124 ± 0.003	0.138949
11	0.101 ± 0.005	0.102808

String/M-theory slightly above at the minimum



Lesson: There might be supergravity theories not realizable as string theories

Other exciting developments:

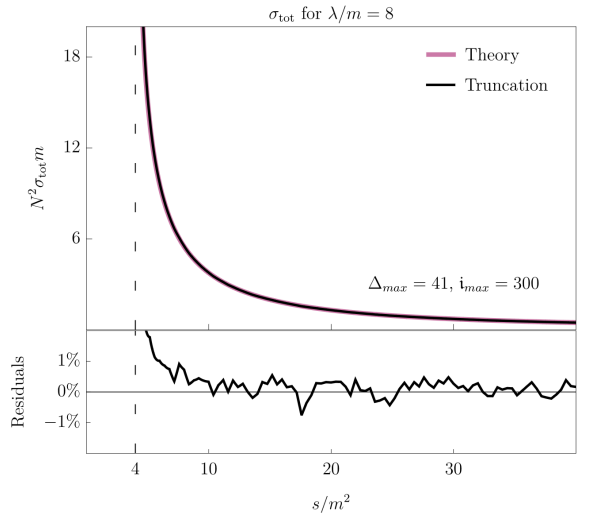


Constructing amplitudes iteratively using elastic unitarity

[Tourkine, Zhiboedov 2303.08839]

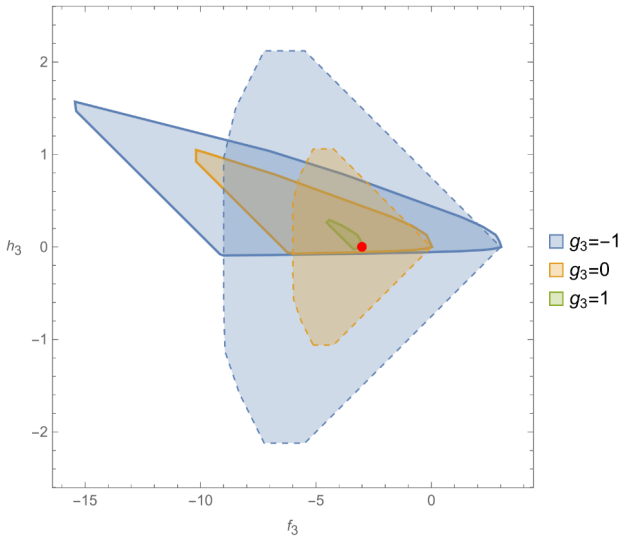
Computing amplitudes with Hamiltonian truncation

[Henning, Murayama, Riva, Thompson, Walters 2209.14306]



Challenge: Study scattering beyond the $2 \to 2$ case

Other exciting developments:



More work on bounds on EFTs from analyticity

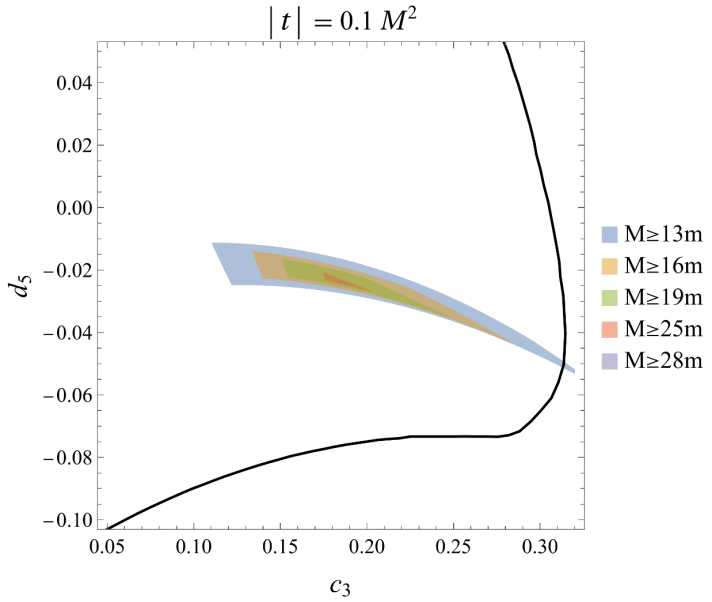
[Snowmass 2203.06805]

[Talk by Carrillo González]

Massive gravity is not positive

[Bellazzini, Isabella, Ricossa, Riva 2304.02550]

Lesson: Is massive gravity in the swampland?

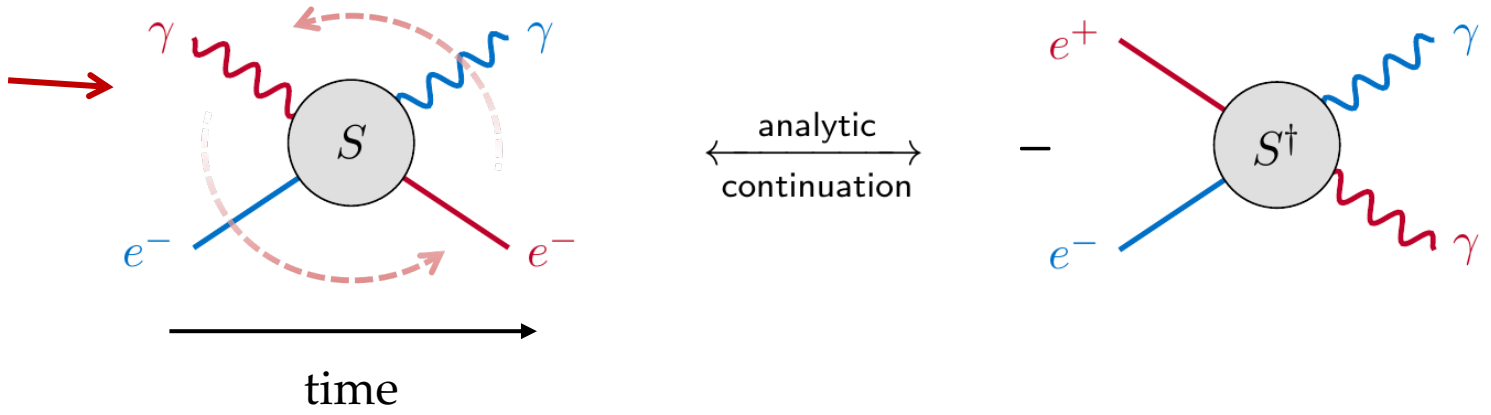


α) CROSSING SYMMETRY

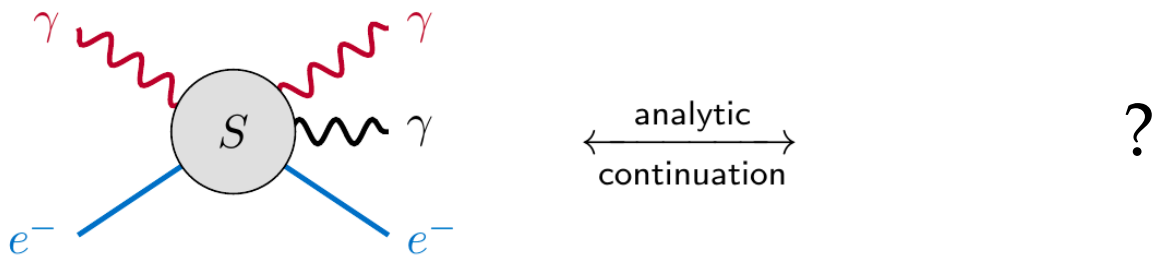
Is a particle equivalent to an anti-particle moving back in time?

Well known story for $2 \to 2$ scattering:

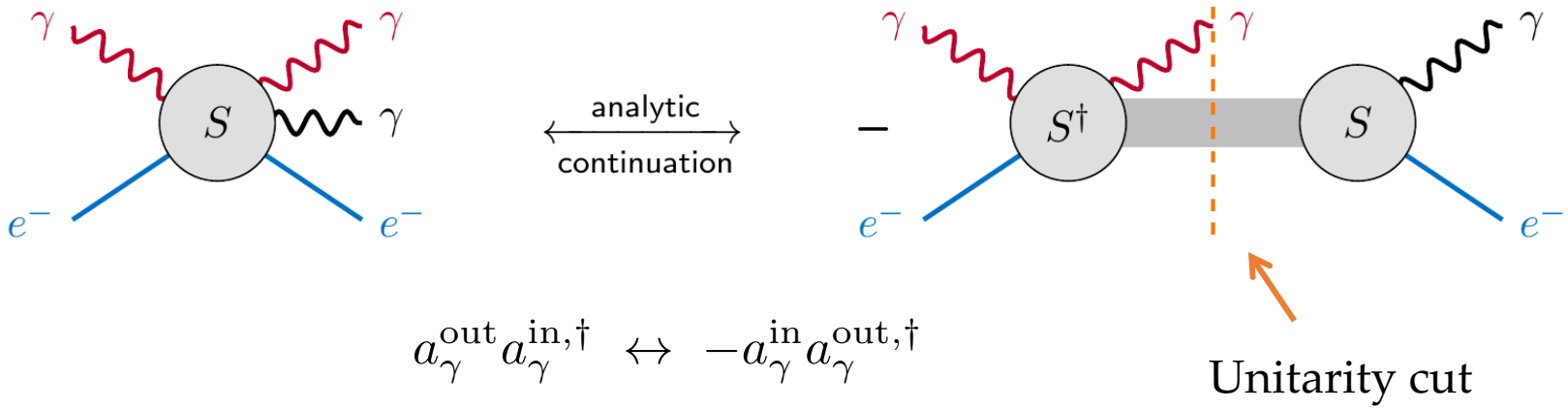
Crossing particles in red



Does the same work for $2 \to 3$?



The result of crossing is **not** yet another scattering amplitude:



$$a_\gamma^{\text{out}} a_\gamma^{\text{in},\dagger} \;\leftrightarrow\; -a_\gamma^{\text{in}} a_\gamma^{\text{out},\dagger}$$

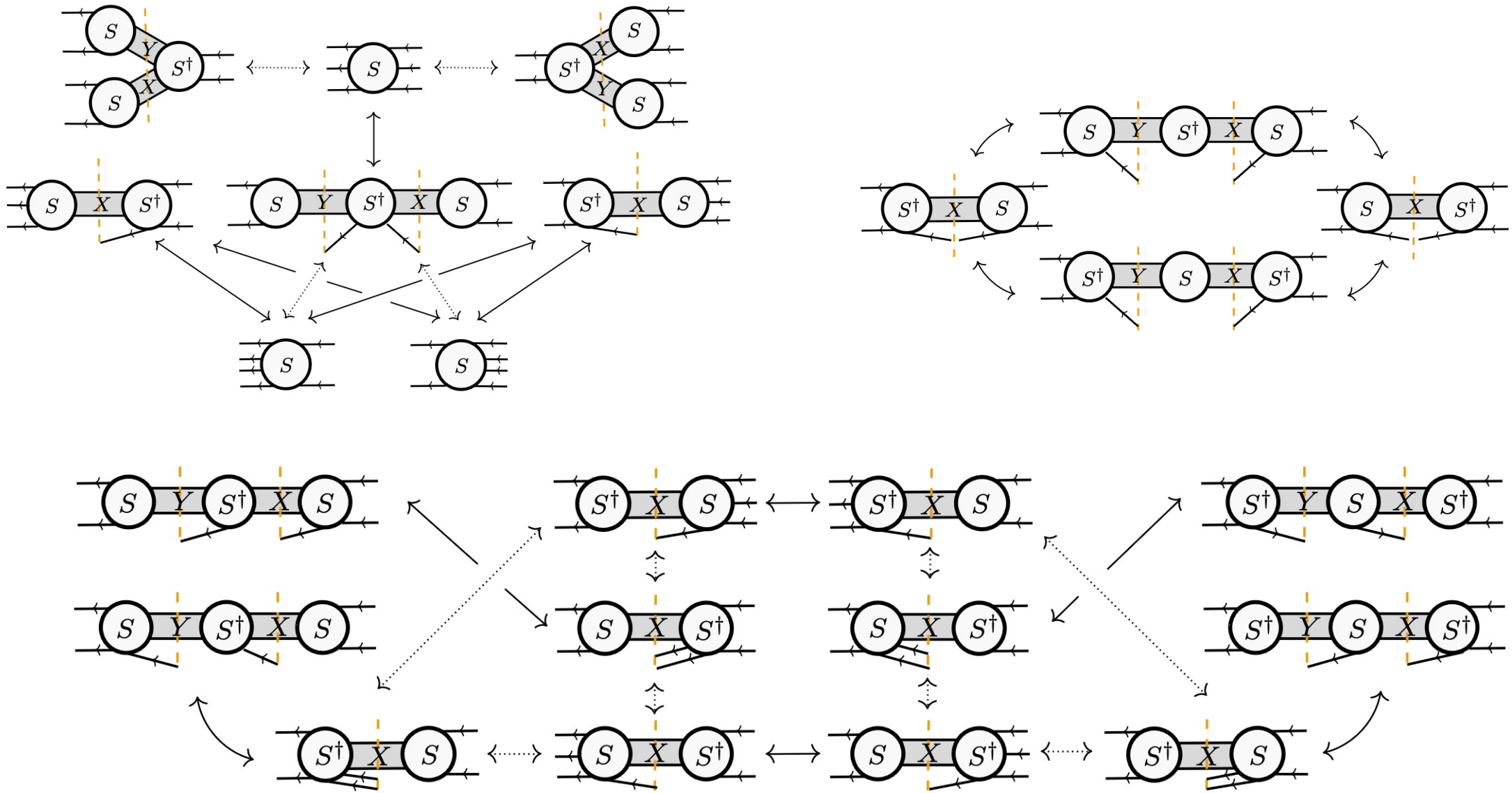
Unitarity cut

Instead, an expectation value of a photon in a Compton scattering background

Lesson: The S-matrix theory is not just about scattering amplitudes!

Challenge: Apply crossing symmetry to computing the RHS

[Caron-Huot, Giroux, Hannesdottir, SM]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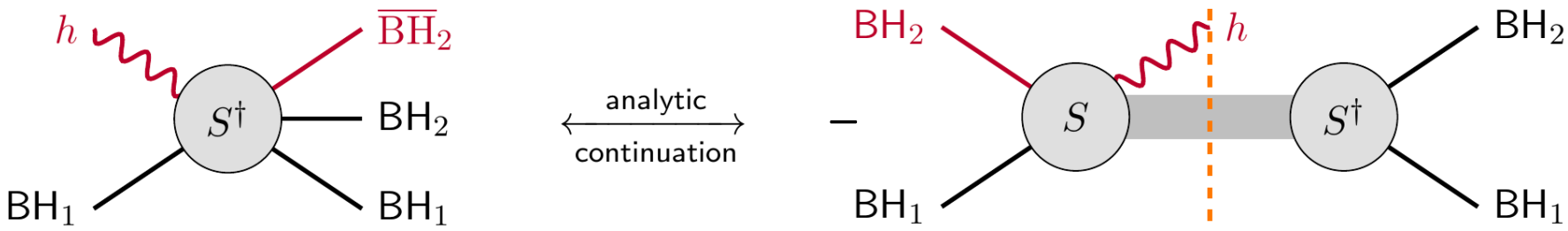
Gravitational radiation in the background of two black holes:

$$a^{\text{in}}_{\overline{\text{BH}_2}}\, a^{\text{out},\dagger}_{h} \;\leftrightarrow\; -a^{\text{out}}_{h}\, a^{\text{in},\dagger}_{\text{BH}_2}$$



[Kosower, Maybee, O'Connell 1811.10950]

Compute the regular time-ordered amplitude

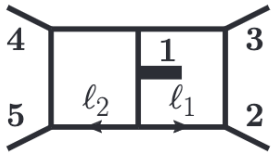
$${}_{\text{in}}\langle \text{BH}_1\, \text{BH}_2\, \overline{\text{BH}_2}|h\, \text{BH}_1\rangle_{\text{out}}$$

Analytic continuation results in

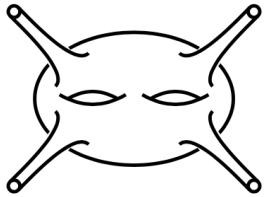
$$-\,{}_{\text{in}}\langle \text{BH}_1\, \text{BH}_2|a^{\text{out}}_{h}|\text{BH}_1\, \text{BH}_2\rangle_{\text{in}}$$
$$= -\,{}_{\text{in}}\langle \text{BH}_1\, \text{BH}_2|S^\dagger a^{\text{in}}_{h} S|\text{BH}_1\, \text{BH}_2\rangle_{\text{in}}$$

Challenge: Need to take analytic features such as **anomalous thresholds** more seriously

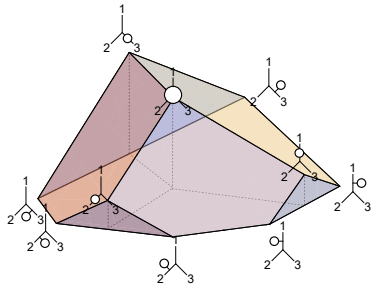
1) Precision frontier



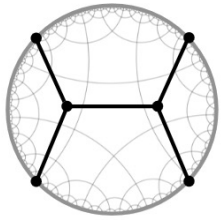
i) String amplitudes



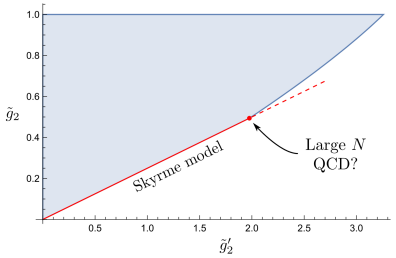
I) Polytopes and tropical geometry



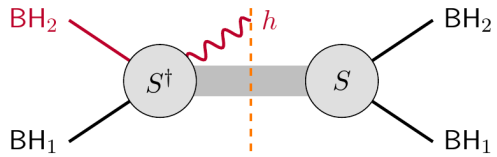
a) Amplitudology in curved space



A) S-matrix bootstrap



α) Crossing symmetry



Thank you!