Bootstrapping the AdS Virasoro-Shapiro amplitude

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From Amplitudes to Gravitational Waves, Nordita
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Based on:

2204.07542, 2209.06223, 2303.08834, 2305.03593 with Luis F. Alday, João Silva

2306.12786 with Luis F. Alday

Strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$

type IIb string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ = $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group

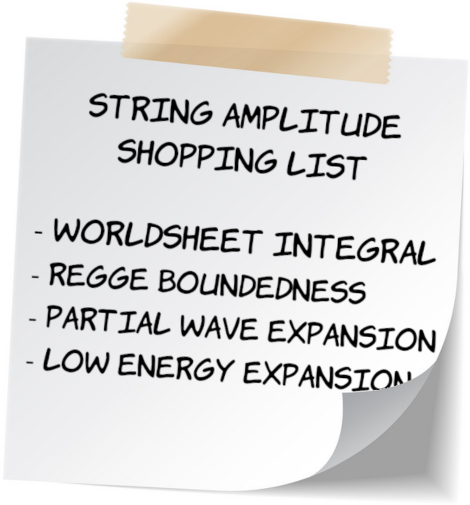
What is the worldsheet theory?

What is the 4 graviton tree level string amplitude?

$$A_4(S,T) = \sum_{k=0}^{\infty} \left(\frac{\alpha'}{R_{\mathrm{AdS}}^2} \right)^k A_4^{(k)}(S,T)$$

Can we bootstrap it?

Part 0
Flat Space Review



I will review these first for the Virasoro-Shapiro amplitude (4 gravitons in the type IIb superstring):

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\,\Gamma(T+1)\,\Gamma(U+1)}$$

$$S = -\frac{\alpha'}{4}(p_1+p_2)^2\,, \qquad T = -\frac{\alpha'}{4}(p_1+p_3)^2$$

$$S+T+U=0$$

World-sheet integral (flat space)

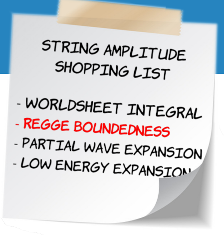
The amplitude is the integral over the world-sheet (sphere)

$$A^{(0)}(S,T) = \int d^2z\ |z|^{-2S-2}|1-z|^{-2T-2} G^{(0)}_{\text{tot}}(S,T,z)$$

$$G^{(0)}_{\text{tot}}(S,T,z) = \frac{1}{3}\left(\frac{1}{U^2} + \frac{|z|^2}{S^2} + \frac{|1-z|^2}{T^2}\right)$$

The integrand is a single-valued function of z!

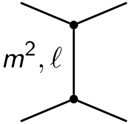
Regge boundedness (flat space)



String amplitudes have soft UV (Regge) bahaviour

$$\lim_{|S|\to\infty} A^{(0)}(S,T) \sim S^{\alpha' T+\alpha_0}$$

and higher spin resonances

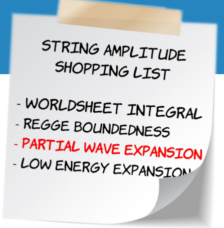


$$= \frac{P_\ell(S)}{T-m^2} \qquad\qquad P_\ell(S) = S^\ell + O(S^{\ell-1})$$

Regge bahaviour places strong constraints on the coefficients $a_{\delta,\ell}$ in

$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T-\delta}$$

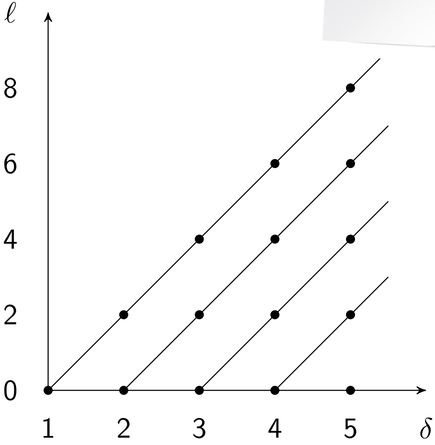
The spectrum (flat space)



The exchanged massive string spectrum is extracted via the partial wave expansion

$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T-\delta}$$

It forms linear Regge trajectories.



Low energy expansion (flat space)

Low energy effective action (supergravity + derivative interactions)
$\rightarrow$ Low energy expansion:

$$A^{(0)}(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \left(\tfrac{1}{2}(S^2+T^2+U^2)\right)^a (STU)^b \alpha^{(0)}_{a,b}$$

Wilson coefficients $\alpha^{(0)}_{a,b}$ are in the ring of single-valued multiple zeta values [Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

Example: $\alpha^{(0)}_{a,0} = \zeta(3+2a)\,, \qquad \alpha^{(0)}_{a,1} = \sum_{\substack{i_1,i_2=0 \\ i_1+i_2=a}}^{a} \zeta(3+2i_1)\zeta(3+2i_2)$

Plan for AdS

How do we determine $A^{(k)}(S,T)$? $\quad A(S,T) = \sum_{k=0}^{\infty} \left(\frac{\alpha'}{R_{\text{AdS}}^2}\right)^k A^{(k)}(S,T)$

1. Ansatz (world-sheet):

$$A^{(k)}(S,T) = \int d^2z \, |z|^{-2S-2}|1-z|^{-2T-2} G_{\text{tot}}^{(k)}(S,T,z)$$

Part 1: Strings in AdS $\rightsquigarrow$ functions in $G_{\text{tot}}^{(k)}(S,T,z)$

2. Fix the parameters of the ansatz (target space):

Part 2: The dual CFT

3. Compare with integrability and localisation results.

Part 1
Strings in AdS

Toy model for strings in AdS

Polyakov action:

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$

$$= S_{\text{flat}} + \frac{1}{R^2_{\text{AdS}}} \underbrace{\lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} V^{\mu\nu}_{\text{graviton}}(q)}_{\equiv \tilde{V}} + \dots$$

AdS metric expanded around flat space:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R^2_{\text{AdS}}} + \cdots$$

$$h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} e^{iq\cdot X}$$

Amplitude:

$$A_4(p_i) \sim \int \mathcal{D}X \mathcal{D}g \, e^{-S_P} V^4_{\text{graviton}} = \int \mathcal{D}X \mathcal{D}g \, e^{-S_{\text{flat}}} \left(1 - \frac{\tilde{V}}{R^2_{\text{AdS}}} + \frac{1}{2}\frac{\tilde{V}^2}{R^4_{\text{AdS}}} + \dots\right) V^4_{\text{graviton}}$$

$$\Rightarrow \quad A_4^{(k)}(p_i) \sim \lim_{q_i\to 0} \left(\frac{\partial}{\partial q_i}\right)^{2k} A^{(0)}_{4+k}(p_i, q_i) + \dots$$

Soft gravitons in flat space

$$A_4^{(k)}(p_i) \sim \lim_{\epsilon\to 0}\left(\frac{\partial}{\epsilon\,\partial q_i}\right)^{2k} A_{4+k}^{(0)}(p_i, \epsilon q_i) + \dots$$

Soft graviton theorem:

$$A_{n+1}(p_1,\dots,p_n,\epsilon q) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon}\frac{\varepsilon_{\mu\nu}p_i^\mu p_i^\nu}{p_i\cdot q} + \frac{\varepsilon\cdot p_i \varepsilon_\mu q_\nu J_i^{\mu\nu}}{p_i\cdot q} + O(\epsilon)\right) A_n(p_1,\dots,p_n)$$

Flat space amplitude with k soft gravitons:

$$A_{4+k}^{(0)}(p_i,\epsilon q_i) \sim \frac{1}{\epsilon^k}A_4^{(0)}(p_i) + \frac{1}{\epsilon^{k-1}}\text{"}\partial_{p_i}\text{"}A_4^{(0)}(p_i) + \dots$$

$$\sim \int d^2z |z|^{-2S-2}|1-z|^{-2T-2}\left(\frac{1}{\epsilon^k} + \frac{1}{\epsilon^{k-1}}\left(\#\log|z|^2 + \#\log|1-z|^2\right) + \dots + \epsilon^{2k}\mathcal{L}_{|w|=3k}(z)\right)$$

$\Rightarrow \quad G_{\text{tot}}^{(k)}(S,T,z) \; \sim$ single-valued multiple polylogs of weight $\leq 3k$

Single-valued multiple polylogarithms

MPLs:

$$L_{a_1 \ldots a_{|w|}}(z) = \int_0^z \frac{dt}{t - a_1} L_{a_2 \ldots a_{|w|}}(t)$$

$$L(z) = 1, \quad a_i \in \{0, 1\}$$

Examples :

$$L_{0^p}(z) = \frac{1}{p!} \log^p(z)$$

$$L_{1^p}(z) = \frac{1}{p!} \log^p(1 - z)$$

$$L_{0^{p-1}1}(z) = -\mathrm{Li}_p(z)$$

$\downarrow z = 1$

MZVs: $\zeta(n_1, n_2, \ldots)$

SVMPLs: [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{\substack{w_1, w_2 \\ |w_1| + |w_2| = |w|}} c_{w, w_1, w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

$$\mathcal{L}_{0^p}(z) = \frac{1}{p!} \log^p |z|^2$$

$$\mathcal{L}_{1^p}(z) = \frac{1}{p!} \log^p |1 - z|^2$$

$$\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1 - \bar{z}) \log |z|^2$$

$\downarrow z = 1$

SVMZVs: $\zeta^{\mathrm{sv}}(n_1, n_2, \ldots)$ [Brown;2013]

Part 2
The Dual CFT

The dual CFT correlator

4 graviton amplitude in $AdS_5 \times S^5$ $\leftrightarrow$ $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$ in $\mathcal{N} = 4$ SYM theory

$$g_s \ll \frac{\alpha'}{R^2_{\text{AdS}}} \ll 1 \quad \Leftrightarrow \quad \frac{1}{N} \ll \frac{1}{\sqrt{\lambda}} \ll 1$$

$\mathcal{O}_2$ = superconformal primary of stress-tensor multiplet

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4) \rangle$$

$\downarrow$ superconformal Ward identity

$$H(U,V) \qquad U = \frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_3)^2(x_2-x_4)^2}, \; V = \frac{(x_1-x_4)^2(x_2-x_3)^2}{(x_1-x_3)^2(x_2-x_4)^2}$$

$\downarrow$ Mellin transform

$$M(s,t)$$

$\downarrow$ Borel transform (flat space limit [Penedones;2010])

$$A(S,T) = \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{\lambda}}\right)^k A^{(k)}(S,T)$$

Integral transforms

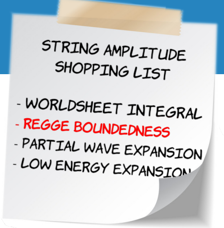
Mellin transform

$$H(U,V) = \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}+\frac{2}{3}} V^{\frac{t}{2}-\frac{4}{3}} \Gamma\left(\frac{4}{3}-\frac{s}{2}\right)^2 \Gamma\left(\frac{4}{3}-\frac{t}{2}\right)^2 \Gamma\left(\frac{4}{3}-\frac{u}{2}\right)^2 M(s,t)$$

Borel transform

$$A(S,T) = 2\lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

Dispersion relation



$M(s,t)$ has only OPE poles:

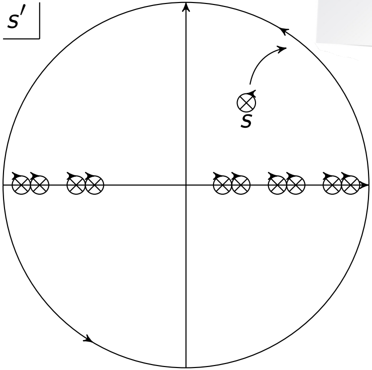
$$\text{poles} \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s' - (\Delta - \ell + 2m)}$$

[Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Regge bounded due to bound on chaos:

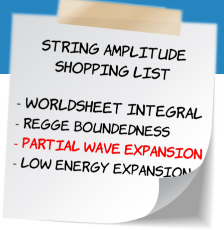
$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}, \; \mathrm{Re}(t) < 2$$

[Maldacena,Shenker,Stanford;2015]

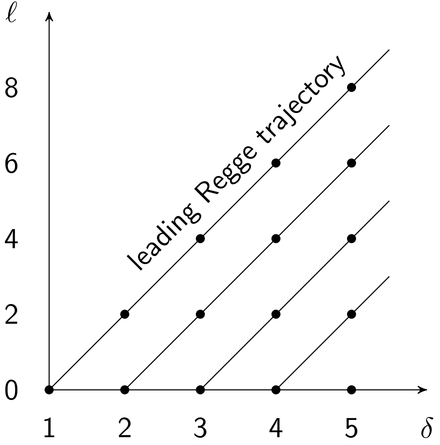


$$M(s,t) = \oint_s \frac{ds'}{2\pi i} \frac{M(s',t)}{(s'-s)} = -\sum_{\text{operators}} \left(\frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s - (\Delta - \ell + 2m)} + \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{u - (\Delta - \ell + 2m)} \right)$$

Spectrum of exchanged operators



Exchanged operators: short single-trace operators of $\mathcal{N} = 4$ SYM theory



known from flat space

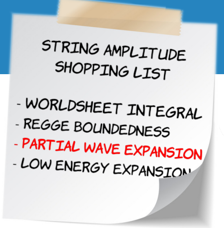
$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data
$\alpha^{(0)}_{a,b}$	+	$\lambda^{-\frac{1}{2}}\alpha^{(1)}_{a,b}$	+	$\lambda^{-1}\alpha^{(2)}_{a,b}$

$$\Delta_{\delta,\ell} = \lambda^{\frac{1}{4}}\Delta^{(0)}_{\delta,\ell} + \lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell} + \lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$$

$$C^2_{\delta,\ell} = C^{2(0)}_{\delta,\ell} + \lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell} + \lambda^{-1}C^{2(2)}_{\delta,\ell}$$

$\alpha^{(1)}_{0,0}$, $\alpha^{(1)}_{1,0}$, $\alpha^{(2)}_{0,0}$ known from localisation

$\Delta^{(1)}_{\delta,\ell}$, $\Delta^{(2)}_{\delta,\ell}$ on leading trajectory known from integrability



The amplitude encodes OPE data of multiple degenerate superprimaries.

We determined the degeneracies starting from type IIb strings in flat 10d:

$$SO(9) \to SO(4) \times SO(5) \overset{KK}{\to} SO(4) \times SO(6)$$

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

Number of superconformal long multiplets with superprimary $\mathcal{O}_{\delta,\ell}$

- $SO(6)$ singlet
- $\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}} + O(\lambda^0)$

Example: $\mathcal{O}_{1,0} = \text{Konishi} \sim \text{Tr}(\phi^I \phi_I)$

The counting was confirmed for $\delta \leq 3$ with quantum spectral curve. [Gromov,Hegedus,Julius,Sokolova;2023]

Dispersion relation $\to$ Residues

Dispersion relation for $M(s,t) \rightsquigarrow A^{(k)}(S,T)$ expanded around $S=\delta=1,2,\dots$:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta,C^{2(0)}_{\delta,\ell})}{(S-\delta)^{3k+1}} + \dots + \frac{R^{(k)}_{1}(T,\delta,C^{2(0)}_{\delta,\ell},\dots,\Delta^{(k)}_{\delta,\ell},C^{2(k)}_{\delta,\ell})}{S-\delta} + \text{reg.}$$

(OPE data)$^{(k-1)}$ fixes most residues of $A^{(k)}(S,T)$!

Next steps (order by order):

- Write world-sheet ansatz for $A^{(k)}(S,T)$.
- Compute its residues and match with the above to fix ansatz.

World-sheet → Residues

Q: How to compute the singularities at $S = \delta = 1, 2, \ldots$ of

$$\int d^2z \, |z|^{-2S-2} |1-z|^{-2T-2} \mathcal{L}_w(z)$$

A: Integrating near $z \sim 0$ using polar coordinates $z = \rho e^{i\phi}$ and

$$\int_0^{\rho_0} d\rho \, \rho^{-2S+2\delta-1} \frac{1}{p!} \log^p(\rho^2) = -\frac{1}{2} \frac{1}{(S-\delta)^{p+1}} + \text{reg.}$$

Leading pole order $3k+1$ (predicted by dispersion relation)
$=$ max order from weight $3k$ SVMPLs (predicted by string model):

$$\mathcal{L}_{0^{3k}}(z) = \frac{1}{(3k)!} \log^{3k} |z|^2 \quad \rightarrow \quad \frac{1}{(S-\delta)^{3k+1}}$$

World-sheet correlator (ansatz)

Ansatz:

$$A^{(k)}(S,T) = B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}(S,T,z)$$

Assumed properties of $G^{(k)}(S,T,z)$:

- uniform transcendentality $3k$ (SVMPLs(z), SVMZVs)
- rational function in S, T with homogeneity $2k-2$
- denominator $= U^n$, $n \leq 2$
- crossing symmetry: $G^{(k)}(S,T,z) = G^{(k)}(T,S,1-z)$

Recall (flat space):

$$G^{(0)}(S,T,z) = \frac{1}{3U^2}$$

World-sheet correlator (solution)

Symmetrised single-valued multiple polylogs:

$$\mathcal{L}^{\pm}_{w}(z) = \mathcal{L}_{w}(z) \pm \mathcal{L}_{w}(1-z) + \mathcal{L}_{w}(\bar{z}) \pm \mathcal{L}_{w}(1-\bar{z})$$

$$k = 1: \quad \text{weight 3 basis} = 4 \text{ symmetric} + 3 \text{ antisymmetric functions}$$

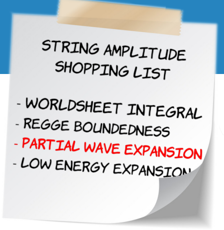
Solution:

$$\begin{aligned} G^{(1)}(S,T,z) = & -\frac{1}{6}\mathcal{L}^{+}_{000}(z) + 0\mathcal{L}^{+}_{001}(z) - \frac{1}{4}\mathcal{L}^{+}_{010}(z) + 2\zeta(3) \\ & + \frac{S-T}{S+T}\left(-\frac{1}{6}\mathcal{L}^{-}_{000}(z) + \frac{1}{3}\mathcal{L}^{-}_{001}(z) + \frac{1}{6}\mathcal{L}^{-}_{010}(z)\right) \end{aligned}$$

$$k = 2: \quad \text{weight 6 basis} = 25 \text{ symmetric} + 20 \text{ antisymmetric functions}$$

We need to input the dimension of 1 operator ($\Delta^{(2)}_{1,0}$ = Konishi) to fix $A^{(2)}(S,T)$ completely.

OPE data



We compute $\forall \delta, \ell \qquad \# \in \mathbb{Q}$

$$k=0: \qquad \langle C^{2(0)} \rangle_{\delta,\ell} = \#$$

$$k=1: \qquad \sqrt{\delta}\langle C^{2(0)}\Delta^{(1)} \rangle_{\delta,\ell} = \#\,, \quad \langle C^{2(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$k=2: \qquad \langle C^{2(0)}(\Delta^{(1)})^2 \rangle_{\delta,\ell} = \#$$

$$\sqrt{\delta}\langle C^{2(0)}\Delta^{(2)} + C^{2(1)}\Delta^{(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$\langle C^{2(2)} \rangle_{\delta,\ell} = \#\zeta(3)^2 + \#\zeta(5) + \#\zeta(3) + \#$$

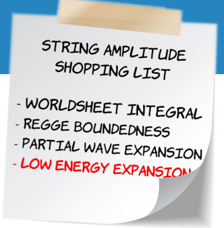
Leading Regge trajectory:

$$\Delta\left(\tfrac{\ell}{2}+1,\ell\right) = 2\sqrt{\tfrac{\ell}{2}+1}\,\lambda^{\frac{1}{4}} - 2 + \frac{3\ell^2+10\ell+16}{4\sqrt{2(\ell+2)}}\lambda^{-\frac{1}{4}}$$

$$- \frac{21\ell^4+144\ell^3+292\ell^2+80\ell-128+96(\ell+2)^3\zeta(3)}{32(2(\ell+2))^{\frac{3}{2}}}\lambda^{-\frac{3}{4}} + O(\lambda^{-\frac{5}{4}})\,,$$

Agrees with integrability result!

[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]



The low energy expansion ($S \sim T \sim 0$) can be computed following [Vanhove,Zerbini;2018]

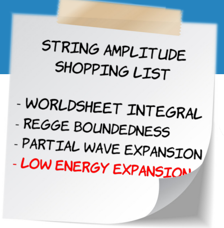
$$\int d^2z |z|^{-2S-2} |1-z|^{-2T-2} \mathcal{L}_w(z)$$

$$= \text{poles} + \sum_{p,q=0}^{\infty} (-S)^p (-T)^q \int \frac{d^2z}{|z|^2|1-z|^2} \underbrace{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)\mathcal{L}_w(z)}_{= \sum_{W \in 0^p ⧢ 1^q ⧢ w} \mathcal{L}_W(z)}$$

$$= \text{poles} + \sum_{p,q=0}^{\infty} (-S)^p (-T)^q \sum_{W \in 0^p ⧢ 1^q ⧢ w} \underbrace{\mathcal{L}_{0W}(1) - \mathcal{L}_{1W}(1)}$$

Single-valued multiple zeta values of weight $1 + p + q + |w|$

Wilson coefficients



$$A^{(k)}(S,T) = \text{SUGRA}^{(k)} + 2\sum_{a,b=0}^{\infty}\left(\tfrac{1}{2}(S^2+T^2+U^2)\right)^a (STU)^b \alpha^{(k)}_{a,b}$$

We compute $\forall a, b$ $\qquad \# \in \mathbb{Q}$

$$\alpha^{(0)}_{a,b} = \sum_{k_i \text{ odd}} \#\zeta(k_1)\ldots\zeta(k_n)$$

$$\alpha^{(1)}_{a,b} = \sum_{k_i \text{ odd}} \#\zeta^{\text{sv}}(k_1,k_2,k_3)\zeta(k_4)\ldots\zeta(k_n) + \ldots$$

$$\alpha^{(2)}_{a,b} = \sum_{k_i \text{ odd}} \#\zeta^{\text{sv}}(k_1,k_2,k_3,k_4,k_5)\zeta(k_6)\ldots\zeta(k_n) + \ldots$$

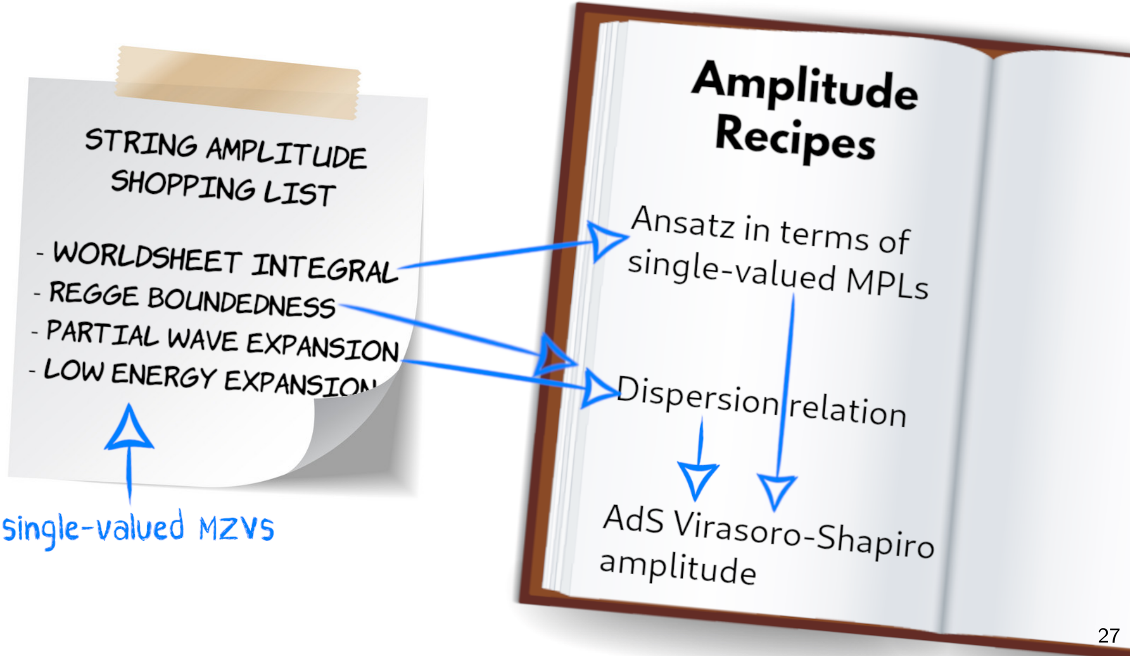
In particular:

$$\alpha^{(1)}_{0,0} = 0\,, \quad \alpha^{(1)}_{1,0} = -\frac{22}{3}\zeta(3)^2\,, \quad \alpha^{(2)}_{0,0} = \frac{49}{4}\zeta(5)\,, \quad \alpha^{(2)}_{1,0} = \frac{4091}{16}\zeta(7)$$

Agrees with localisation result!

[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

Summary


STRING AMPLITUDE
SHOPPING LIST
- WORLDSHEET INTEGRAL
- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- LOW ENERGY EXPANSION
single-valued MZVs
Amplitude
Recipes
Ansatz in terms of
single-valued MPLs
Dispersion relation
AdS Virasoro-Shapiro
amplitude

Future directions

- Open strings / AdS Veneziano amplitude
  - Generalizations of KLT relations / single-valued map?
  - Problem: no strong coupling OPE data known for consistency checks. Integrability?
- Compute $A^{(k)}(S, T)$ directly from string theory?
  - Ramond-Ramond background flux...
  - Pure spinors?

Thank you!

Questions?