A measure for chaos in string scattering

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Talk @ “From Amplitudes to Gravitational Waves”
24-28 July 2023 NORDITA

INFN Sezione di Roma Tor Vergata

Università di Roma Tor Vergata

Based on work in collaboration with
Maurizio Firrotta, Jacob Sonnenschein and Dorin Weissman: 2207.13112 and 2303.17233
see also
Rosenhaus 2003.07381, Gross, Rosenhaus 2103.15301, Rosenhaus 2112.10269
Firrotta, Rosenhaus 2207.01641, Firrotta 2301.04069
For time-honoured DDF and its revival
Del Giudice, Di Vecchia, Fubini, Annals Phys. 70, 378 (1972).
Hindmarsh, Skliros 1006.2559, 1107.0730
MB, Firrotta 1902.07016, Addazi, MB, Firrotta and Marcianò 2008.02206, Aldi, Firrotta 1912.06177, Aldi, MB, Firrotta 2010.04082, 2101.07054
Future developments
MB, Di Russo 'HmEST' *w.i.p.*, Di Vecchia, Firrotta *w.i.p.*

Plan

- Motivation
- Probing and measuring chaos with particles, waves, strings and ‘black-holes’
- Digression: RMT (Random Matrix Theory) and β-ensemble
- HES (Highly Excited String) and DDF (Del Giudice, Di Vecchia, Fubini) operators
- Chaos in the decay of HES into two light strings
    - Amplitude
    - Statistical analysis
- Chaos in 4-point amplitudes with HES
    - HES dressing factor wrt Veneziano amplitude ...
    - high energy: fixed-angle vs Regge regime
    - chaotic behavior, transition to 'regular' behavior
- Conclusions and outlook

... No chaos in Veneziano, neither in Remmen ...

Motivation

Chaotic behaviour is common in a wide range of processes, including humans
Energy spectrum of quantum Hamiltonian systems $\{E_n\}$ (*e.g.* RMT) or better spacings

$$\delta_n = E_{n+1} - E_n$$

or even better, ratios

$$r_n \equiv \frac{E_{n+1} - E_n}{E_n - E_{n-1}} = \frac{\delta_{n+1}}{\delta_n} \quad , \quad \tilde{r}_n = min\{r_n, \frac{1}{r_n}\}$$

For our purposes: analogy with dependence of scattering amplitudes $\mathcal{A}(\alpha)$ or better log derivatives

$$F(\alpha) \equiv \frac{d}{d\alpha} \log \mathcal{A}(\alpha)$$

on some kinematical (angular) variable α

$$\{z_n\} = \{\alpha : F(\alpha) = 0\}$$

spacings

$$\delta_n = z_n - z_{n+1}$$

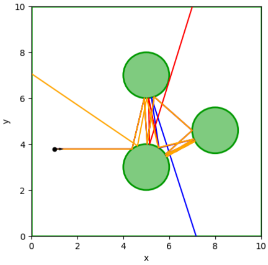
ratio's

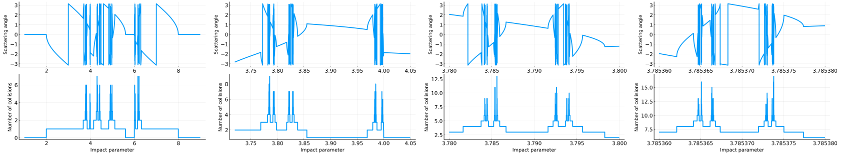
$$r_n \equiv \frac{z_{n+1} - z_n}{z_n - z_{n-1}} = \frac{\delta_{n+1}}{\delta_n} \quad , \quad \ldots \quad \tilde{r}_n$$

Classical example: pinball scattering

From classical to quantum chaos

Pinball scattering: high-sensitivity on initial condition, classical ‘deterministic’ chaos





Quantum Chaos: quantum version of Sinai billiard (square with disk removed) ... ergodic [Bohigas, Giannoni, Schmit, ...], Hadamard/Artin billiard ... deterministic chaos (Riemann surfaces with $g \geq 2$ or with cusps)

Chaos in the S-matrix ... ‘leaky torus’ [Gutzwiller] ... ζ function!

Riemann hypothesis: all (infinite number) non-trivial zero's on critical line $z_n = 1/2 + iy_n$, normalized spacings

$$\bar{\delta}_n = \frac{y_n - y_{n-1}}{2\pi} \log \frac{y_n}{2\pi}$$

Probability Distribution Function (PDF): Wigner surmise

$$p_W(\bar{\delta}) = \frac{32}{\pi^2} \bar{\delta}^2 e^{-\frac{4}{\pi}\bar{\delta}^2}$$

GUE distribution (2×2, $\beta = 2$) of ratio's $r_n = \bar{\delta}_{n+1}/\bar{\delta}_n$

$$f_{GUE}(r) = \frac{81\sqrt{3}}{4\pi} \frac{(r + r^2)^2}{(1 + r + r^2)^4}$$

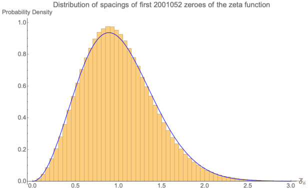
with $r_{peak} = \frac{1}{\sqrt{2}}$ and $\langle r \rangle = 4/\pi \approx 1.273$, well approximated by Log-Normal

$$f_{\rm LN}(r) = \frac{1}{\sqrt{2\pi\sigma^2} r} \exp\left(-\frac{[\log(r) - \mu]^2}{2\sigma^2}\right)$$

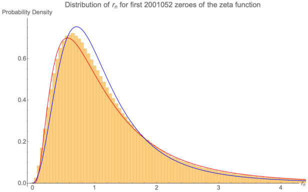
with $r_{peak} = \exp(\mu - \sigma^2)$ and $\langle r \rangle = \exp(\mu + \frac{1}{2}\sigma^2)$, for $\mu = 0$ and $2\sigma^2 = \pi^2 - 8$

Distribution of the first $N = 2,001,052$ zero's of the ζ function

Table available online: [A. Odlyzko, "Tables of zeros of the Riemann zeta function," http://www.dtc.umn.edu/ odlyzko/zeta tables]



Wigner surmise: distribution of 'normalized' spacings $\bar{\delta}$ between consecutive zero's of Riemann ζ function

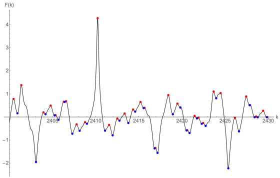


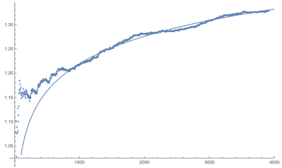
GUE (blue) and Log-normal (orange) fits of ratio's r_n

Leaky torus: upper half plane [M. C. Gutzwiller, Physica D: Nonlinear Phenomena 7, 341 (1983)]
(a): $x = -1$, (b): x=1, (c): $(x - 1/2)^2 + y^2 = 1/4$, (d): $(x + 1/2)^2 + y^2 = 1/4$
with (a)=(c), (b)=(d)
Phase shift of waves from $y = \infty$ to $y = y_{out}$

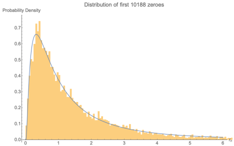
$$F(k) = \frac{Im[\zeta(1 + 2ik)]}{Re[\zeta(1 + 2ik)]}$$

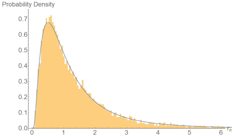
$\sim$ phase of ζ along $z = 1$, distribution of local extrema $F'(z_n) = 0$
Sample of $N = 22,618$ extrema from $k_1 \approx 3.19$ to $k_N \approx 12,927$: $\langle r \rangle_{min} = 1.394$, $\langle r \rangle_{max} = 1.418$, $\langle r \rangle_{all} = 1.944$



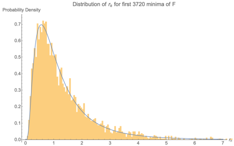


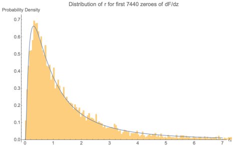
Left: Function $F(k)$.
Right: Average value of the ratio $\langle r \rangle$ as a function of number of zero's N of dF/dk





Left: Distribution of the ratio r (first 10188 zero's of dF/dk)
Right:Distribution of the ratio r (first 3720 maxima of $F(k)$)





Left: Distribution of the ratio r (first 3720 minima of $F(k)$)
Right: Distribution of the ratio r (first 7440 zero's of dF/dk)

Large number of nuclear resonances ... impossible to diagonalize the Hamiltonian.
Statistical approach: Porter-Thomas distribution, Wigner surmise $\sim$ eigenvalues of a random matrix ... RMT ... excellent agreement with experimental data!
Three universality classes: GOE, GUE or GSE (for gaussian matrices), COE, CUE, CSE (for circular ensembles $\lambda = e^{i\theta}$) [Dyson, Wishart, Metha, Gaudin, Berry ...]
Level repulsion: Coulomb gas $V(x_i, x_j) = \log |x_i - x_j|$, β ensemble

$$P_{\beta,N}(\lambda) = C_N e^{-\frac{\beta}{2}\sum_i \lambda_i^2} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta$$

Special cases: GUE $\beta = 2$, GOE $\beta = 1$, GSE $\beta = 4$
For 3×3 matrices and ratio $r = (\lambda_3 - \lambda_2)/(\lambda_2 - \lambda_1)$

$$f_\beta(r) = \frac{3^{\frac{3+3\beta}{2}} \Gamma(1 + \frac{\beta}{2})^2}{2\pi\Gamma(1+\beta)} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+\frac{3}{2}\beta}}$$

For $N > 3$ mild dependence on N.

Hard to find systems with a large number of 'states' at weak coupling ... strings ... black holes
Black Holes: scrambling, thermalization, ... information loss ...
Scrambling time $\sim$ Ehrenfest time, breakdown of semiclassical approximation
Holography [Festuccia, Liu 0506202] Butterfly effect [Shenker, Stanford 1306.0622]
Exponential growth of (OTOC) out-of-time-order correlator ... , [Kulaxizi, Ng, Parnachev 1812.03120, Karlsson, Kulaxizi, Parnachev, Tadíc, 1904.00060]
Quantum Lyaponuv-like exponent ... (holographic) bound on chaos [Maldacena, Shenker, Stanford 1503.01409]

$$\lambda_L < 2\pi\kappa_B T_H/\hbar$$

horizon: large red-shift $\sim$ exponentially large time delay vs photon-rings, chaotic behavior of critical geodesics
NB: ‘merger’ when photon-rings coincide NOT horizons [Christodoulou, Ruffini]
‘classical’ Lyapunov exponent ... chaos at the rim of BH and fuzzball shadows [MB, Grillo, Morales 2002.05574, MB, Consoli, Grillo, Morales 2011.0434]

$$\lambda_L < \frac{C_d}{b_{min}} \approx -Im\omega_{QNM}$$

valid for (near)extremal ‘gravitating objects’ (BHs, branes, ... ECOs, fuzzballs)

Why strings ?

(Open bosonic) strings: Regge resonances

$$\alpha' M^2 = N - 1$$

Very narrow at $g_s =$, broadening and mixing effects even at small g_s ... RMT?
Highly excited strings (HES): large N and many different harmonics $\sim$ random walks
String/BH correspondence: transition when string inside its 'horizon' [Horowitz, Polchinski; Damour, Veneziano; Susskind, ...]

$$2GM = \ell_s = \sqrt{\alpha'}$$

Since $G \approx g_s^2 \alpha'$ and $\sqrt{\alpha'} M \approx \sqrt{N}$ need $g_s^2 = 1/\sqrt{N}$... weak coupling
Test of the correspondence [Amati, Russo]: emission from an ensemble of excited strings at mass/level N, get 'expected' black-body spectrum for (low-energy) emitted quanta
More recently [Firrotta, Rosenhaus; Firrotta] *e.g.* decay amplitude of a 'micro-state' at level N into a 'micro-state' at level $N' < N$ with tachyon/photon (low-mass) emission with

$$E_k << M_{N'} < M_N$$

... thermalization at

$$T_{eff} = T_{Hag}/\sqrt{N} \quad , \quad 2\pi \sqrt{\frac{c}{6}}\, T_{Hag} = \frac{1}{\sqrt{\alpha'}}$$

Highly excited strings and DDF operators

Focus on open bosonic strings ... closed string: KLT/double copy
Spectrum $\alpha' M_N^2 = N - 1$ with $S \leq N$
Exponentially growing degeneracy ... Hagedorn / 'CHardy-Ramanujan' / Dedekind

$$d_N \approx e^{2\pi\sqrt{\frac{c}{6}N}}$$

Hard to identify BRST invariant vertex operators for $N > 3$ [Stieberger, Taylor; MB, Lopez, Richter; Schlotterer; ...]
DDF [Del Giudice, Di Vecchia, Fubini] approach
Choose null momentum q ($q^2 = 0$) and p ($\alpha' p^2 = 1$) such that $2\alpha' pq = 1$
Then $p_N = p - Nq$ on-shell at level N, 'transverse' DDF operators

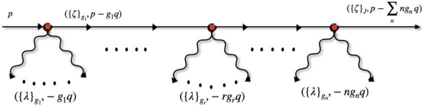
$$A_n^i(q) = \oint \frac{dz}{2\pi} \partial X^i e^{inqX} \quad , \quad [A_n^i, A_m^j] = n\delta^{ij}\delta_{n+m}$$

Most general BRST invariant state

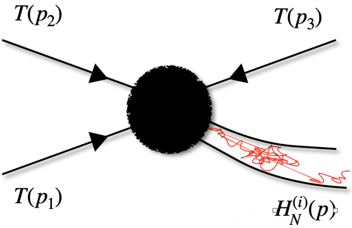
$$|\{n_k\} : N = \sum_k k n_k, p_N\rangle = \prod_{k=1}^{\infty} A_{-k}^{i_k}(q)|0, p\rangle$$

Transverse 'covariant' polarizations: $\zeta_k^\mu = \lambda_k^i(\delta_i^\mu - 2\alpha' p_i q^\mu)$ with $\zeta_k \cdot p = 0 = \zeta_k \cdot q$

$$\left|H_N^{(i)}(\{\zeta\})\right\rangle = \left|\{n_k\} : N=\sum_k k n_k,\ p_N=p-Nq\right\rangle$$



Physical picture: tachyon absorbing/emitting photons



4-point HTTT amplitude

From low-mass to typical states ...

Low-mass: tachyon ($N=0$), vector boson ($N=1$), tensor boson ($N=2$: $n_1=2$ or $n_2=1$), ... integer partitions

$$N=\sum_{k=1}^{\infty} k\, n_k\,, \qquad J=\sum_{k=1}^{\infty} n_k$$

Leading Regge trajectory $n_1=N=J$ ... very special / a-typical

Typical state $\gamma\langle J\rangle_N=\sqrt{N}\log N$ with $\gamma=\pi\sqrt{\frac{2}{3}}$, Gumbel distribution

$$d_N(J)=\gamma\exp\left(-\gamma(J-\langle J\rangle_N)-e^{\gamma(J-\langle J\rangle_N)}\right)$$

Coherent states [Skliros, Hindmarsch; Copeland; MB, Firrotta; Aldi, Addazi, Marcianò; ...] ... normal ordering

$$|\mathcal{C},\lambda_n;p\rangle=e^{\sum_{k=1}^{\infty}\frac{1}{k}\lambda_k\cdot A_{-k}}|p\rangle \quad , \quad V_{\mathcal{C}}=e^{\sum_k\frac{1}{k}\hat{\zeta}_k\cdot\mathcal{P}_k+\sum_{k,n}\frac{1}{2kn}\hat{\zeta}_k\cdot\hat{\zeta}_n\mathcal{S}_{k,n}}e^{ipX}$$

where $\hat{\zeta}_k^{\mu}=e^{-ikq}\zeta_k^{\mu}$ and

$$\zeta_k\cdot\mathcal{P}_k=\sum_{h=1}^{k}\frac{i}{(h-1)!}\mathcal{Z}_{k-h}(u_\ell^{(k)})\zeta_k\cdot\partial^h X \quad , \quad \mathcal{S}_{k,n}=\sum_{h=1}^{k}h\mathcal{Z}_{k-h}(u_\ell^{(k)})\mathcal{Z}_{n+h}(u_\ell^{(n)})=\mathcal{S}_{n,k}$$

with $u_\ell^{(k)}=\frac{-ik}{(\ell-1)!}q\cdot\partial^\ell X$ and cycle index polynomial $\mathcal{Z}_k(u_\ell)=\sum_{n_\ell:\sum_\ell \ell n_\ell=k}\prod_{\ell=1,k}\frac{u_\ell^{n_\ell}}{n_\ell!\,\ell^{n_\ell}}$

Decay of a HES into two light particles (tachyons)

Simple, yet 'generic', HES (Highly Excited String) at level $N(>> 1)$

$$|H_N^{(J)}\rangle = \prod_{k=1}^{N}(\lambda\cdot A_{-k}(q))^{n_k}|0,p\rangle = \prod_{k=1}^{N}\left(\lambda\cdot\frac{\partial}{\partial\mathcal{J}_k}\right)^{n_k}|\mathcal{C},\mathcal{J}_k;p\rangle$$

with $\lambda_k = \lambda_\ell = \lambda$ complex null polarisation $\lambda\cdot\lambda = 0 = p\cdot\lambda = q\cdot\lambda$
Decay amplitude $\sim$ 3-point function on the disk

$$\mathcal{A}(p_1,p_2,p_3) = \langle cV_T(p_1)cV_T(p_2)cV_{HES}(p_3)\rangle$$

where c ghost ($h=-1$) and V's BRST invariant vertex operators.
If $H_N^{(S)}$ had definite spin S

$$\mathcal{A}_{H_S TT} = C_S(\{n_k\})[\lambda\cdot(p_1-p_2)]^S_{\lambda\otimes s=H_S}$$

In the rest frame $\vec{p}_2 = -\vec{p}_1$, Legendre/Gegenbauer polynomial ... NO chaotic behaviour
Generic partitions of N, 'random' superposition of many different 'spins' $N\geq S\geq J$ ...
chaotic behavior of angular distribution.

Chaotic behavior of angular distribution of decay products

Decay amplitude [Gross, Rosenhaus; MB, Firrotta, Sonnenschein, Weissmann]

$$\mathcal{A}_{H_N^{(J)}\to TT} = (\sin\alpha)^J \prod_{m=1}^{\infty}\left[\sin(\pi m\cos^2\frac{\alpha}{2})\frac{\Gamma(m\cos^2\frac{\alpha}{2})\Gamma(m\sin^2\frac{\alpha}{2})}{\Gamma(m)}\right]^{n_m}$$

where $\cos\alpha = 2\alpha' q\cdot p_T$ $(\alpha\sim\pi-\alpha)$
Consider logarithmic derivative

$$F(\alpha)\equiv\frac{d}{d\alpha}\log\mathcal{A}(\alpha) = J\cot\alpha - \frac{\pi}{2}\sin\alpha\sum_{m=1}^{N}n_m\sum_{k=1}^{m-1}\frac{m}{m-k-m\cos^2\frac{\alpha}{2}}$$

Setting $z=\cos^2\frac{\alpha}{2}$, look for solutions of $F(z)=0$: extrema ('peaks' of $|\mathcal{A}|$)
Result: ratios r_n of the spacings between consecutive peaks of the amplitude distributed according to β-ensemble

Statistical analysis

Selection of 'generic' state at level N: technical difficulties in selecting un-biased random states *i.e.* non-trivial algorithm that generate random partitions at a given level N such that all partitions be equally likely

a) Large N ~10,000 ... sufficient number of zeros for single amplitude

b) Intermediate N ~100 many different states ~ union of many sets $\{r_n\}_{N(J)}$

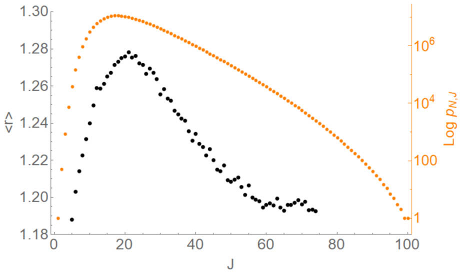
Fit with β-ensemble or log-normal distributions:

$$\beta(N) = \beta_0 + \frac{\beta_1}{N} + \ldots$$

with $\beta_0 = 1.68$ up to slow log terms

Average $\langle r_n \rangle$ increases slowly (logarithmically) with N

Mild dependence on J at fixed N



Tables for HTT decay

N	J	Total number of states	Points in sample	Per state	Average $\langle r_n \rangle$	Fitted β
50	11	17,475	46,354	24	1.206	3.36
75	15	552,767	69,247	34	1.247	2.81
100	18	11.1×10^6	92,251	46	1.271	2.55
150	23	1.90×10^9	139,428	70	1.307	2.26
200	28	158×10^9	184,705	90	1.333	2.09
300	37	295×10^{12}	276,244	138	1.357	1.96
400	45	184×10^{15}	370,123	186	1.372	1.88
800	70	1.08×10^{26}	728,048	362	1.400	1.76
1600	109	4.22×10^{38}	1,446,008	720	1.413	1.72

Dependence of $\langle r \rangle$ and β on N
for samples of 2000 states at each N and $J = \langle J \rangle_N$

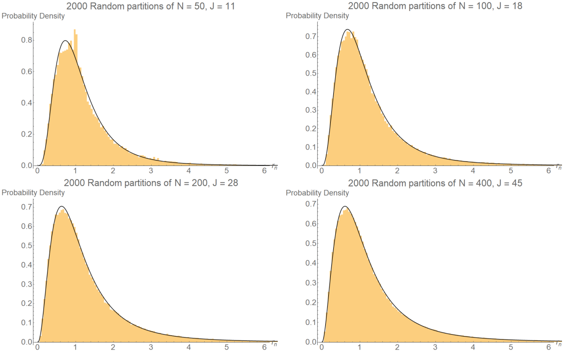
N	Total number of states	Points in sample	Per state	Average $\langle r_n \rangle$	Fitted β
50	204,226	215,980	22	1.194	3.58
60	966,467	261,619	26	1.213	3.27
80	15.8×10^6	352,526	34	1.244	2.87
100	191×10^6	441,100	44	1.266	2.62
150	40.9×10^9	668,831	66	1.301	2.32
200	3.97×10^{12}	886,007	88	1.325	2.15

Dependence of $\langle r \rangle$ and β on N
for samples of 10,000 random partitions of N and J Gumbel-distributed

Plots for HTT decay

J	Total number of states	Points in sample	Per state	Average $\langle r_n \rangle$	Fitted β
6	143,247	155,162	80	1.203	3.60
10	2.98×10^6	126,008	64	1.241	2.95
14	8.86×10^6	105,502	54	1.263	2.65
18	11.1×10^6	92,251	46	1.271	2.55
22	9.24×10^6	83,405	42	1.276	2.52
26	6.32×10^6	76,211	38	1.272	2.57
30	3.91×10^6	70,650	30	1.262	2.69
50	204,226	51,287	26	1.209	3.38
70	5604	31,060	16	1.197	3.50

Dependence of $\langle r \rangle$ and β on J for $N = 100$. For each J: 2000 random states.



Simplest case: HES and 3 tachyons $\mathcal{A}(T,T,T,H)$
Use coherent states in DDF approach 'as' generating function

$$\mathcal{A}_{gen}(T,T,T,\mathcal{C};\mathcal{J}_n) = \int_0^1 dz\, z^{-\frac{s}{2}-2}(1-z)^{-\frac{t}{2}-2}\, e^{\mathcal{J}_n\left(\mathcal{T}_n^{(2)}(z)+\mathcal{T}_n^{(3)}(z)\right)}$$

where

$$\mathcal{T}_n^{(2)}(z) = zp_2\frac{(nq{\cdot}p_3)_{n-1}}{\Gamma(n)}\,{}_2F_1(1+nq{\cdot}p_2, 1{-}n; 2{-}n(1{+}q{\cdot}p_3)|z)$$

$$\mathcal{T}_n^{(3)}(z) = p_3\frac{(nq{\cdot}p_3)_n}{nq{\cdot}p_3\Gamma(n)}\,{}_2F_1(nq{\cdot}p_2, 1{-}n; 1{-}n(1{+}q{\cdot}p_3)|z)$$

Project onto specific amplitude(s)

$$\mathcal{A}(T(p_1), T(p_2), T(p_3), H_N^{(J)}(q,p)) = \prod_n\left(\zeta{\cdot}\frac{d}{d\mathcal{J}_n}\right)^{g_n}\mathcal{A}_{gen}(T,T,T,\mathcal{C})\Big|_{\mathcal{J}_n=0}$$

with $N=\sum_n ng_n$, $J=\sum_n g_n$ and $\zeta_n=\zeta_m=\zeta$, as before

Dressing Factor and ... chaos

$$\mathcal{A}^{HES}_{gen}(s,t) = \mathcal{A}_{Ven}(s,t) e^{\sum_n \mathcal{J}_n \mathcal{O}_n\left(\frac{d}{d\xi}\right) + \sum_{n,m} \mathcal{J}_n \mathcal{J}_m \mathcal{M}_{n,m}\left(\frac{d}{d\xi}\right)} {}_1F_1(-\alpha' s - 1; -\alpha' s - \alpha' t - 2 | \xi)\Big|_{\xi=0}$$

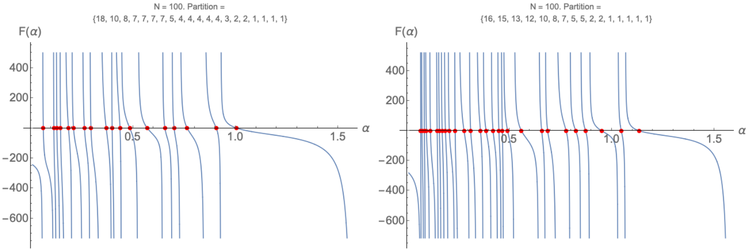
Veneziano amplitude ...

$$\mathcal{A}_{Ven}(s,t) = \int_0^1 dz\, z^{-\alpha' s - 2}(1-z)^{-\alpha' t - 2} = \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' s - \alpha' t - 2)}$$

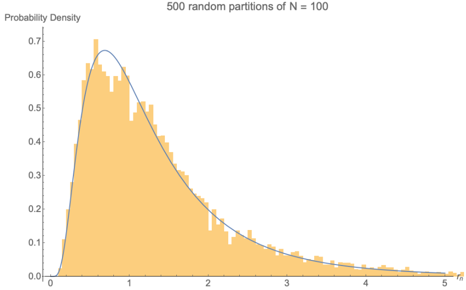
$\times$ Dressing Factor:

$$\mathcal{D}_{HES} = \sum_\ell C_\ell \frac{(-\alpha' s - 1)_\ell}{(-\alpha' s - \alpha' t - 2)_\ell}$$

a) Veneziano (or first Regge trajectory) ... NO chaos
b) dressing factor ... chaotic behavior
c) high energy ($\mathcal{M}_{n,m}$ sub-leading): fixed angle regime vs Regge regime
d) transition from chaotic to 'regular' behavior



Log-derivative of HTTT amplitude. Two generic ‘nearby’ partitions.



Distribution of r_n for 500 random partitions of $N = 100$, with log-normal fit

Chaotic behavior in the Regge regime (1)

Regge $\alpha' s >> \alpha'|t| >> 1$

$$\mathcal{A}_{gen}(T, T, T, \mathcal{C}) = \int_0^1 dz\, z^{-\frac{s}{2}-2}(1-z)^{-\frac{t}{2}-2}\, e^{\mathcal{J}_n\left(\mathcal{T}_n^{(2)}(z)+\mathcal{T}_n^{(3)}(z)\right)}$$

captured by leading behavior around $z \simeq 1$

$$\mathcal{T}_n^{(3)}\Big|_{z=1} = (-)^{n+1} p_3 \frac{\Gamma(n+nq\cdot p_1)}{\Gamma(n)\Gamma(1+nq\cdot p_1)}$$

and

$$\mathcal{T}_n^{(2)}\Big|_{z=1} = (-)^{n+1} p_2 \frac{\Gamma(n+nq\cdot p_1)}{\Gamma(n)\Gamma(1+nq\cdot p_1)}$$

and the amplitude simplifies to

$$\mathcal{A}_{Regge} = (-)^N \prod_n \left(\frac{\Gamma(n+nq\cdot p_1)}{\Gamma(n)\Gamma(1+nq\cdot p_1)}\zeta\cdot p_1\right)^{g_n} \int_0^1 dz\,(1-z)^{-t/2-2} e^{-(s/2-2)(1-z)}$$

that after integration yields

$$\mathcal{A}_{Regge} = (-)^N (\zeta\cdot p_1)^J \Gamma\left(-\frac{t}{2}-1\right) s^{\frac{t}{2}+1} \prod_n \left(\frac{\Gamma(n+nq\cdot p_1)}{\Gamma(n)\Gamma(1+nq\cdot p_1)}\right)^{g_n}$$

Setting $t = -\left(s - \sum_j M_j^2\right) \sin^2\left(\frac{\theta}{2}\right)$

$$\frac{\Gamma(n + nq{\cdot}p_1)}{\Gamma(n)\Gamma(1 + nq{\cdot}p_1)} = \frac{1}{\Gamma(n)}\Gamma\left(n - \frac{n}{\sin\theta + 1}\right)\Gamma\left(\frac{n}{\sin\theta + 1}\right)\sin\left(\frac{n\pi}{\sin\theta + 1}\right)$$

for $s >> |t|$: $\theta << 1$, $\frac{1}{1+\sin\theta} \simeq 1 - \sin\theta$ and

$$\frac{\Gamma(n + nq{\cdot}p_1)}{\Gamma(n)\Gamma(1 + nq{\cdot}p_1)} \simeq \frac{(-)^{n+1}}{\Gamma(n)}\Gamma(n\sin\theta)\Gamma(n - n\sin\theta)\sin(n\pi\sin\theta)$$

and finally, using $\zeta{\cdot}p_1 \simeq \sqrt{s}$,

$$\mathcal{A}_{Regge} = (-\sqrt{s})^J\Gamma\left(-\frac{t}{2} - 1\right)s^{\frac{t}{2}+1}\prod_n\left(\frac{\Gamma(n\sin\theta)\Gamma(n - n\sin\theta)}{\Gamma(n)}\sin(n\pi\sin\theta)\right)^{g_n}$$

Barring overall dependence on s, very similar to 2-body decay after $\cos^2\frac{\alpha}{2} \leftrightarrow \frac{1}{1+\sin\theta}$

Chaotic behavior in the high energy fixed angle regime (1)

In the generating function

$$\mathcal{A}_{gen}(T, T, T, \mathcal{C}) = \int_0^1 dz\, z^{-\frac{s}{2}-2}(1-z)^{-\frac{t}{2}-2} \prod_n W_n(\mathcal{J}_n;\, z)$$

with $s >> 1$, $t >> 1$ with s/t fixed, factor

$$W_n(\mathcal{J}_n;\, z) = e^{\mathcal{J}_n\left(\mathcal{T}_n^{(2)}(z)+\mathcal{T}_n^{(3)}(z)\right)}$$

slowly varying, saddle point at $z^* = \frac{s}{s+t}$ that yields

$$\mathcal{A}_{gen}^{f.a} \simeq \prod_n W_n(\mathcal{J}_n;\, \frac{s}{s+t})\, e^{-s\log s - t\log t + (s+t)\log(s+t)}$$

so that

$$\mathcal{A}^{f.a} \simeq \prod_n \left(\mathcal{T}_n^{(2)}\left(\frac{s}{s+t}\right) + \mathcal{T}_n^{(3)}\left(\frac{s}{s+t}\right)\right)^{g_n} e^{-s\log s - t\log t + (s+t)\log(s+t)}$$

Chaotic behavior in the high energy fixed angle regime (2)

Using kinematics in the fixed-angle regime

$$q{\cdot}p_2 = -\frac{1}{1+\sin\theta}\,, \quad q{\cdot}p_3 = \frac{1-\sin\theta}{1+\sin\theta} = -2q{\cdot}p_2 - 1$$

$$\zeta{\cdot}p_2 = -\frac{\sqrt{s}}{2} + \frac{\sqrt{s}}{2}\frac{\cos\theta}{1+\sin\theta}\,, \quad \zeta{\cdot}p_3 = -\sqrt{s}\frac{\cos\theta}{1+\sin\theta}$$

one has

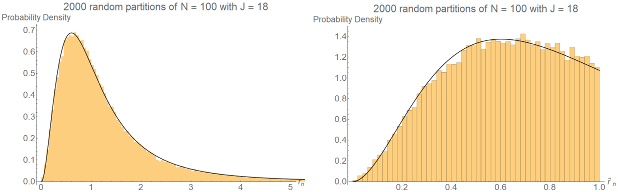
$$\mathcal{T}_n^{(2)}(s,\theta) = \frac{\sqrt{s}\left(\frac{\cos\theta}{1+\sin\theta}-1\right)\left(n\frac{1-\sin\theta}{1+\sin\theta}\right)_{n-1}}{2\Gamma(n)[\cos^2(\frac{\theta}{2})+\frac{M_{tot}^2}{s}\sin^2(\frac{\theta}{2})]}\,{}_2F_1(1-\frac{n}{1+\sin\theta}, 1-n; 2-\frac{2n}{1+\sin\theta}|\frac{1}{\sigma})$$

with $\sigma = \cos^2(\frac{\theta}{2})+\frac{M_{tot}^2}{s}\sin^2(\frac{\theta}{2})$ and

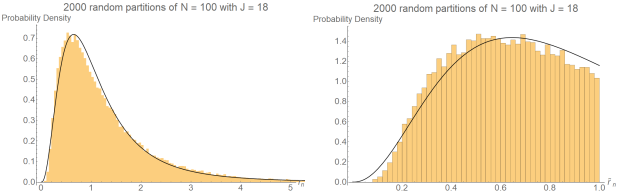
$$\mathcal{T}_n^{(3)}(s,\theta) = -\frac{\sqrt{s}}{\Gamma(n)}\frac{\cos\theta\left(n\frac{1-\sin\theta}{1+\sin\theta}\right)_n}{n(1-\frac{2N}{s}-\sin\theta)}\,{}_2F_1(-\frac{n}{1+\sin\theta}, 1-n; 1-\frac{2n}{1+\sin\theta}|\frac{1}{\sigma})$$

Finally

$$\mathcal{A}^{f.a} \simeq \prod_n \left(\mathcal{T}_n^{(2)}(s,\theta) + \mathcal{T}_n^{(3)}(s,\theta)\right)^{g_n} e^{-s\,f(\theta)}$$

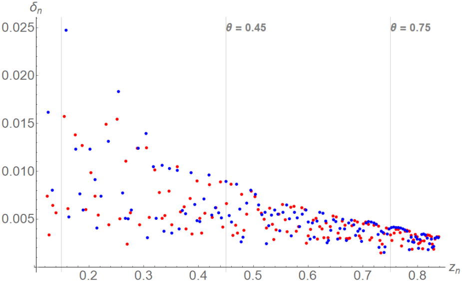


Dressing factor in fixed-angle regime: distributions of r and $\tilde{r}$ for 2000 random partitions of $N = 100$ and $J = 18$

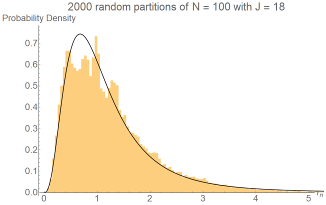


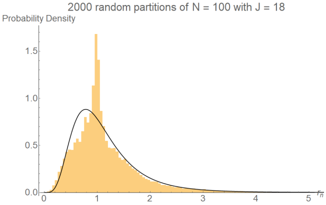
Dressing factor in Regge regime: distributions of r and $\tilde{r}$ for 2000 random partitions of $N = 100$ and $J = 18$

Transition from chaos to regular behavior



Spacings δ_n as a function of z_n for two random states of $N = 100$
transition from random to regular behavior





Distribution of spacing ratios r_n in the ranges $\theta \in (0.15, 0.45)$ (left) and $\theta \in (0.15, 0.75)$ (right). In the latter, narrow peak at $r = 1$ on top of chaotic distribution.

Summary

- For 2-body decay processes: distribution of spacings of peaks well modelled by β-ensemble, with the parameter β depending on the level N and the helicity J of HES state.
  For $N = 50 - 1600$, β decreasing from 3.4 to 1.7, while $\langle r \rangle$ slow-monotonously increasing with N.
- For 4-point scattering amplitude: Veneziano (non-chaotic) times dressing factor ('chaotic'), depending on HES state.
  High-energy: fixed-angle limit vs Regge regime.
  For HES states with $N = 100$ GUE-like distributions for r_n with β around 2.
- Transition from chaotic to regular spacings as range of scattering angle from small to large.
- Chaotic behavior completely disappears for leading Regge trajectory or nearby states, *i.e.* for HES with $N \approx J$

Outlook

- Clarify (origin of) dependence of β on N and J ... more statistics
- More amplitudes with one HES and amplitudes with two or more HES [Di Vecchia, Firrotta *w.i.p.*]
- Chaotic behaviour in other kinematical variables ... Coon amplitude and Remmen amplitude
- Coherent states as proxy's of 'spinning BHs' ... (neutral) fuzzballs ... top stars [Bah, Berti, Heidmann, Spinney; MB, Di Russo, Grillo, Morales, Sudano]
- HmEST ... [MB, Di Russo *w.i.p.*]
- Higher-loops

Appendix 1: Random partitions of a large integer

As discussed in the talk, the number of partitions of an integer N grows exponentially in $\sqrt{N}$.
Since we cannot probe the full space of states, we need a reliable method of picking representative, generic states in a random way.
Picking a partition of a large integer N at random, which each partition having an equal probability of being chosen, is a non-trivial task. We present here one algorithm that accomplishes this goal.
We represent a partition as a list n_m, $m = 1, 2, \ldots N$, where n_m is the number of times that m occurs in the partition.
Rely on probabilistic algorithm presented in [Arratia:2016]. It relies on an observation by Fristedt [Fristedt:1993] on the asymptotic distributions of $\{n_m\}$ for large N, namely that each n_m has the geometric distribution

$$P(n_m = k) = (1 - p_m)^k p_m$$

with

$$p_m = 1 - \exp\left(-\frac{m\pi}{\sqrt{6N}}\right)$$

One can generate a random partition of N by drawing values of $\{n_m\}$, $m = 1, 2, \ldots, N$ from the above distribution, until one reaches one that corresponds to a partition of N. That is, until we get a set of $\{n_m\}$ that satisfy the constraint $\sum_m mn_m = N$. The result of [Fristedt:1993] implies that the partitions of N that will be reached by this algorithm will be uniformly distributed.
The downside of the algorithm is that it needs to reject many sets of $\{n_m\}$ until it reaches one that satisfies the constraint, with the expected number of rejections being $\mathcal{O}(N^{3/4})$. By use of probabilistic algorithms one can improve the number of rejections to $\mathcal{O}(N^{1/4})$ or even $\mathcal{O}(1)$ [Arratia:2016].
The simpler, $\mathcal{O}(N^{1/4})$ algorithm is as follows:

1. Draw $\{n_m\}$ for $m \geq 2$, with n_m distributed according to (**??**).
2. Set $k \equiv N - \sum_{m=2}^{N} mn_m$. If $k < 0$ restart from step 1.
3. Draw a random variable $u \in (0, 1)$ from the uniform continuous distribution. If $u < e^{-\frac{k\pi}{\sqrt{6N}}}$, reject the partition and return to step 1.
4. Set $n_1 = k$ to finish.

Step 3, where some partitions are rejected at a specifically chosen probability, assures that the probability to output a given partition is as before.
We can use a modification of the above algorithm to generate a partition of a given length J. We modify only step 1, where we start by choosing $\{n_m\}$ such that $n_J \geq 1$ and $n_{m>J} = 0$. Then, the result after step 4 will be a partition of N where the maximum summand in the partition is $m_{\max} = J$. Then, taking the conjugate partition, we get a partition of N into J parts.
We have used several methods of picking random partitions. One is the brute force method: generate a list of all possible partitions of a given N (and J when that is constrained), then, select random elements from the list with equal probability.
This is the simplest method at smaller N, but becomes impractical quickly as one increases N. For unconstrained partitions of N we have used Mathematica's built-in (as part of the Combinatorica package) RandomPartition function.

To produce constrained partitions, i.e. of large N with fixed J, we have used the algorithm described above in the cases where the brute force method was unavailable.

Appendix 2: Kinematical setup

$$p_1 = (E_1\,,\, p_{in}\,,\, 0\,,\, \vec{0}), \quad p_2 = (E_2\,,\, -p_{in}\,,\, 0\,,\, \vec{0})$$

$$p_3 = -(E_3\,,\, p_{out}\cos\theta\,,\, p_{out}\sin\theta\,,\, \vec{0}), \quad p = -(E_4\,,\, -p_{out}\cos\theta\,,\, -p_{out}\sin\theta\,,\, \vec{0})$$

$$q = \frac{(1,\,0,\,1,\,\vec{0})}{E_4 + \sin\theta\; p_{out}}\;, \quad \lambda = \frac{(0,\,1,\,0,\,\vec{\Lambda})}{\sqrt{1+|\vec{\Lambda}|^2}}$$

where

$$E_1 = \frac{s + M_1^2 - M_2^2}{2\sqrt{s}} = \frac{\sqrt{s}}{2}\;, \quad E_2 = \frac{s + M_2^2 - M_1^2}{2\sqrt{s}} = \frac{\sqrt{s}}{2}$$

$$E_3 = \frac{s + M_3^2 - M_4^2}{2\sqrt{s}} = \frac{s - 2N}{2\sqrt{s}}\;, \quad E_4 = \frac{s + M_4^2 - M_3^2}{2\sqrt{s}} = \frac{s + 2N}{2\sqrt{s}}$$

and

$$p_{in}^2 = \frac{F(s,\,M_1^2,\,M_2^2)}{4s} = 2 + \frac{s}{4}\;; \quad p_{out}^2 = \frac{F(s,\,M_3^2,\,M_4^2)}{4s} = 2 + \frac{s}{4}\left(1 - \frac{2N}{s}\right)^2$$

relevant scalar products

$$q\cdot p_1 = -\frac{E_1}{\sin\theta\; p_{out} + E_4} = \frac{-1}{1 + \frac{2N}{s} + 2\sin\theta\sqrt{\frac{2}{s} + \frac{1}{4}\left(1 - \frac{2N}{s}\right)^2}} = q\cdot p_2$$

$$q\cdot p_3 = \frac{E_3 - p_{out}\sin\theta}{E_4 + p_{out}\sin\theta} = \frac{1 - \frac{2N}{s} - 2\sin\theta\sqrt{\frac{2}{s} + \frac{1}{4}\left(1 - \frac{2N}{s}\right)^2}}{1 + \frac{2N}{s} + 2\sin\theta\sqrt{\frac{2}{s} + \frac{1}{4}\left(1 - \frac{2N}{s}\right)^2}}$$

and

$$\lambda\cdot p = p_{out}\cos\theta = \sqrt{s}\cos\theta\sqrt{\frac{2}{s} + \frac{1}{4}\left(1 - \frac{2N}{s}\right)^2} = -\lambda\cdot p_3$$

$$\lambda\cdot p_1 = p_{in} = \sqrt{s}\sqrt{\frac{2}{s} + \frac{1}{4}} = -\lambda\cdot p_2$$

where for convenience $\vec{\Lambda} = \vec{0}$
Since $\zeta\cdot p_j = \lambda\cdot p_j - \lambda\cdot p\; q\cdot p_j$, it follows that