Large Deviations in the Early Universe

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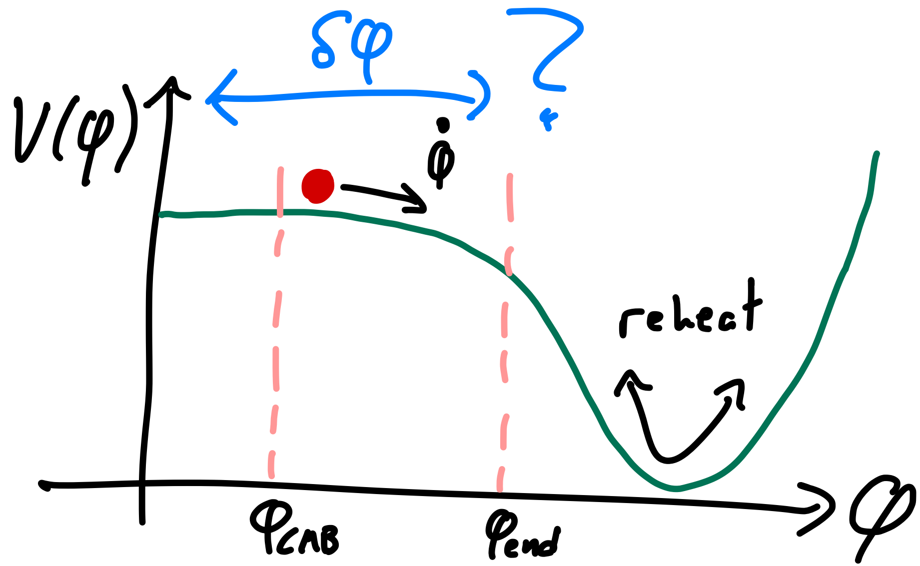

From Amps to Grav Waves
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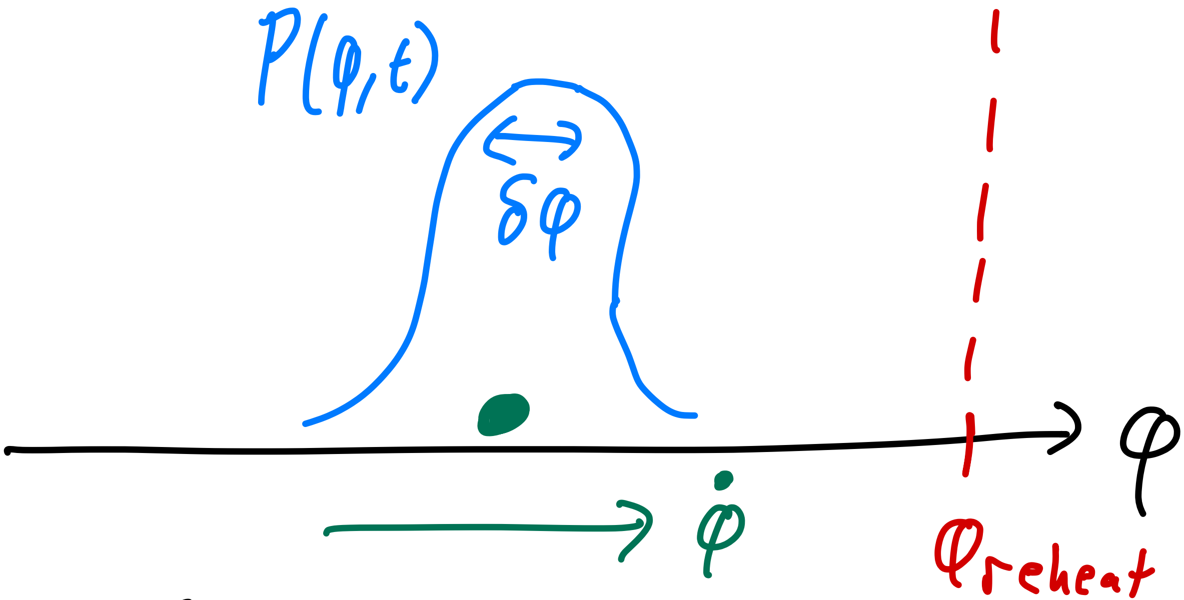
July 26, 2023

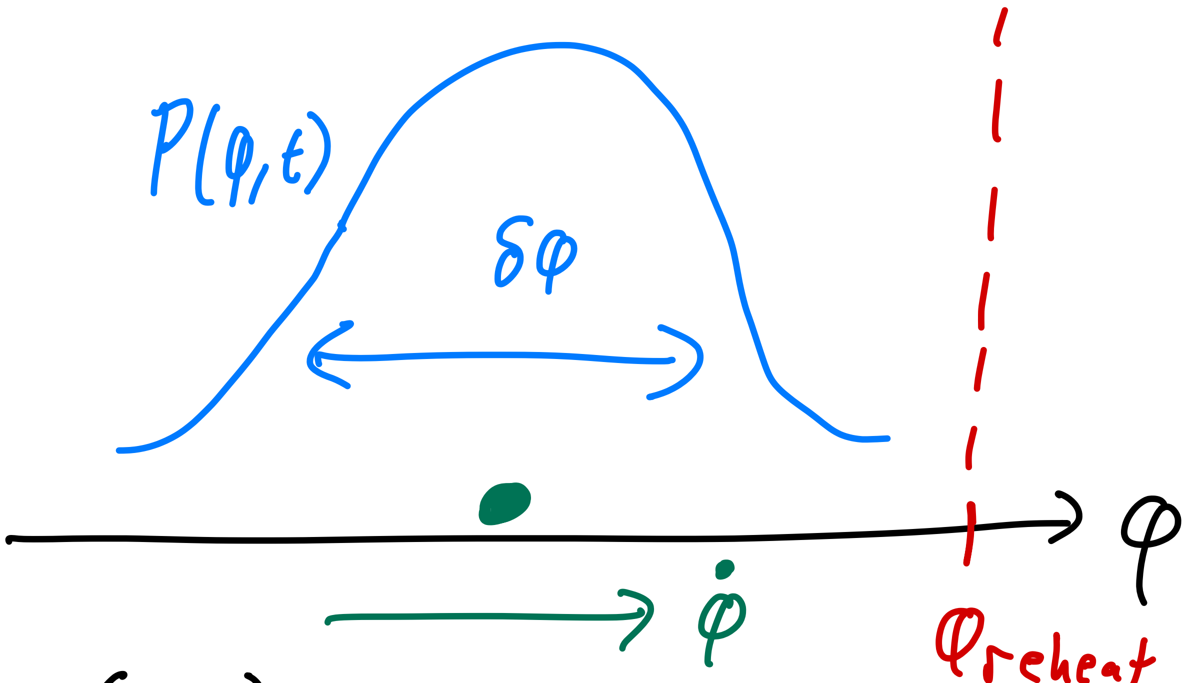
Inflaton Fluctuations



$$(\Delta\varphi)_{classical} > (\Delta\varphi)_{noise}$$

Small $\delta\varphi \Rightarrow$ universal end to inflation

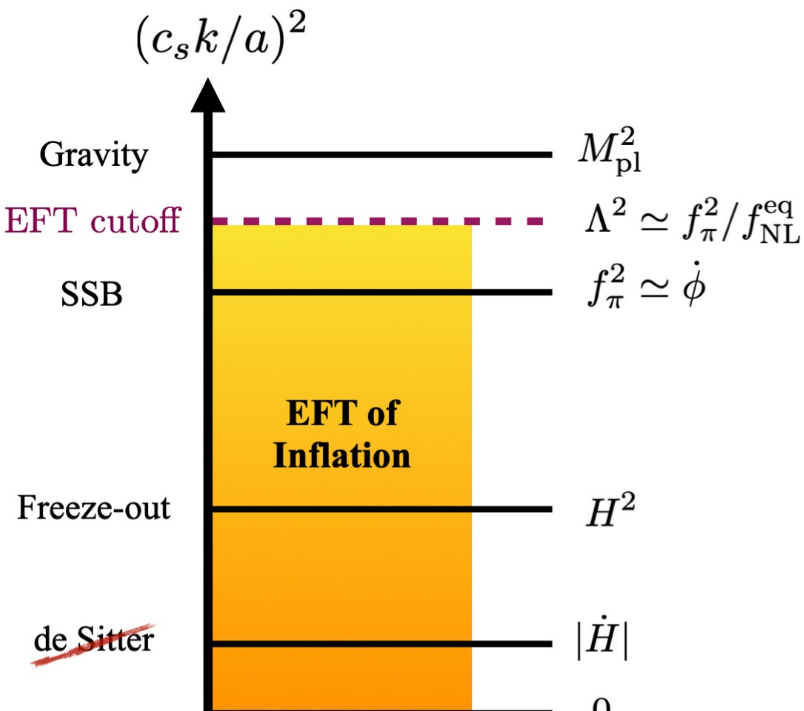
Eternal Inflation



$$(\Delta\varphi)_{classical} \sim (\Delta\varphi)_{noise}$$

Some patches never reheat!

EFT of Inflation



Gaussian

$\Delta t \sim 1/H$

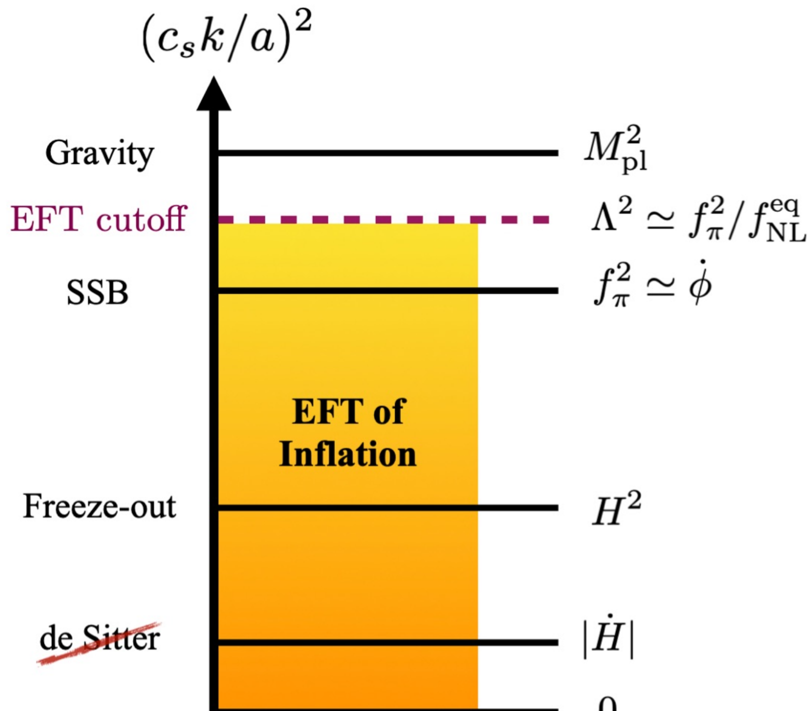
$(\Delta\phi)_{cl} \sim \dot\phi/H = f_\pi^2/H$

$(\Delta\phi)_{\rm noise} \sim H$

Eternal Inflation

$\Rightarrow f_\pi \sim H$

EFT of Inflation



Non-Gaussian tails

$$\Delta t \sim 1/H$$

$$(\Delta\phi)_{cl} \sim \dot\phi/H \sim f_\pi^2/H$$

$$(\Delta\phi)_{tail} \sim f_\pi^2/H$$

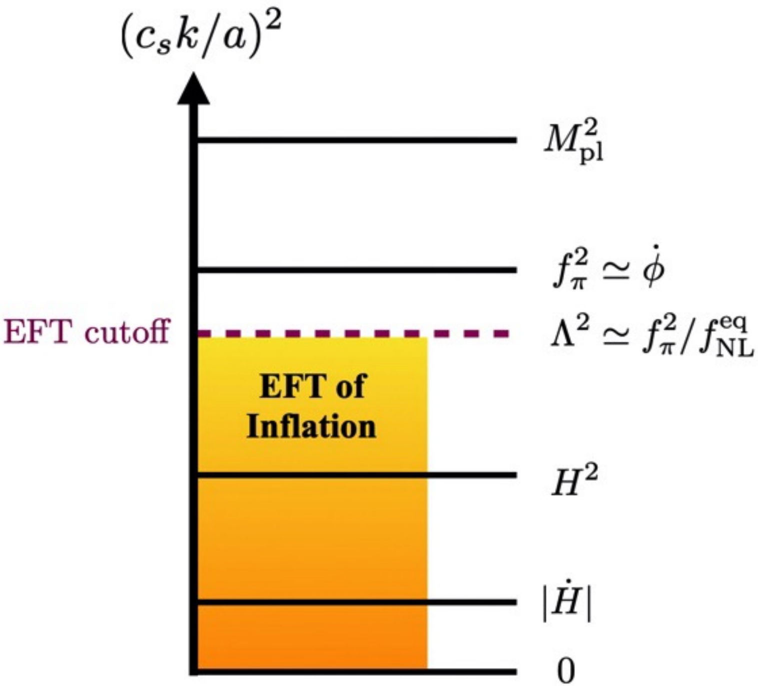
Eternal Inflation

$$\Rightarrow f_\pi \gg H$$

Energy scale associated with tail:

$$(\dot\phi)_{tail} \sim H(\Delta\phi)_{tail} \sim f_\pi^2$$

EFT of Inflation: $f_\pi < \Lambda$



$$f_{NL}^{eq} \sim f_\pi^2/\Lambda^2$$

Current constraint:

$$f_{NL}^{eq} = -26 \pm 47$$

$\Rightarrow \Lambda^2 \ll f_\pi^2$ allowed

Only need $\Lambda > H$ for EFT consistency

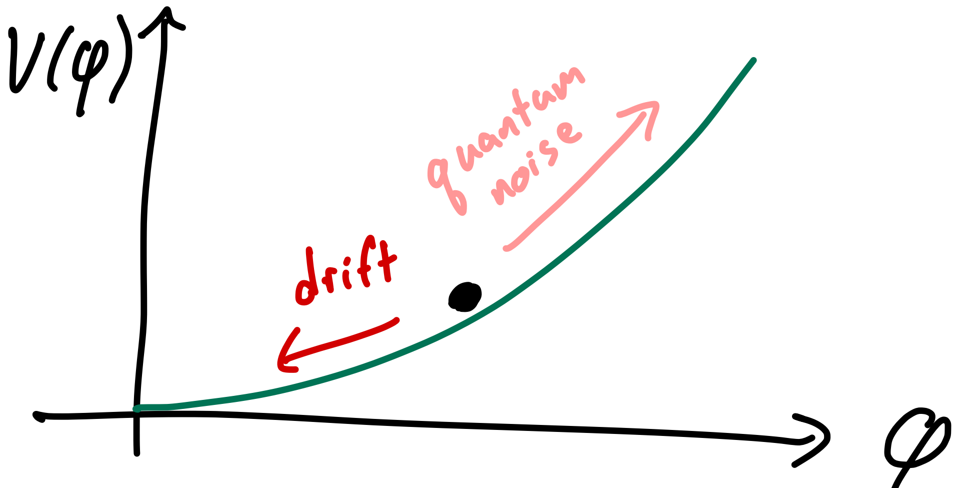
Energy scale associated with tail:

$$(\dot\phi)_{tail} \sim H(\Delta\phi)_{tail} \sim f_\pi^2 \gg \Lambda^2$$

Can we reliably compute tail in these models??

Starobinsky's Stochastic Inflation

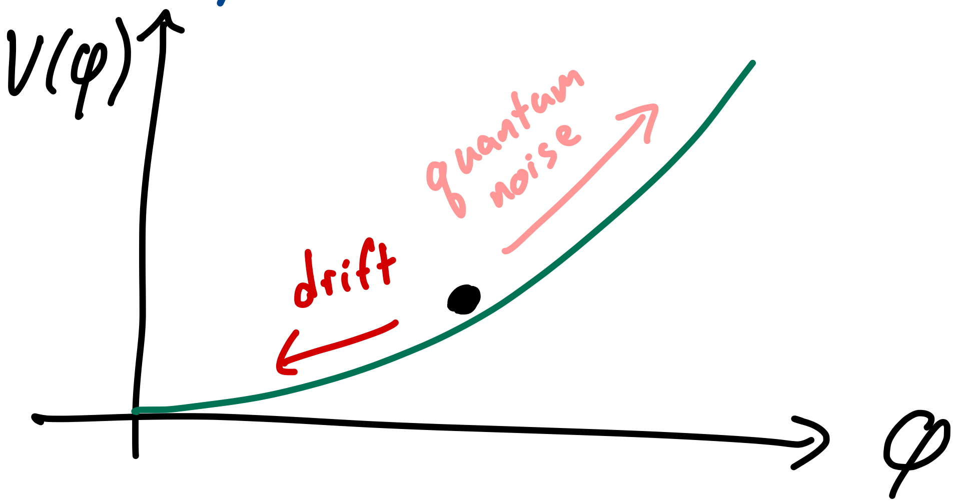
Massless scalar field in dS



$\Rightarrow$ Focker - Planck equation:

$$\frac{\partial}{\partial t} P(\varphi, t) = \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \varphi^2} P(\varphi, t) + \frac{1}{3H} \frac{\partial}{\partial \varphi} \left[V'(\varphi) P(\varphi, t) \right]$$

Starobinsky's Stochastic Inflation



Gaussian noise

Tree-level potential

Systematic corrections?

Leading Order

Probability distribution $P(\varphi, t)$

$$\frac{\partial}{\partial t} P(\varphi,t) = \underbrace{\frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \varphi^2} P(\varphi,t)}_{\text{noise}} + \underbrace{\frac{1}{3H} \frac{\partial}{\partial \varphi}\left[V'(\varphi) P(\varphi,t)\right]}_{\text{drift}}$$

H: Hubble and $V' = \frac{\partial V}{\partial \varphi}$

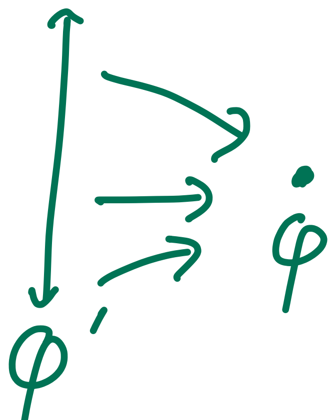
Fixed point solution $\left(\partial P/\partial t = 0\right)$

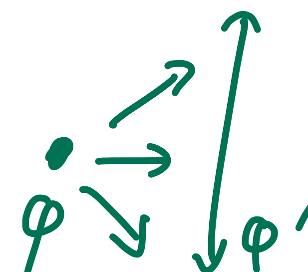
$$P_{eq} \sim \exp\left(-8\pi V/3H^4\right)$$

Beyond Leading Order

Assume "Markovian" fluctuations (no memory)

$$\frac{\partial}{\partial t}P(\varphi,t)=\int d\varphi'\left[P(\varphi';t)\,W(\varphi|\varphi') - P(\varphi,t)\,W(\varphi'|\varphi)\right]$$





$W(\varphi|\varphi')$ is transition rate $\varphi' \to \varphi$

Beyond Leading Order

Perform Kramers-Moyal "local" expansion

$$\frac{\partial}{\partial t} P(\varphi, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varphi^n} \Omega_n(\varphi) P(\varphi, t)$$

w/ $\Omega_n(\varphi) = \int d\Delta\varphi \, (-\Delta\varphi)^n \, W(\varphi + \Delta\varphi \,|\, \Delta\varphi)$

Beyond Leading Order

$$\frac{\partial}{\partial t} P(\varphi, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varphi^n} \Omega_n(\varphi) P(\varphi, t)$$

$\Omega_n(\varphi)$ has polynomial expansion

$$\Omega_n(\varphi) = \sum_{m=0}^{\infty} \frac{1}{m!} \Omega_n^{(m)} \varphi^m$$

LO Stochastic Inflation

$$V = \sum_{\ell} \frac{1}{\ell!} c_\ell \varphi^\ell \Rightarrow \Omega_1^{(m)} = \frac{1}{3H} c_{m+1} \quad \Big| \quad \Omega_2^{(0)} = \frac{H^3}{8\pi^2}$$

Beyond Leading Order

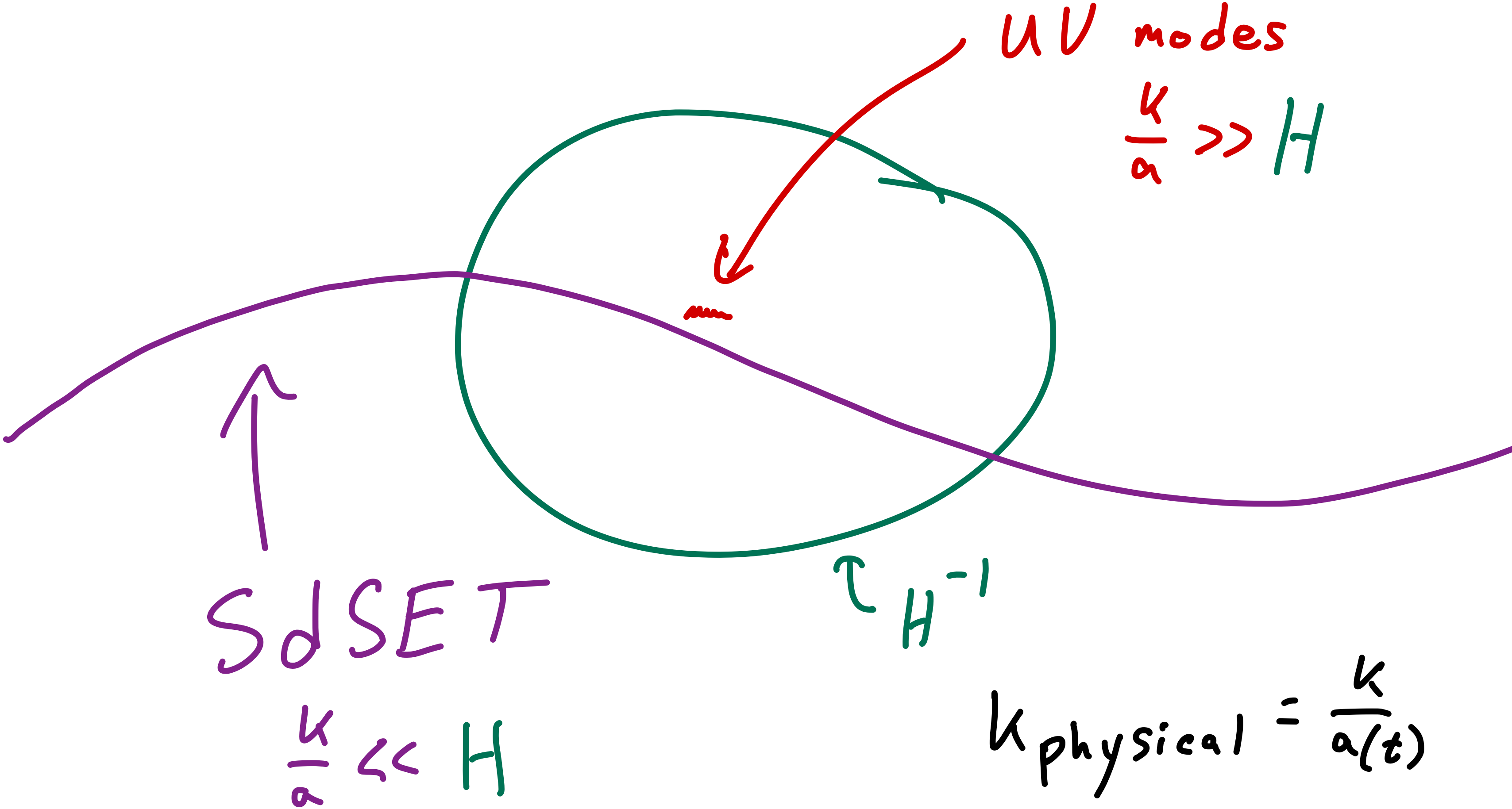
Generic structure

$$\frac{\partial}{\partial t} P(\varphi, t) = \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varphi^n} \left[\sum_{m=0}^{\infty} \frac{1}{m!} \Omega_n^{(m)} \varphi^m P(\varphi, t) \right]$$

Higher order noise

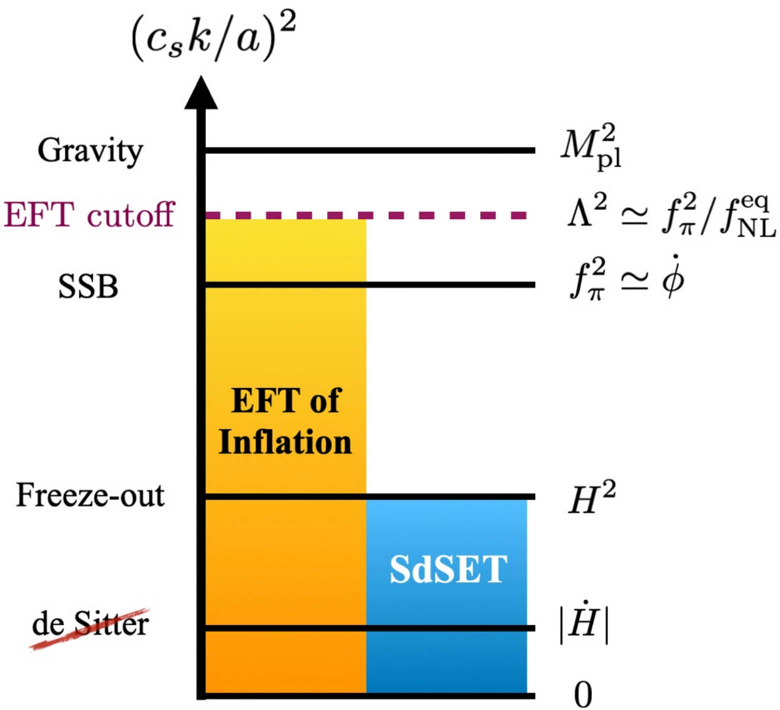
$$+ \frac{1}{3H} \frac{\partial}{\partial \varphi} \left[V'(\varphi) P(\varphi, t) \right]$$

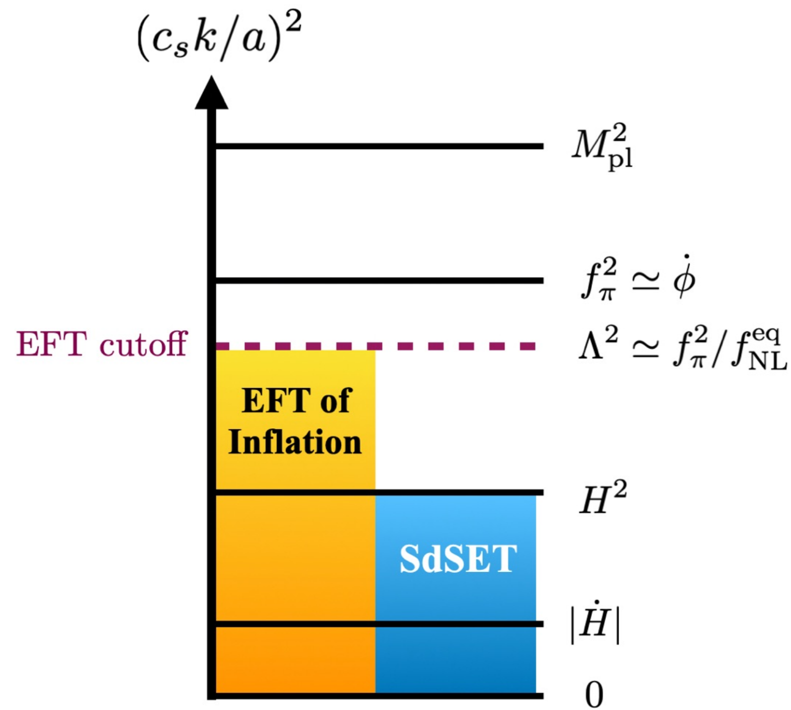
Soft de Sitter Effective Theory



$$k_{\text{physical}} = \frac{k}{a(t)}$$

Soft de Sitter Effective Theory





SdSET Fields

Two IR degrees of freedom

- "Growing" mode φ_+ ← Correlators of interest
- "Decaying" mode φ_-

w/ $\phi_s = H\left((aH)^{-\alpha}\varphi_+ + (aH)^{-\beta}\varphi_-\right)$

Time dependence factorizes :)

Defining SdSET

- DOF φ_+ and φ_-
- Power counting $\sim \dfrac{k}{\Lambda_{UV}}$ w/ $\Lambda_{UV} = aH$
- Symmetries (1) "spacetime"
  (2) "reparametrization"
- Initial conditions

* Very close analogy w/ Heavy Quark EFT

Light Scalars in dS

Composite operators

$$\mathcal{O}_n = \varphi_+^n \sim \left(k/aH\right)^{n\alpha} \to \mathcal{O}(1)$$

RG mixing expected

Contract any two legs

$$\langle \mathcal{O}_n \ldots \rangle \supset \langle \mathcal{O}_{n-2} \ldots \rangle \binom{n}{2} \frac{C_\alpha^2}{2} \int \frac{d^3p}{(2\pi)^3} \frac{H^{2-2\alpha}}{p^{3-2\alpha}}$$

Light Scalars in dS

$$\int \frac{d^3p}{(2\pi)^3} \frac{H^{2-2\alpha}}{p^{3-2\alpha}}$$ is scaleless and diverges as $\alpha \to 0$

Isolate UV divergence

$$p^2 \to p^2 + k_{IR}^2$$

$$\langle \mathcal{O}_n \ldots \rangle \supset \langle \mathcal{O}_{n-2} \cdots \rangle \binom{n}{2} \frac{c_\alpha^2}{4\pi^2} \left(-\frac{1}{2\alpha} - \gamma_E - \log \frac{aH}{k_{IR}} \right)$$

Dynamical RG $\Leftrightarrow$ Stocastic Inflation

Resum time dependent logs:

$$\frac{\partial}{\partial \underline{t}} \langle \mathcal{O}_n \ldots \rangle = -\frac{n}{3} \sum_{m \geq 1} \frac{c_m}{m!} \langle \mathcal{O}_{n-1} \mathcal{O}_m \ldots \rangle + \frac{n(n-1)}{8\pi^2} \langle \mathcal{O}_{n-2} \ldots \rangle$$

(Starobinsky; Starobinsky, Yokoyama)

Is equivalent to a Fokker-Planck eq for $P(\varphi, t)$ w/ $\langle \varphi^n \rangle = \int d\varphi \, P(\varphi, t) \, \varphi^n$ (Baumgart & Sundrum)

Stochastic Inflation for Inflation

Work with scalar metric fluctuation

$$\zeta \simeq H\left(t - \varphi/f_\pi^2\right)$$

Has non-linearly realized shift symmetry

$$\Rightarrow \frac{\partial}{\partial t} P(\zeta, t) = {\sum_{n\geq 2}}' (-1)^n \frac{\gamma_n}{n!} \frac{\partial^n}{\partial \zeta^n} P(\zeta, t)$$

$\uparrow$
$H=1$

Determine γ_n by computing operator mixing

$$\frac{\partial}{\partial t} \langle \zeta^N \rangle = {\sum_n}' \gamma_n \binom{N}{n} \langle \zeta^{N-n} \rangle$$

Non-Gaussian Corrections

$$\gamma_2 : \quad \langle \zeta^2 \rangle = \int \frac{d^3k}{(2\pi)^3} \underbrace{\langle \zeta(\vec{k})\, \zeta(\vec{k}') \rangle}_{\frac{\Delta_\zeta}{k^3}(2\pi)^3 \delta^3(\vec{k}-\vec{k}')} = 2\gamma_2 \log \frac{aH}{\underbrace{k}_{\text{IR reg}}}$$

$$\gamma_3 : \quad \langle \zeta^3 \rangle = \int \frac{d^3k_1\, d^3k_2\, d^3k_3}{(2\pi)^3} \underbrace{\langle \zeta(k_1)\, \zeta(k_2)\, \zeta(k_3) \rangle}_{\text{Two contributions}}$$

$\dot{\zeta}^3$ $\qquad$ $\dot{\zeta}\, \partial_i \zeta\, \partial^i \zeta$

$$\gamma_3 = \frac{\Delta_\zeta^2}{32\pi^4} \left(\left(1 - \frac{1}{c_s^2}\right)\left(9 + c_s^2\right) + \frac{c_3}{c_s^2} \right)$$

Non-Gaussian Corrections

$$\frac{\partial}{\partial t} P(\zeta, t) = \frac{\gamma_2}{2} \frac{\partial^2}{\partial \zeta^2} P(\zeta, t) - \frac{\gamma_3}{3!} \frac{\partial^3}{\partial \zeta^3} P(\zeta, t)$$

Assuming P is nearly Gaussian:

$$\langle \zeta \zeta \rangle \sim \Delta_\zeta$$

$$\zeta \sim \Delta_\zeta^{1/2}$$

$$\partial_\zeta \sim \Delta_\zeta^{-1/2}$$

$$\Delta_\zeta \sim \frac{H^4}{f_\pi^4}$$

$$\Rightarrow$$

$$\gamma_2 \frac{\partial^2}{\partial \zeta^2} \sim \mathcal{O}(1)$$

$$\gamma_3 \frac{\partial^3}{\partial \zeta^3} \sim \frac{\Delta_\zeta^{1/2}}{c_s^2} \sim \frac{H^2}{\Lambda^2} \ll 1$$

$$\Lambda = f_\pi c_s$$

Gaussian Distribution

Solve $\frac{\partial}{\partial t} P_G(\xi, t) = \frac{\Delta_\xi}{4\pi^2} \frac{\partial^2}{\partial \xi^2} P_G(\xi, t)$

with boundary condition

$$P_G(\xi, \xi_i; t=0) = \delta(\xi - \xi_i) \quad \text{and} \quad P_G(\varphi_r[\xi]) = 0$$

$$\Rightarrow P_G(\varphi[\xi] < 0, \xi_i; t)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \left[e^{-(\xi-\xi_i)^2/(2\sigma^2 t)} - e^{-4\xi_i/2\sigma^2} e^{-(\xi+\xi_i)^2/2\sigma^2 t} \right]$$

$$\sigma^2 = 2\gamma_2 = \frac{\Delta_\xi}{2\pi}$$

Onset of Eternal Inflation

Probability of reheating at time t:

$$P_{R,G}(t) = -\frac{d}{dt}\int_{-\infty}^{0} d\varphi\, P_G(\varphi;t) \sim e^{-t/2\sigma^2}$$

Volume of reheating surface:

$$\langle V\rangle_G = L^3\int_0^{\infty} dt\, e^{3t}\, P_{R,G}(t) \simeq L^3\int_0^{\infty} dt\, e^{t(3-1/2\sigma^2)}$$

$$\Rightarrow \langle V\rangle_G \to \infty \quad \text{when} \quad \sigma^2 = \frac{\Delta_S}{2\pi^2} > \frac{1}{6} \Rightarrow \text{Eternal inflation}$$

Non-Gaussian Distribution

Solve $\frac{\partial}{\partial t} P_{NG}(\xi, t) = \left(\frac{\Delta_4}{8\pi^2} \frac{\partial^2}{\partial \xi^2} - \gamma_3 \frac{\partial^3}{\partial \xi^3} \right) P_{NG}(\xi, t)$

Boundary conditions: $P_{NG}(\xi, \xi_i ; t=0) = \delta(\xi - \xi_i)$

$P_{NG}(\varphi_r[\xi]) = 0$

Solution: $P_{NG} = \exp\left(- \frac{\gamma_3 t}{3!} \frac{\partial^3}{\partial \xi^3} \right) P_G$ + images

Non-Gaussian Onset of Eternal Inflation

Probability of reheating at time t:

$$P_{R,NG}(t) \sim \exp\left[-t\left(\frac{1}{2\sigma^2} - \frac{\gamma_3}{3!}\frac{1}{\sigma^6}\right)\right]$$

Volume of reheating surface:

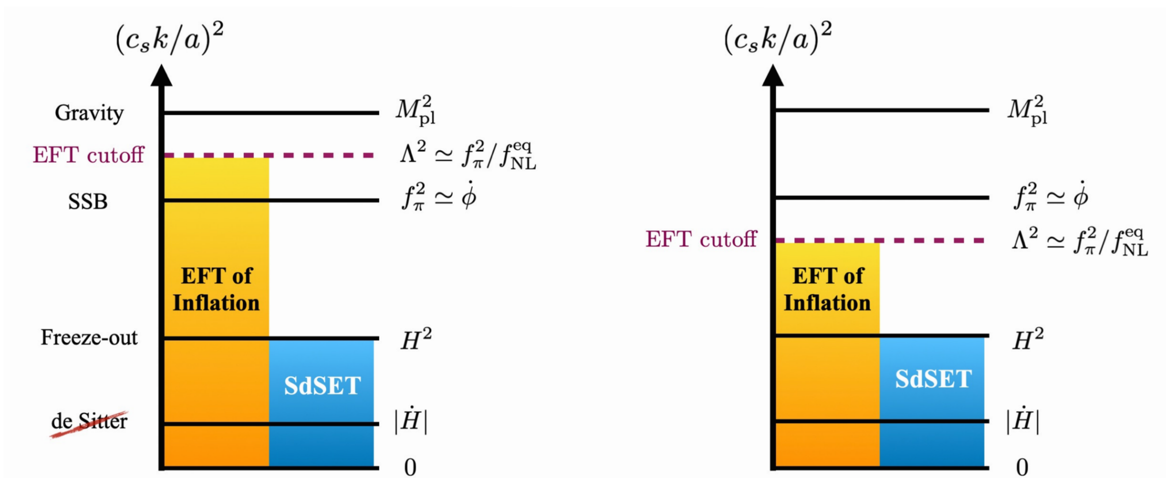
$$\langle V\rangle_{NG} = L^3\int_0^\infty dt\, e^{3t}\, P_{R,NG}(t)$$

$\langle V\rangle_{NG} \to \infty$ when $\frac{1}{2\sigma^2} - \frac{\gamma_3}{3!}\frac{1}{\sigma^6} < 3$

$$\Rightarrow \frac{1}{2} - \frac{1}{48}\frac{f_\pi^2}{\Delta^2}\left((c_s^2-1)(9+3c_s^2)+c_3\right) < \frac{3\Delta_\xi}{2\pi^2}$$

Interpretation

$$\underbrace{\frac{1}{2}}_{\text{Gaussian}} - \underbrace{\frac{1}{48}\frac{f_\pi^2}{\Lambda^2}\left((c_s^2-1)(9+3c_s^2)+c_3\right)}_{\text{non-Gaussian}} < \frac{3\Delta_\zeta}{2\pi^2}$$

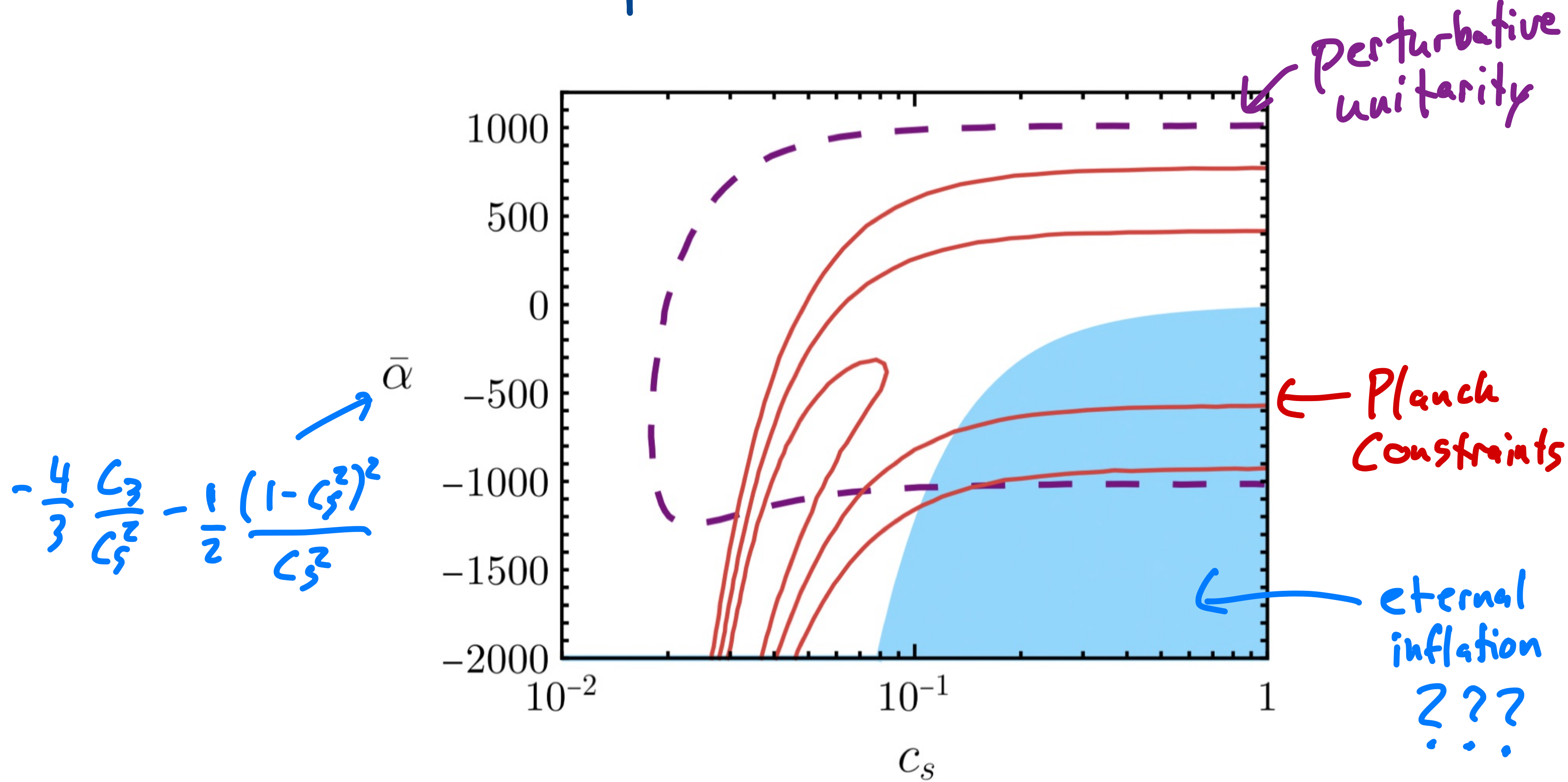


$\Lambda \gg f_\pi$

Correction is out of control

$\Lambda \ll f_\pi$

Implications



Do these models really predict eternal inflation??

Or is there a suprising breakdown of EFT??

Random Walks and RG

Toy models of fluctuating fields

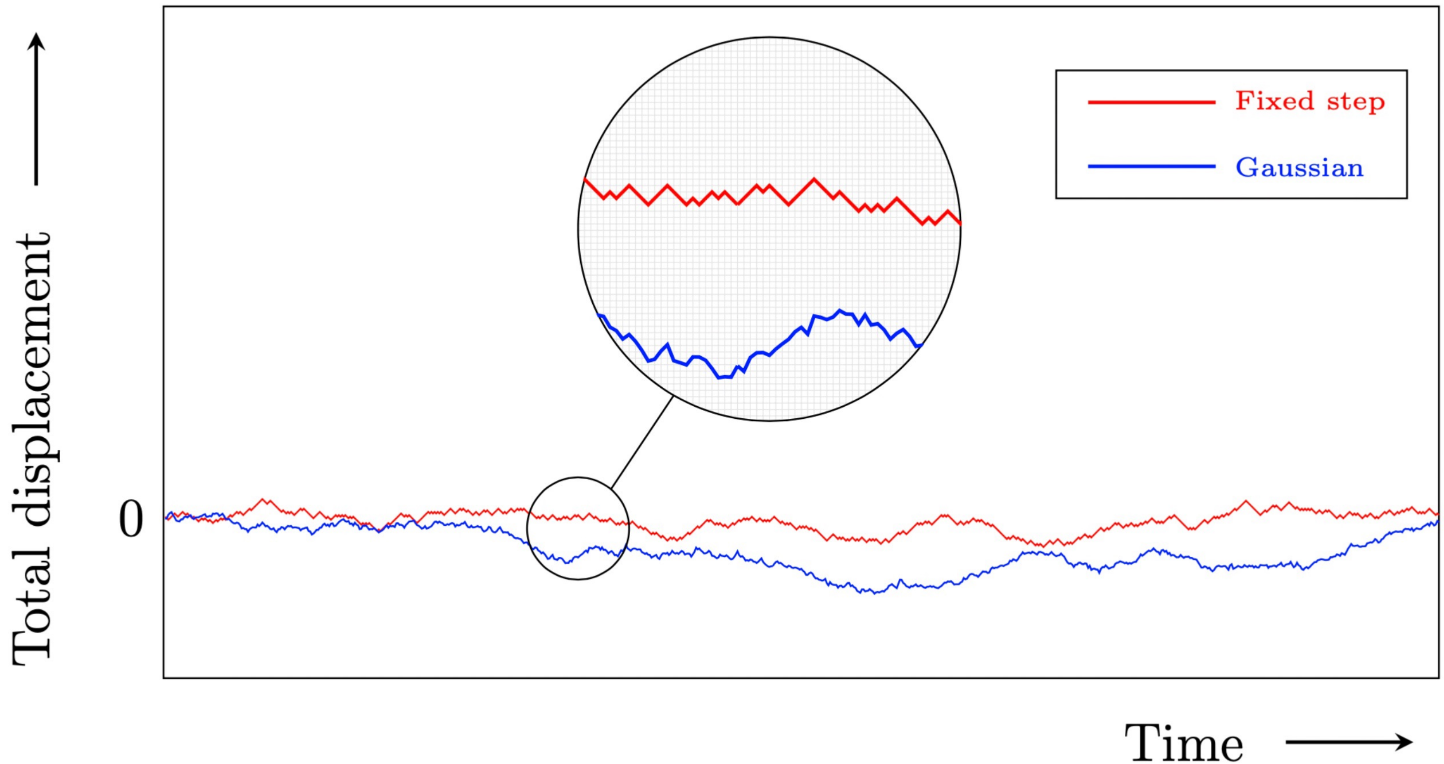
Independent and identically distributed random variable x.

1) Gaussian $\qquad P_g(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$

2) Fixed step $\qquad P_f(x) = \frac{1}{2}\delta(|x|-1)$

$\langle x \rangle = 0$ and $\langle x^2 \rangle - \langle x \rangle^2 = 1$

Same Coarse Grained Predictions



Compute Displacement

How far does walker move in N steps?

$$X = \sum_{i=1}^{N} x_i \Rightarrow P(X) = \int \prod_{i=1}^{N} dx_i \, p(x_i) \, \underbrace{\delta\left(X - \sum_{i=1}^{N} x_i\right)}_{\int \frac{dk}{2\pi} e^{-ik(X - \Sigma x_i)}}$$

$$\Rightarrow P(X) = \int \frac{dk}{2\pi} e^{-ikX} \underbrace{\int \prod_{i=1}^{N} dx_i \, P(x_i) \, e^{ikx_i}}_{\langle e^{ikx} \rangle^N}$$

Large Step Limit

1) Gaussian: $\langle e^{ikx} \rangle_g = e^{-k^2/2}$

$$\Rightarrow \quad P_g(X) = (2\pi N)^{-1/2} \exp\left(-X^2/2N\right)$$

2) Fixed step: $$P_f(X) = \frac{N!}{\left(\frac{N-X}{2}\right)! \left(\frac{N+X}{2}\right)!} 2^{-N}$$

Combinatorics

Stirling's approx $$\Rightarrow \quad P_f(X) \simeq \exp\left(-X^2/2N - X^4/12N^3 + \dots\right)$$

Gaussians!

Central Limit Theorem

Assume expansion $\log\langle e^{ikx}\rangle = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!}\langle x^n\rangle_c$

connected correlators

w/ $\langle x\rangle_c = \langle x\rangle$, $\langle x^2\rangle_c = \langle x^2\rangle - \langle x\rangle^2$, ...

$= \sigma_0^2$

$$P(\bar{X}) = \int \frac{dk}{2\pi} e^{-ik\bar{X}} \exp\left(ikN - \frac{1}{2}Nk^2\sigma_0^2 + \frac{i^3}{3!}Nk^3\langle x^3\rangle_c + \dots\right)$$

If integral support dominated by Gaussian:

$$k \lesssim 1/\sqrt{N}\sigma_0$$

Central Limit Theorem

Gaussian support: $k \lesssim 1/\sqrt{N}\sigma_0$

$$P(X) = \int \frac{dk}{2\pi} e^{-ikX} \exp\left(\overbrace{ikN}^{\sqrt{N}} - \overbrace{\tfrac{1}{2} N k^2 \sigma_0^2}^{O(1)} + \overbrace{\tfrac{i^3}{3!} N k^3 \langle x^3 \rangle_c}^{1/\sqrt{N}} + \dots\right)$$

Rescale (coarse grain): $k \to k/\sqrt{N}$, $X \to \sqrt{N} X$

$$\Rightarrow P(X) \to \frac{1}{\sqrt{N}} \int \frac{dk}{2\pi} \exp\left(ik\left(\sqrt{N}\langle x \rangle - X\right) - \tfrac{1}{2} k^2 \sigma_0^2 + O\left(\tfrac{1}{\sqrt{N}}\right)\right)$$

$$\simeq \exp\left(-X - \sqrt{N}\langle x \rangle\right)^2 / 2\sigma_0^2$$

Central Limit Theorem is RG

"UV insensitive"

In $N \to \infty$ limit:

$\langle x \rangle$: relevant

σ_0: marginal

$\langle x^n \rangle_G$ w/ $n > 2$: irrelevant

Large Deviation Principle

Consider probability of finding random walker distance beyond N after N steps:

$$P(X > N) = ?$$

Expect UV dependence:

Fixed step walker: $P_f(X > N) = 0$

Gaussian walker: $P_g(X > N) \neq 0$

Large Deviation Principle

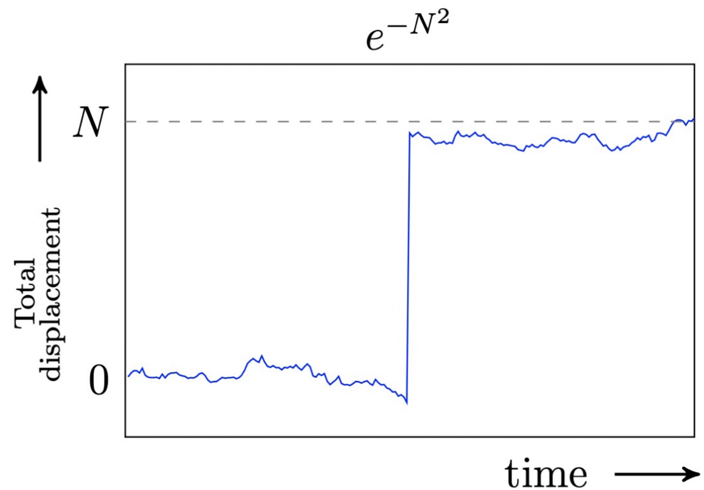
Compute $\langle e^{\theta X} \rangle$ w/ $\theta > 0$

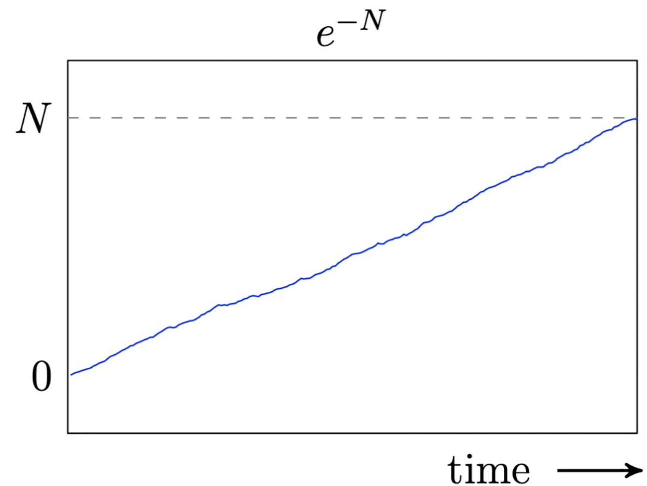
Naively expect $\langle e^{\theta X} \rangle \sim e^{\theta \mathcal{O}(\sqrt{N})}$

But $\langle e^{\theta X} \rangle = \langle e^{\theta x} \rangle^{N} = e^{N\theta^2/2}$

Central limit theorem fails!

Large Deviation Principle





CLT

LDP

Large Deviation Principle

Define "sample mean" $\tilde{X} = X/N$

A distribution satisfies the LDP if

$$P(X) \simeq \exp(-N\, I(\tilde{X}))$$

where $I(\tilde{X})$ is the "rate function"

$$I_g(\tilde{X}) = \tilde{X}^2/2$$

$$I_f(\tilde{X}) = \frac{1}{2}\left[(1-\tilde{X})\ln(1-\tilde{X}) + (1+\tilde{X})\ln(1+\tilde{X})\right]$$

Cramér's Theorem

Distribution of $\tilde{X}$ satisfies LDP w/

$$I(\tilde{X}) = \sup_{\theta}\left[\theta\tilde{X} - \lambda(\theta)\right] \quad \text{w/} \quad \lambda(\theta) = \ln\langle e^{\theta x}\rangle$$

Proof: Assume LDP holds $P(\tilde{X}) \simeq e^{-NI(\tilde{X})}$

Then $\langle e^{\theta X}\rangle = \langle e^{N\theta\tilde{X}}\rangle \simeq \int d\tilde{X}\, e^{N(\theta\tilde{X} - I(\tilde{X}))}$

$$\simeq e^{N \sup_{\tilde{X}}\left[\theta\tilde{X} - I(\tilde{X})\right]}$$

Saddle point

Cramér's Theorem

So we have $\langle e^{\theta X} \rangle \simeq e^{N \sup_{\tilde{X}} \left[\theta \tilde{X} - I(\tilde{X})\right]}$

Also $\langle e^{\theta X} \rangle = \left(\langle e^{\theta x}\rangle\right)^N = e^{N\lambda(\theta)}$

$\Rightarrow \quad \lambda(\theta) = \sup_{\tilde{X}} \left[\theta \tilde{X} - I(\tilde{X})\right]$

Legendre transform: $I(\tilde{X}) = \sup_{\theta} \left[\theta \tilde{X} - \lambda(\theta)\right]$

LDP $\Rightarrow$ New saddle!

From LDP → CLT

If $I(\tilde{X})$ is convex and has single global minimum $\tilde{X}_0$

$$\Rightarrow P(\tilde{X}) \simeq \exp\left(-\frac{1}{2} N I''(\tilde{X}_0)(\tilde{X}-\tilde{X}_0)^2\right)$$

Good approximation for small deviations

⇒ Central Limit Theorem

Eternal Inflation and the LPD

Fokker-Planck for Gaussian theory

$\Rightarrow \quad P_G \sim \exp(-\xi^2/2\sigma^2 t)$

$\Rightarrow$ Typical fluctuations $\xi \sim \sqrt{t}$

Eternal inflation dominated by $\xi \sim t$

Eternal Inflation and the LPD

Eternal inflation dominated by $\xi \sim t$

General solution to F-P eq:

$$P = \exp\left(\cdots \frac{\partial^n}{\partial \xi^n} \right) P_G$$

Write $P = \int \frac{dk}{2\pi} e^{-ik\xi} \rho(k)$

w/ $\rho(k) = \left(\exp\left(-k^2 \frac{\sigma^2}{2} t + \sum_{n \geq 2} (ik)^n \frac{\gamma_n}{n!} t \right) \right)$

$$\equiv \langle \exp(ik\xi) \rangle$$

Eternal Inflation and the LPD

When $\xi = \alpha t$ for some constant α

as $t \to \infty$ steepest descents $\Rightarrow$

$$P \simeq \exp\left(-t\, I(\alpha)\right)$$

w/ $$I(\alpha) = \Big(-i\alpha k_*(\alpha) - k_*^2(\alpha)\frac{\sigma^2}{2} + \sum_{n>2} \left(i k_*(\alpha)\right)^n \frac{\gamma_n}{n!}\Big)$$

and $$\Big(i \sum (i k_*(\alpha))^n \frac{\gamma_{n+1}}{n!}\Big) - k_* \sigma^2 = i\alpha$$

Cramér's Thm!

Eternal Inflation is UV Sensitive

Phase transition to eternal inflation probes the tail of the distribution

Sensitive to new saddle of the path integral

Breakdown of EFT of Inflation

$$\alpha f_\pi^2 > \Lambda^2$$

Outlook

Fluctuations of fields in dS governed by RG in SdSET

Typical fluctuations governed by coarse grained RG $\Rightarrow$ CLT

Rare fluctuations governed by LDP $\Rightarrow$ UV sensitive