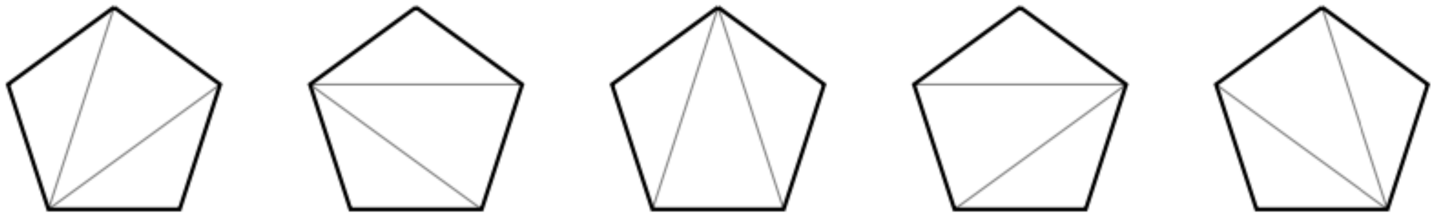
Differential Equations for Cosmological Correlators



Hayden Lee

University of Chicago

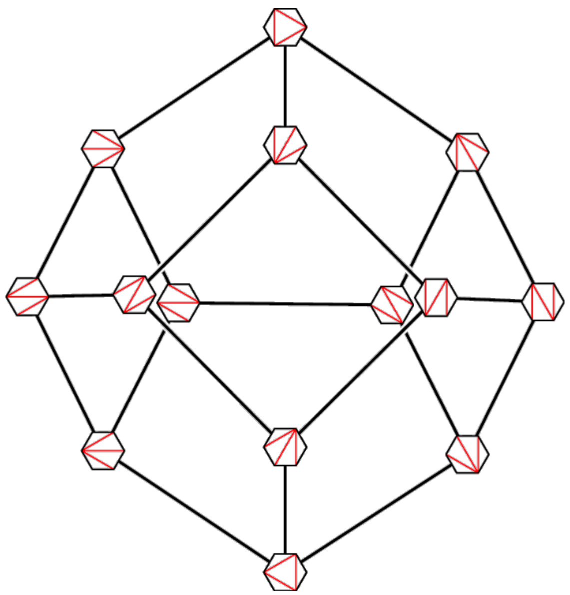
w/ N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, G. Pimentel

[to appear]

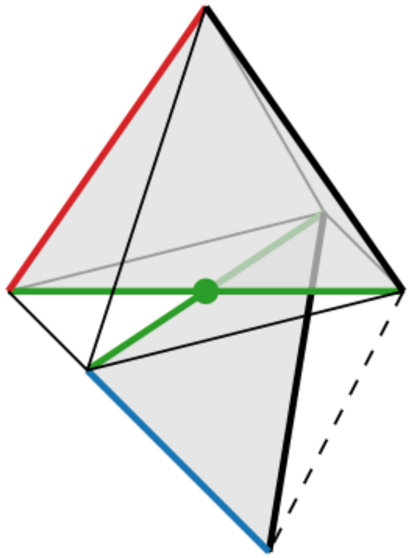
Nordita [7/26/2023]

Over the past decade, we have seen scattering amplitudes emerge from new mathematical structures in boundary kinematic space.

amplituhedron



associahedron



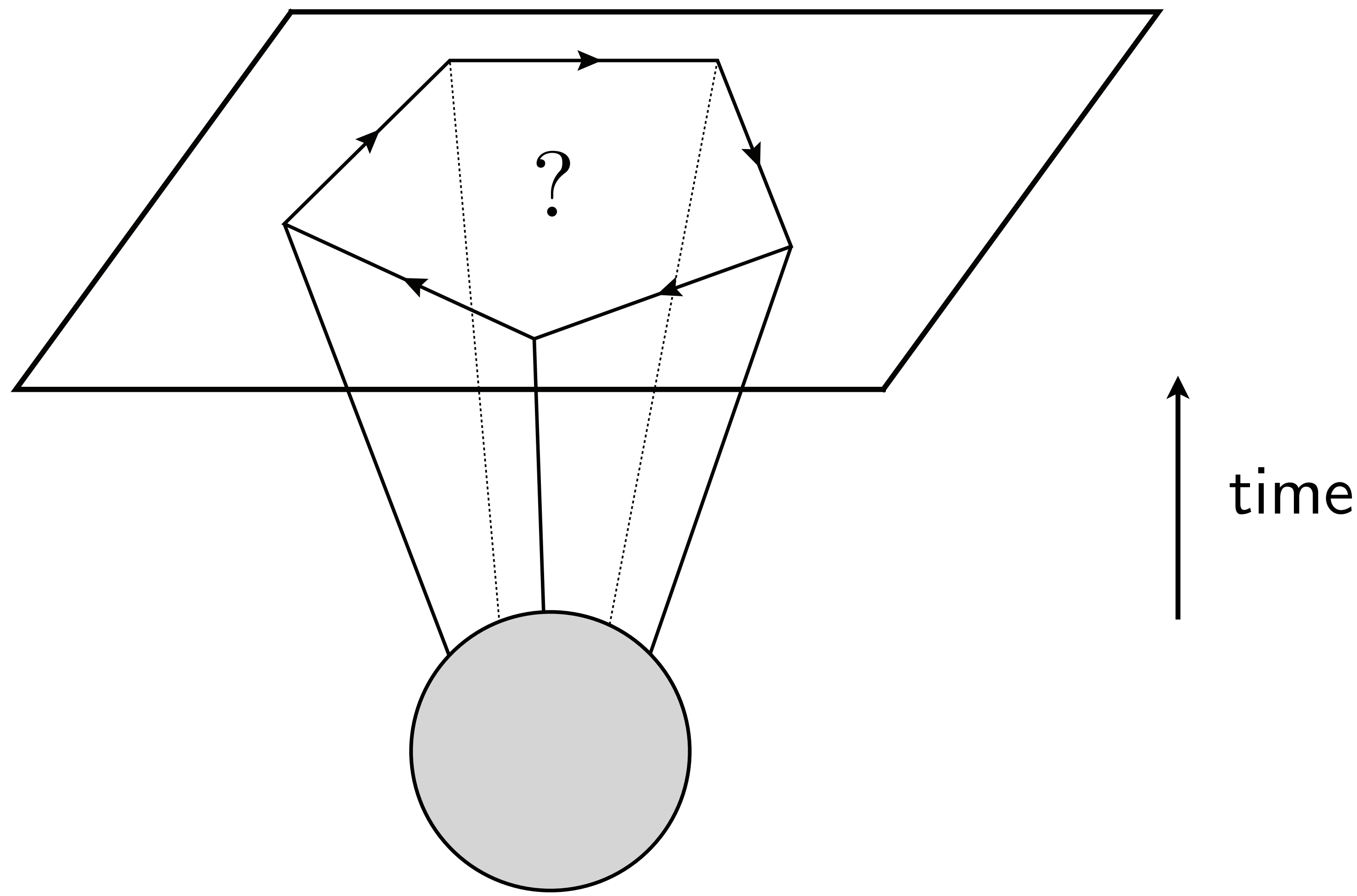
cosmological polytopes

conceptual advantage: focuses directly on observables

practical advantage: simplifies calculations

We can't directly observe the pre-Big Bang evolution of the universe,

but instead must infer it from spatial correlations on the future boundary.

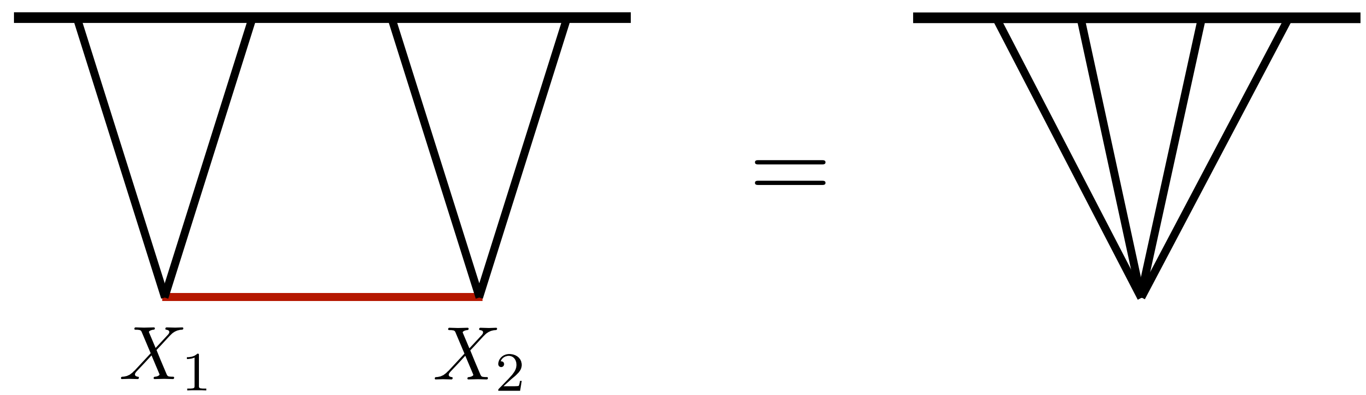


How can we see "time evolution" from boundary correlators?

In de Sitter space, conformal symmetry implies that

boundary correlators satisfy interesting, local differential equations.

$$(\Delta_{X_1} - \mu^2)$$



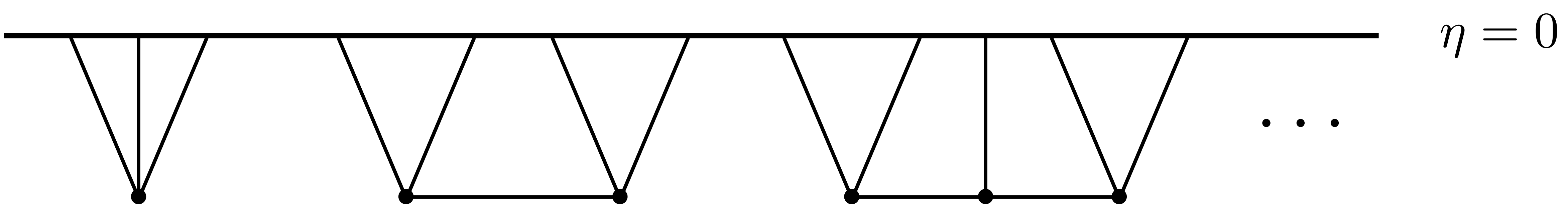
$$\Delta_X = (X^2 - 1)\partial_X^2 + 2X\partial_X$$

Arkani-Hamed, Maldacena [2015]
Arkani-Hamed, Baumann, HL, Pimentel [2018]

Is there a deeper reason for their existence beyond de Sitter?

Cosmological Wavefunction

Consider the wavefunction of conformally-coupled scalars in FRW.



with

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \frac{\lambda}{3!}\phi^3 \right]$$

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)\,, \quad a(\eta) \propto \frac{1}{\eta^{1+\varepsilon}}$$

$\varepsilon = 0$: dS

$\varepsilon = -1$: flat

$\varepsilon = -2$: radiation

$\varepsilon = -3$: matter

Cosmological Wavefunction

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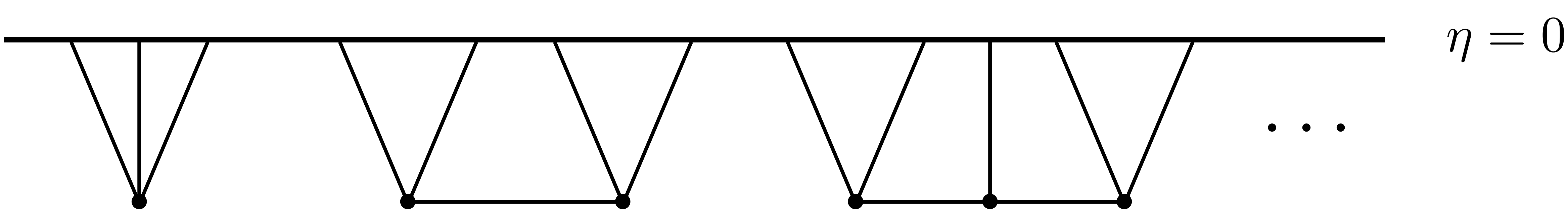
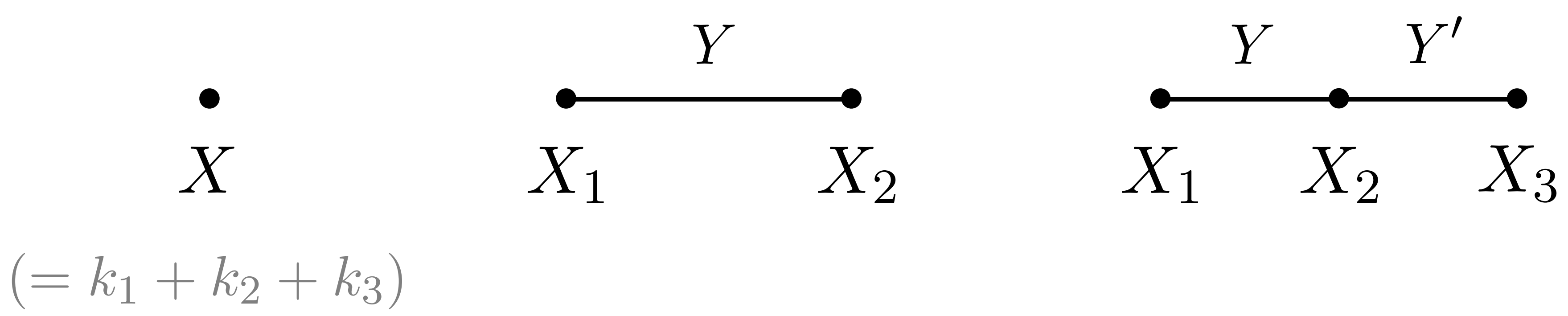


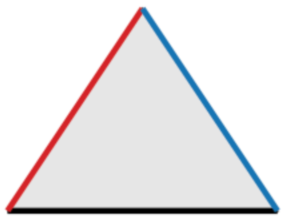
We can associate these Feynman diagrams with graphs, where bulk-to-boundary propagators are truncated.



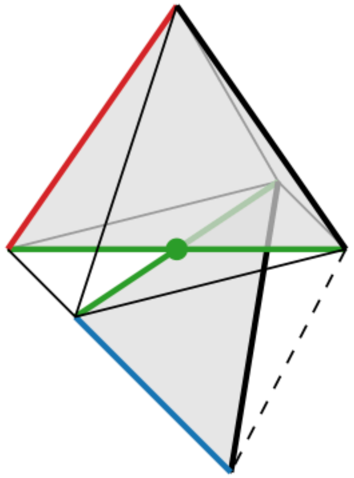
Wavefunction in Minkowski

Flat-space wavefunction is described by rational functions with simple poles.

$$= \frac{1}{(X_1 + Y)(X_2 + Y)(X_1 + X_2)} \Leftrightarrow$$



$$= \frac{X_1 + 2X_2 + X_3 + Y + Y'}{(X_1 + Y)(X_2 + Y + Y')(X_3 + Y')(X_{12} + Y)(X_{23} + Y')X_{123}} \Leftrightarrow$$



It also has an interesting geometric origin (“cosmological polytopes”).

Arkani-Hamed, Benincasa, Postnikov [2017]

Wavefunction in FRW

Wavefunction in FRW is related to the flat-space one as

$$\psi_{\mathrm{FRW}} = \int_0^\infty dx_1 \cdots dx_n \, (x_1 \cdots x_n)^\varepsilon \, \psi_{\mathrm{flat}}(X_i \to x_i + X_i)$$

twist

For example, a two-site graph has the integral representation

$$= \int_0^\infty dx_1 dx_2 \, \frac{(x_1 x_2)^\varepsilon}{(x_1 + X_1 + 1)(x_2 + X_2 + 1)(x_1 + x_2 + X_1 + X_2)}$$

This integral can be computed using the method of differential equations.

Family of Integrals

Consider a family of integrals with the same singularities.

$$I_{\vec{n}} = \int (x_1 x_2)^{\varepsilon}\, \Omega_{\vec{n}}\,, \quad \Omega_{\vec{n}} = \frac{dx_1 \wedge dx_2}{T_1^{m_1} T_2^{m_2} B_1^{n_1} B_2^{n_2} B_3^{n_3}}$$

$$T_1 = x_1\,, \quad B_1 = x_1 + X_1 + 1$$

$$T_2 = x_2\,, \quad B_2 = x_2 + X_2 + 1\,, \quad B_3 = x_1 + x_2 + X_1 + X_2$$

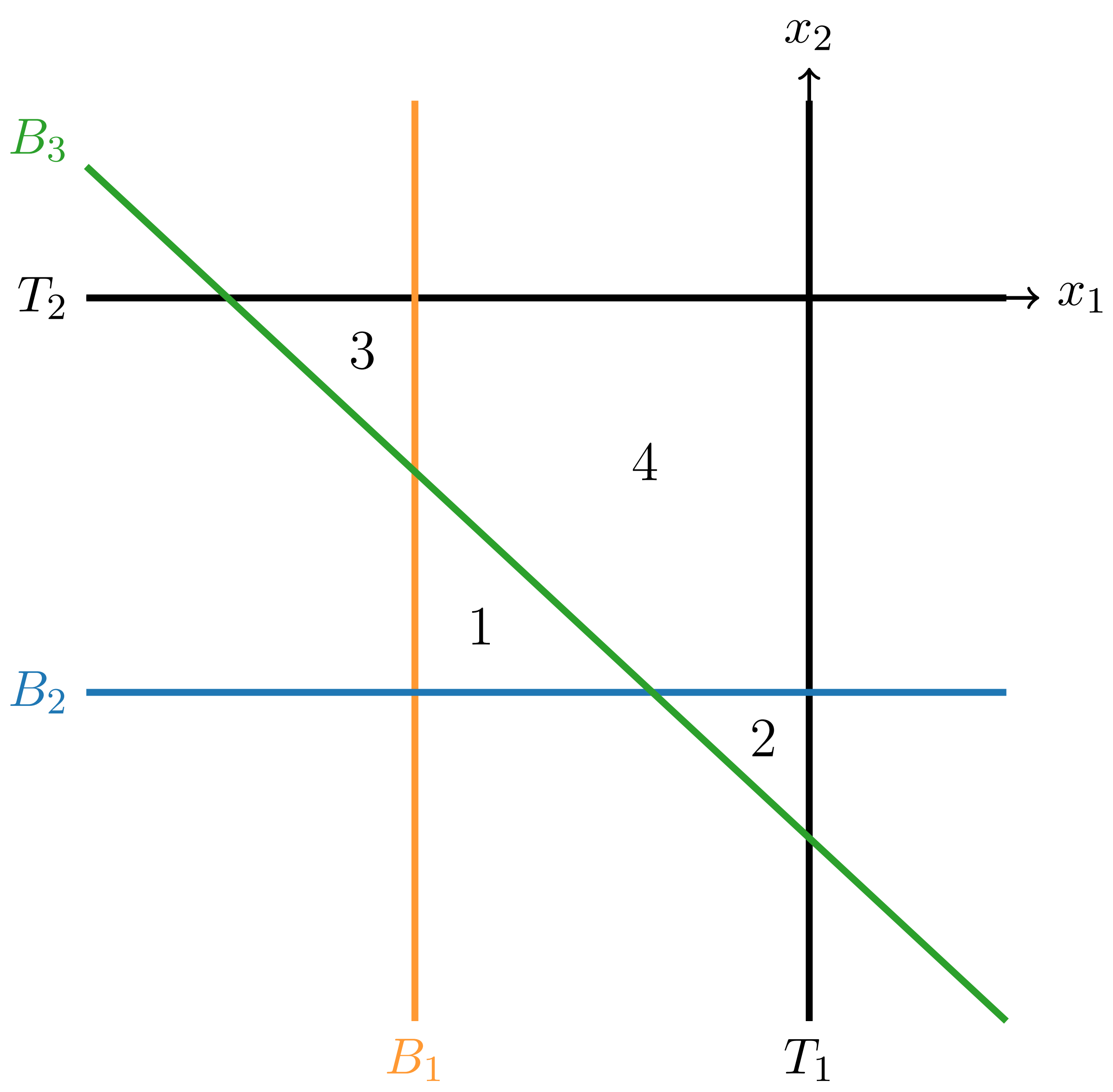
These integrals form a finite-dimensional vector space.

(Twisted) cohomology provides a geometric way to determine the size of this vector space.

Aomoto, Kita [2011]
Mastrolia, Mizera [2018]

Master Integrals

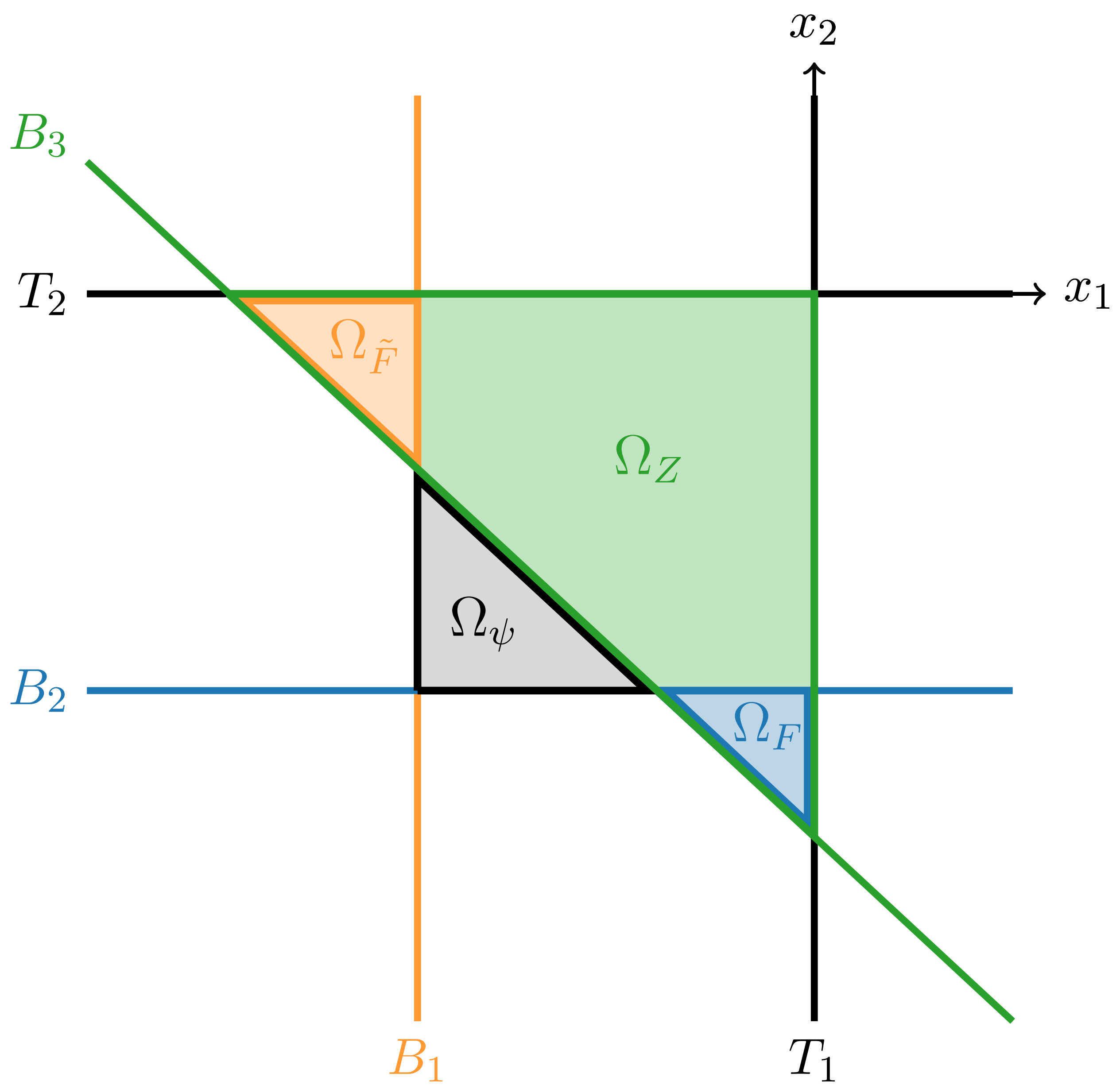
The number of independent master integrals equals the number of bounded regions defined by the singular divisors of the integrand.



$$T_1 = x_1$$
$$T_2 = x_2$$
$$B_1 = x_1 + X_1 + 1$$
$$B_2 = x_2 + X_2 + 1$$
$$B_3 = x_1 + x_2 + X_1 + X_2$$

Master Integrals

A good choice for the basis of integrals is given by the canonical forms of bounded regions.



$$\vec{I} = \begin{bmatrix} \psi \\ F \\ \tilde{F} \\ Z \end{bmatrix} = \int (x_1 x_2)^\varepsilon \begin{bmatrix} \Omega_\psi \\ \Omega_F \\ \Omega_{\tilde{F}} \\ \Omega_Z \end{bmatrix}$$

$$\Omega_{\Delta_{123}} = d\log\left(\frac{L_1}{L_3}\right) \wedge d\log\left(\frac{L_2}{L_3}\right)$$

Differential Equations

Taking the differential leads to first-order differential equations:

$$\mathrm{d}\vec{I} = \varepsilon A \vec{I}$$

$$\mathrm{d} = \sum_i \mathrm{d}X_i \frac{\partial}{\partial X_i} \qquad \mathrm{d}A = A \wedge A = 0$$

with

letters

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathrm{d}\log(X_1 + X_2) + \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 + 1) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 - 1)$$

$$+ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 + 1) + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 - 1)$$

The Solution

The solution after imposing appropriate boundary conditions is

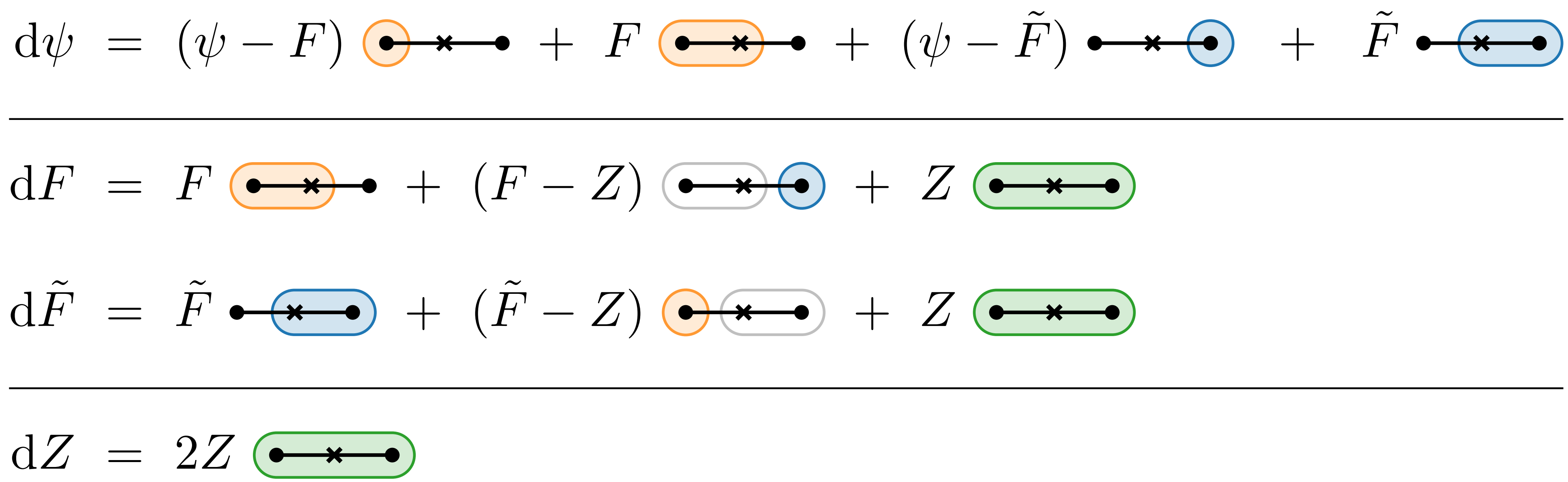
$$\psi = c_A(\varepsilon)(1+X_1)^{\varepsilon}(1+X_2)^{\varepsilon}$$
$$+ c_B(\varepsilon)(X_1+X_2)^{2\varepsilon}\left(1 - {}_2F_1\left[\begin{matrix}1,\varepsilon\\ 1-\varepsilon\end{matrix}\,\middle|\,\frac{1-X_2}{1+X_1}\right] - {}_2F_1\left[\begin{matrix}1,\varepsilon\\ 1-\varepsilon\end{matrix}\,\middle|\,\frac{1-X_1}{1+X_2}\right]\right)$$

This satisfies a local, inhomogeneous second-order equation.

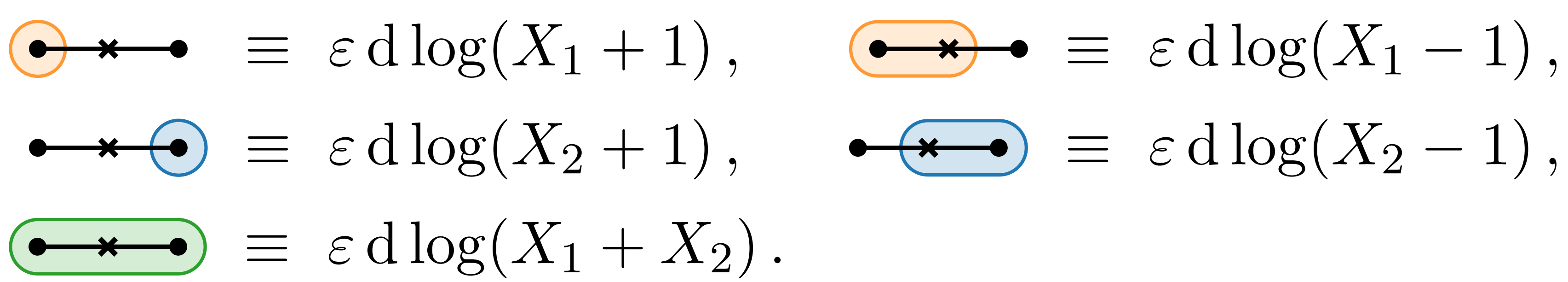
$$\left[(X_1^2-1)\partial_{X_1}^2 + 2(1-\varepsilon)X_1\partial_{X_1} - \varepsilon(1-\varepsilon)\right]\psi = \frac{1}{(X_1+X_2)^{1-2\varepsilon}}$$

Graphical Representation

We may also express the differential equations graphically as

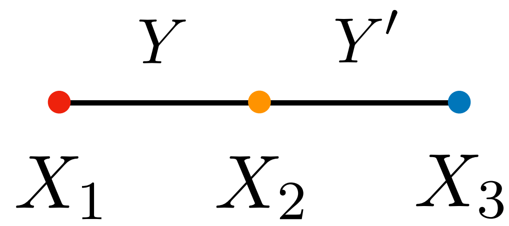


with

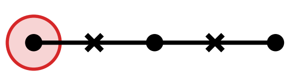


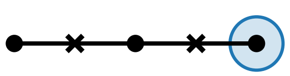
More Complex Example

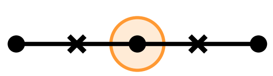
A similar pattern holds for a three-site graph:

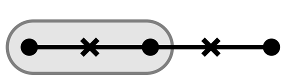


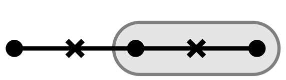
The letters are given by connected tubings of a marked graph.

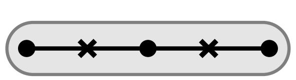
 $\equiv \varepsilon \, \mathrm{d}\log(X_1 + Y)$

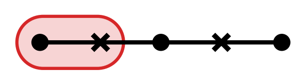
 $\equiv \varepsilon \, \mathrm{d}\log(X_3 + Y')$

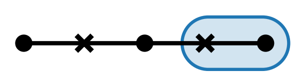
 $\equiv \varepsilon \, \mathrm{d}\log(X_2 + Y + Y')$

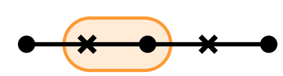
 $\equiv \varepsilon \, \mathrm{d}\log(X_1 + X_2 + Y')$

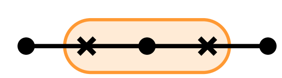
 $\equiv \varepsilon \, \mathrm{d}\log(X_2 + X_3 + Y)$

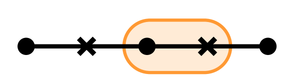
 $\equiv \varepsilon \, \mathrm{d}\log(X_1 + X_2 + X_3)$

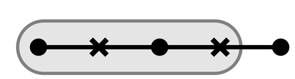
 $\equiv \varepsilon \, \mathrm{d}\log(X_1 - Y)$

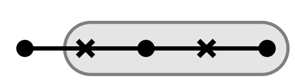
 $\equiv \varepsilon \, \mathrm{d}\log(X_3 - Y')$

 $\equiv \varepsilon \, \mathrm{d}\log(X_2 - Y + Y')$

 $\equiv \varepsilon \, \mathrm{d}\log(X_2 - Y - Y')$

 $\equiv \varepsilon \, \mathrm{d}\log(X_2 + Y - Y')$

 $\equiv \varepsilon \, \mathrm{d}\log(X_1 + X_2 - Y')$

 $\equiv \varepsilon \, \mathrm{d}\log(X_2 + X_3 - Y)$

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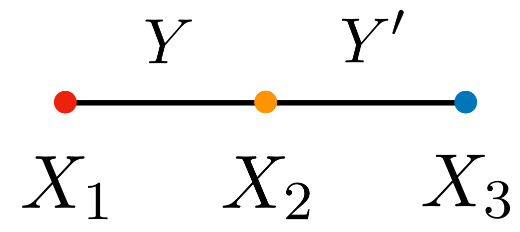
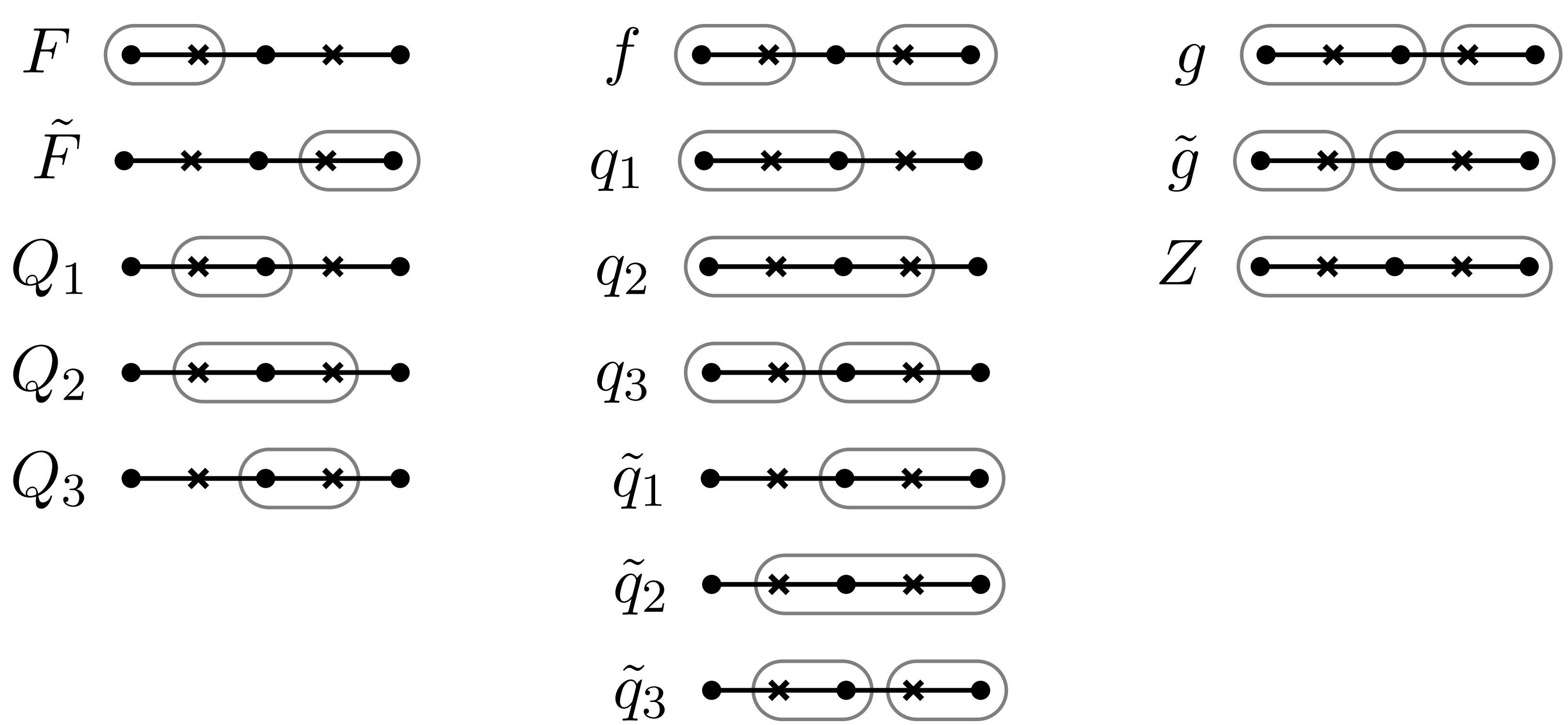
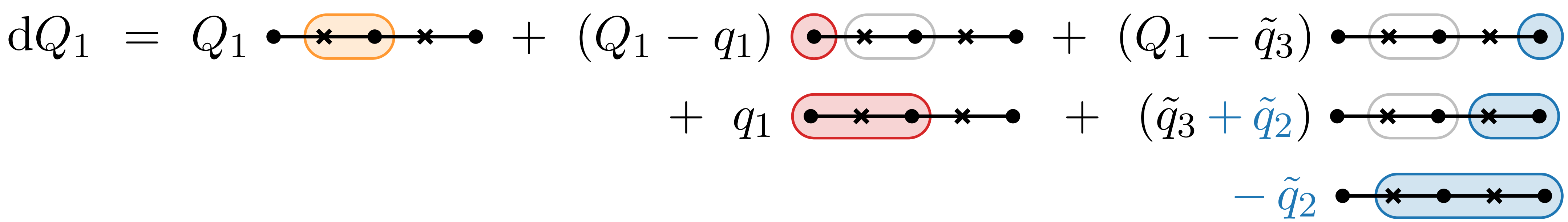


The sources are given by (disjoint) tubings that enclose at least one cross.



Time Evolution as Boundary Flow

Taking the differential of one of the source functions gives



The equation can be predicted using simple graphical rules.

Time Evolution as Boundary Flow

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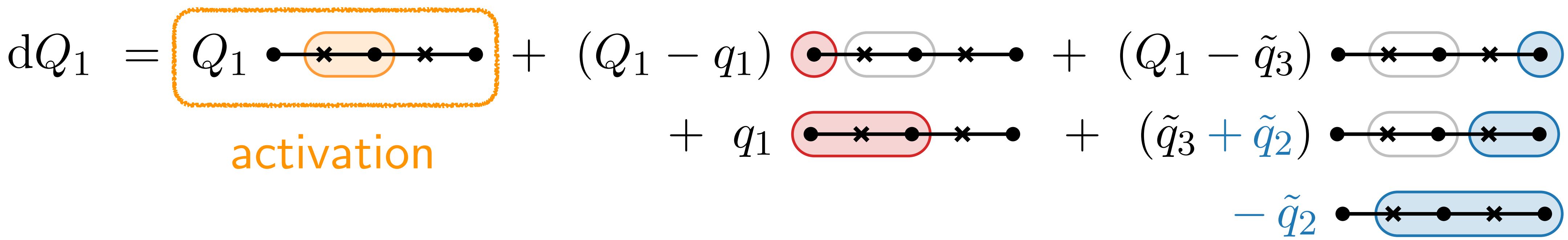

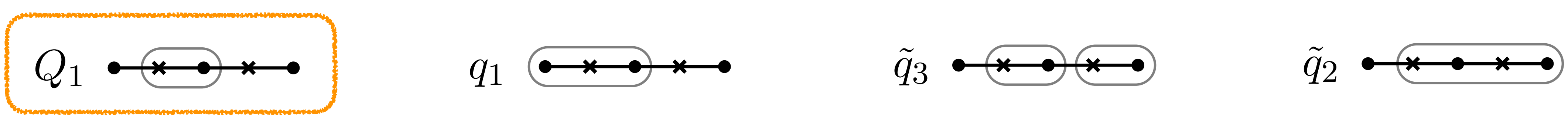

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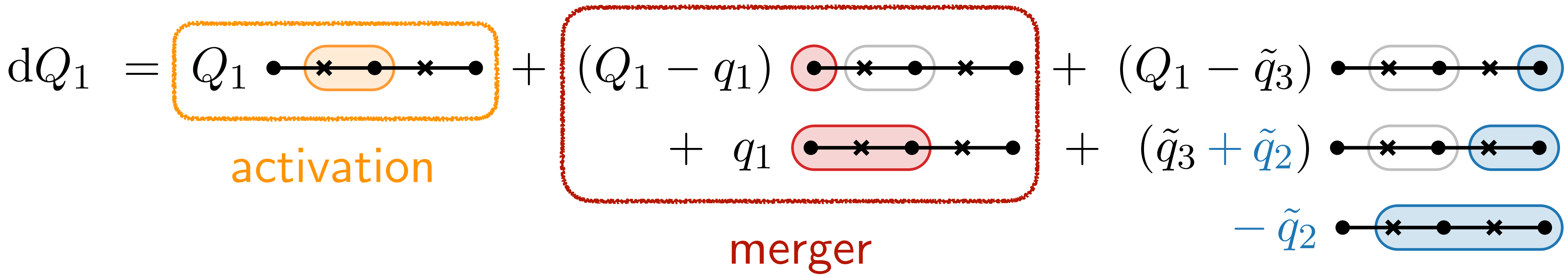

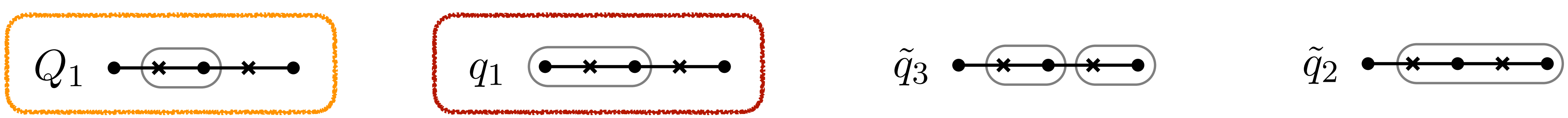

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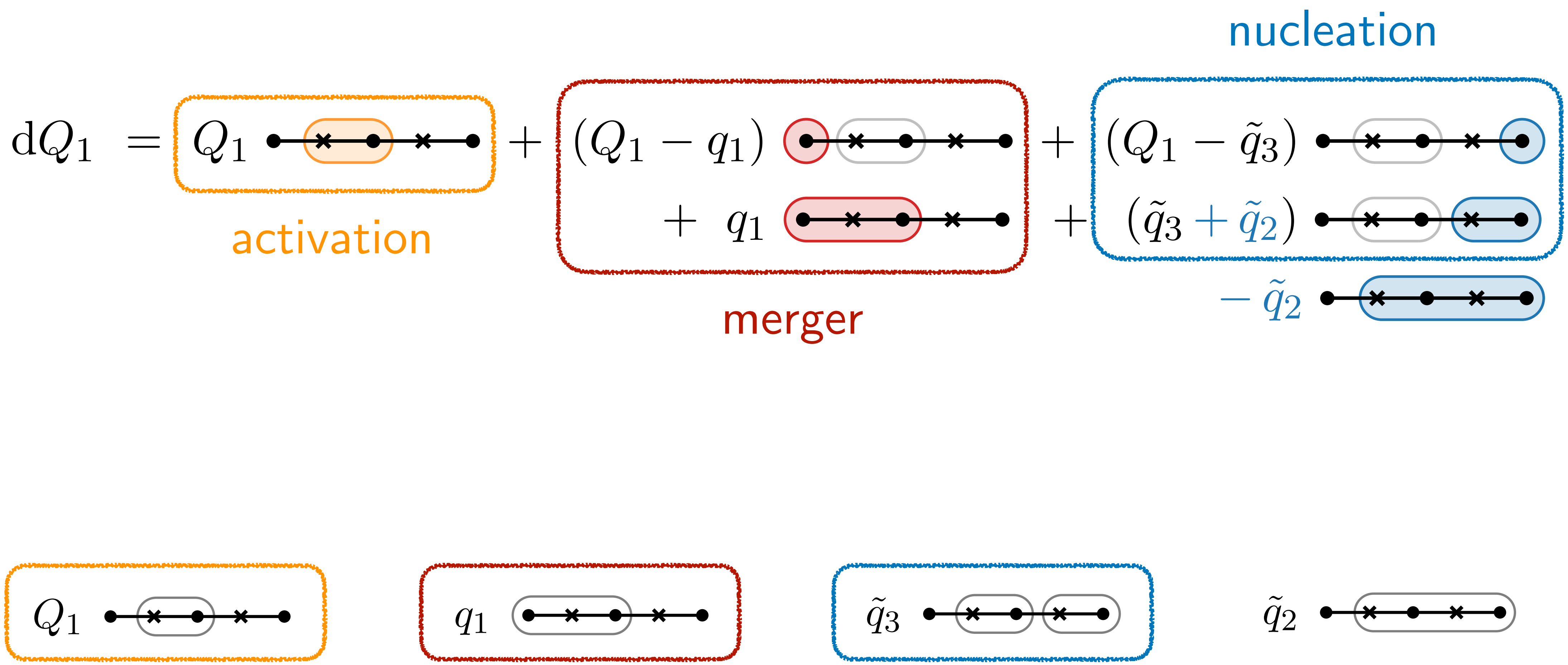

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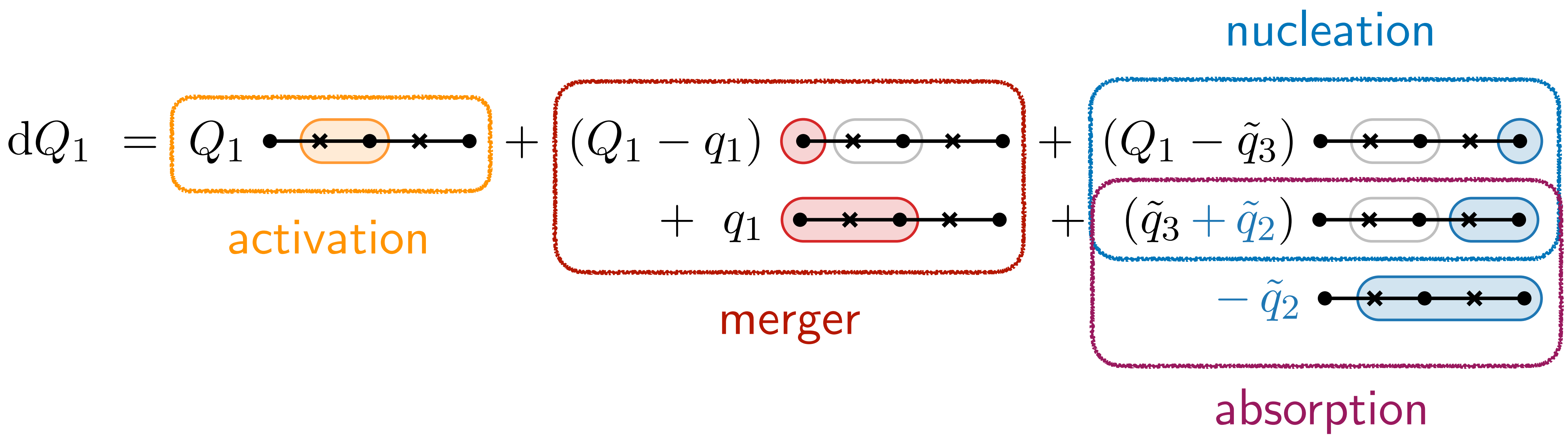

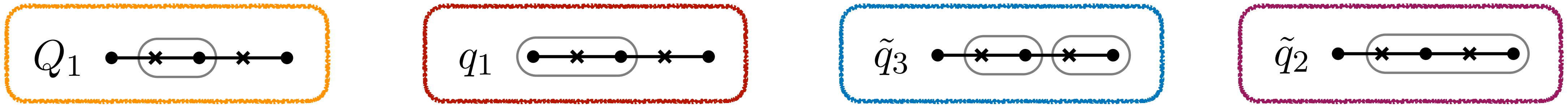

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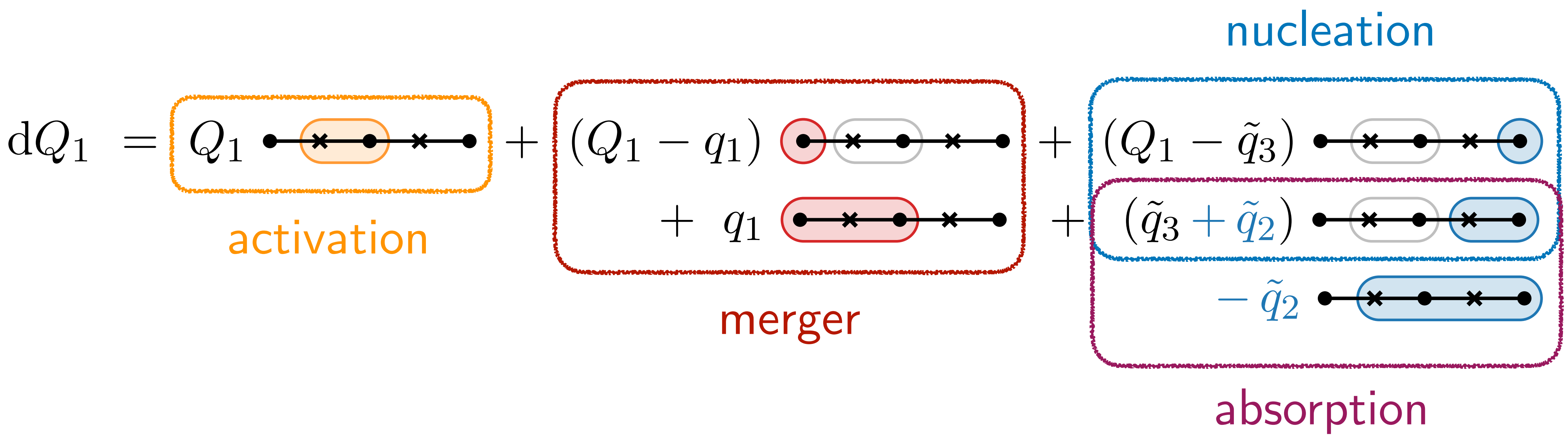
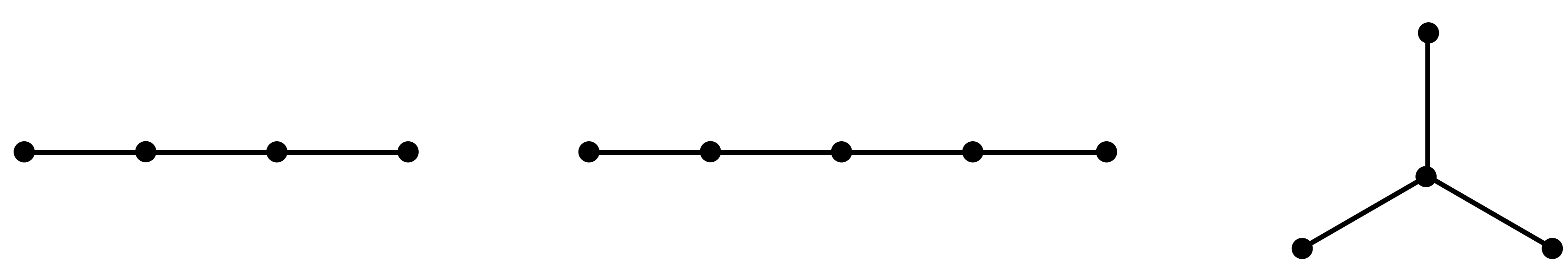


These graphical rules are universal, which can be used to predict the differential equations for arbitrary tree graphs!



Space of Functions

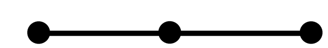
The functions arising at tree level have an iterative structure.

power law

power law

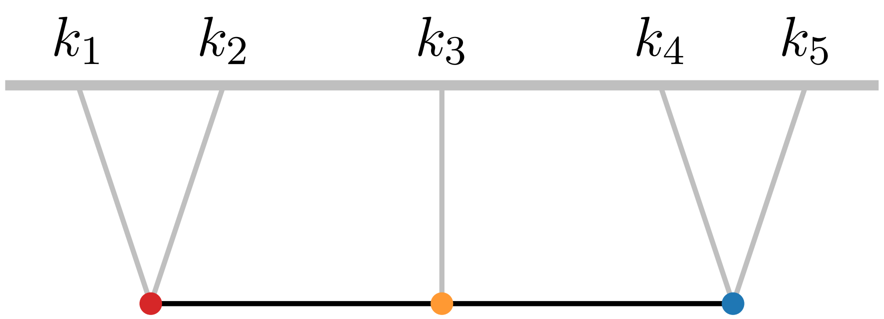
hypergeometric $= \int$ power law

Kampé de Fériet $= \int$ hypergeometric

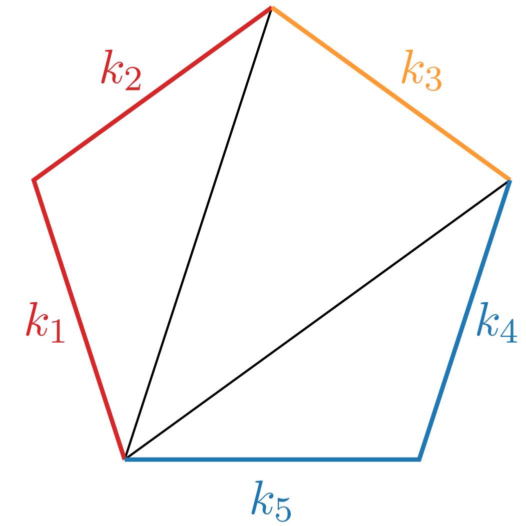


Beyond Single Graphs

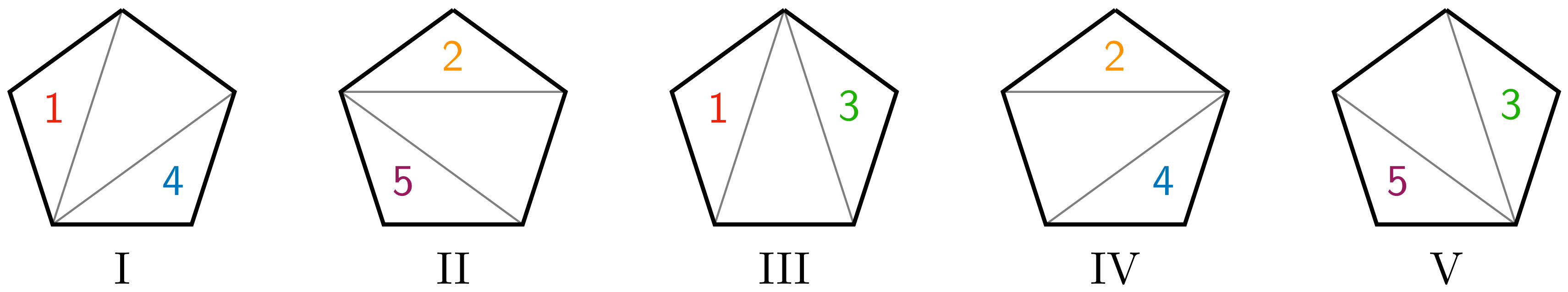
A graph corresponds to a specific triangulation of a kinematic polygon.



$\Leftrightarrow$

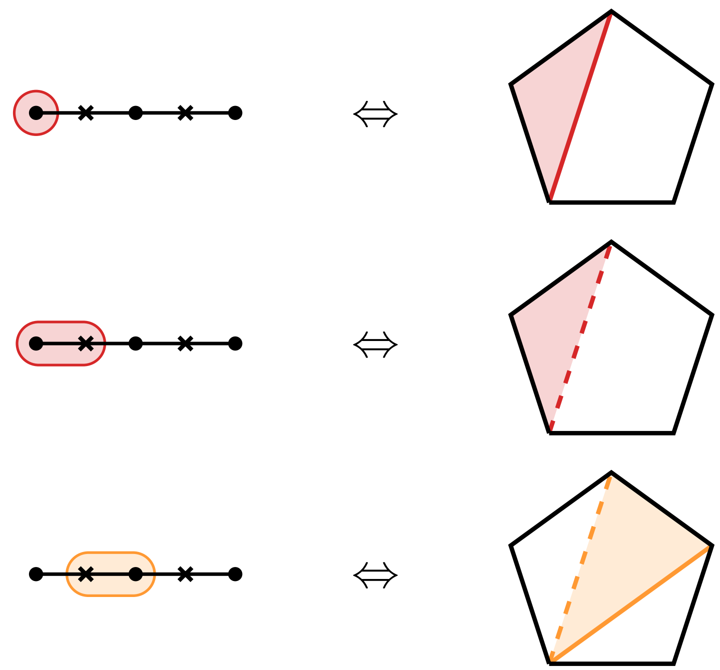


If the scalars have a color charge (e.g. bi-adjoint ϕ^3), then there are additional triangulations corresponding to distinct permutations.



Beyond Single Graphs

Letters are now represented by subpolygons with (dashed) internal edges.

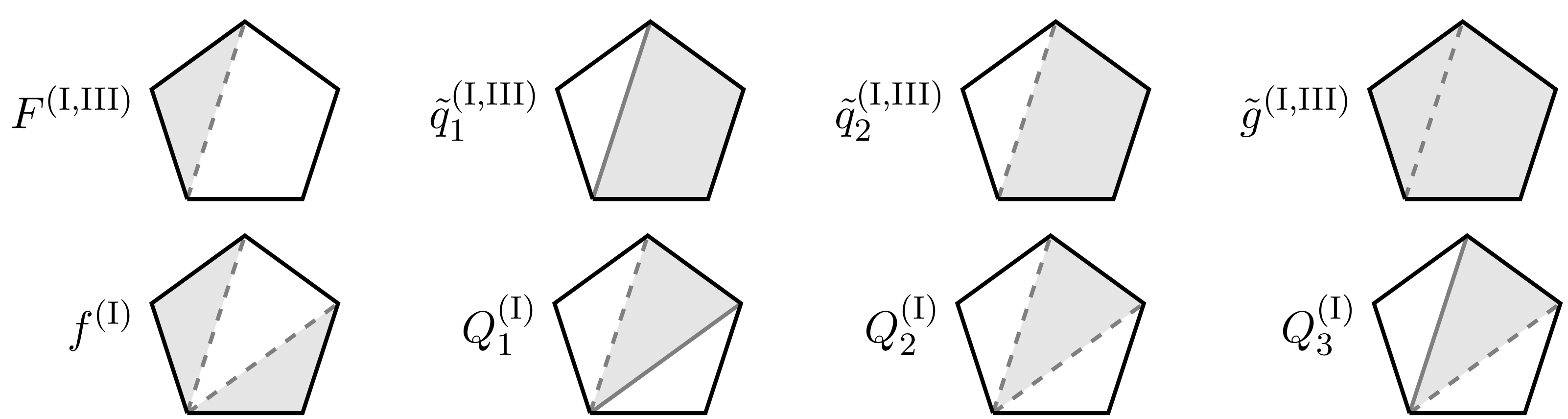


$\equiv \mathrm{d}\log(k_{12} + y_{12})$

$\equiv \mathrm{d}\log(k_{12} - y_{12})$

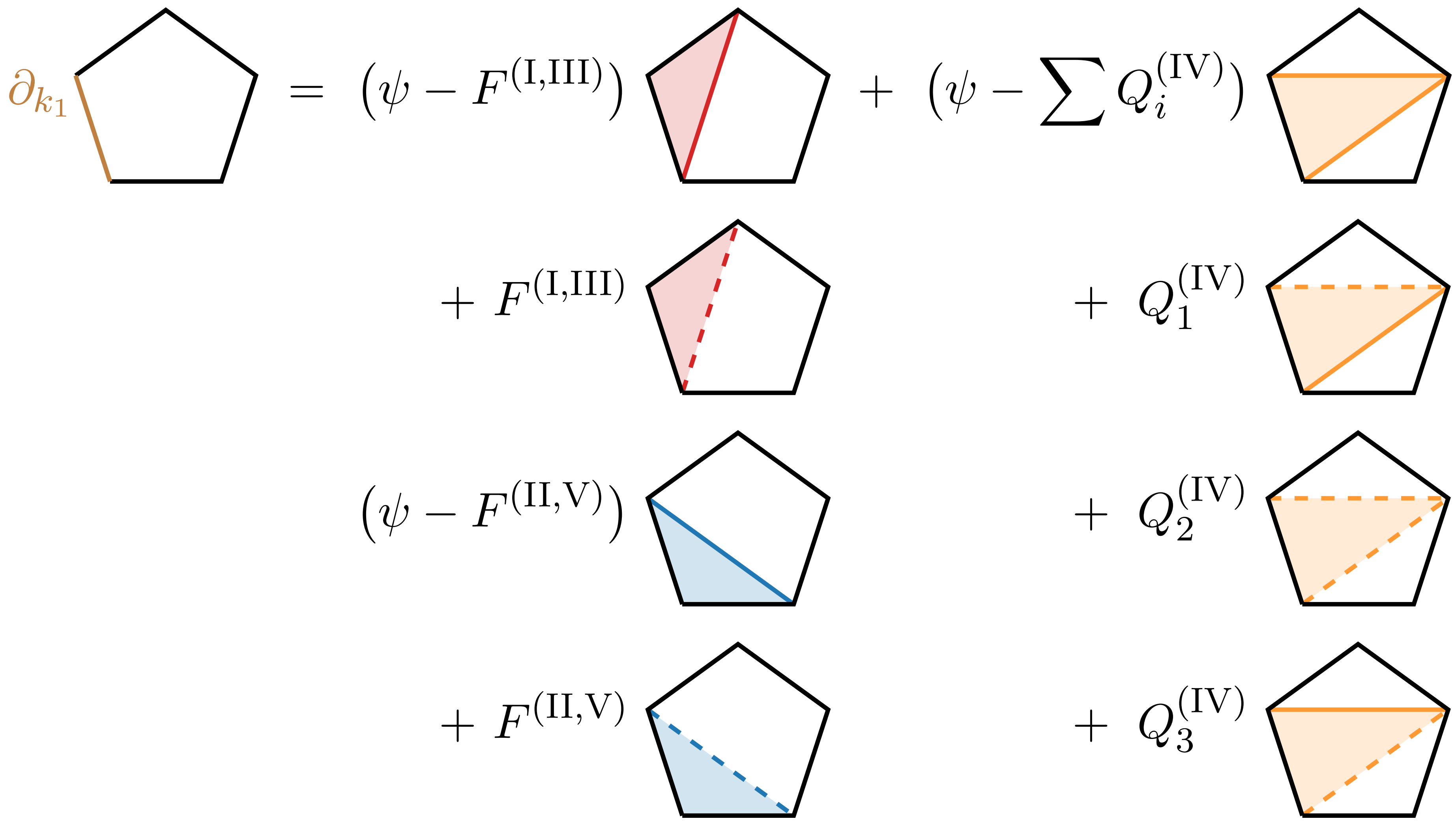
$\equiv \mathrm{d}\log(k_3 - y_{12} + y_{45})$

Sources are given by subpolygons with at least one dashed internal edge.



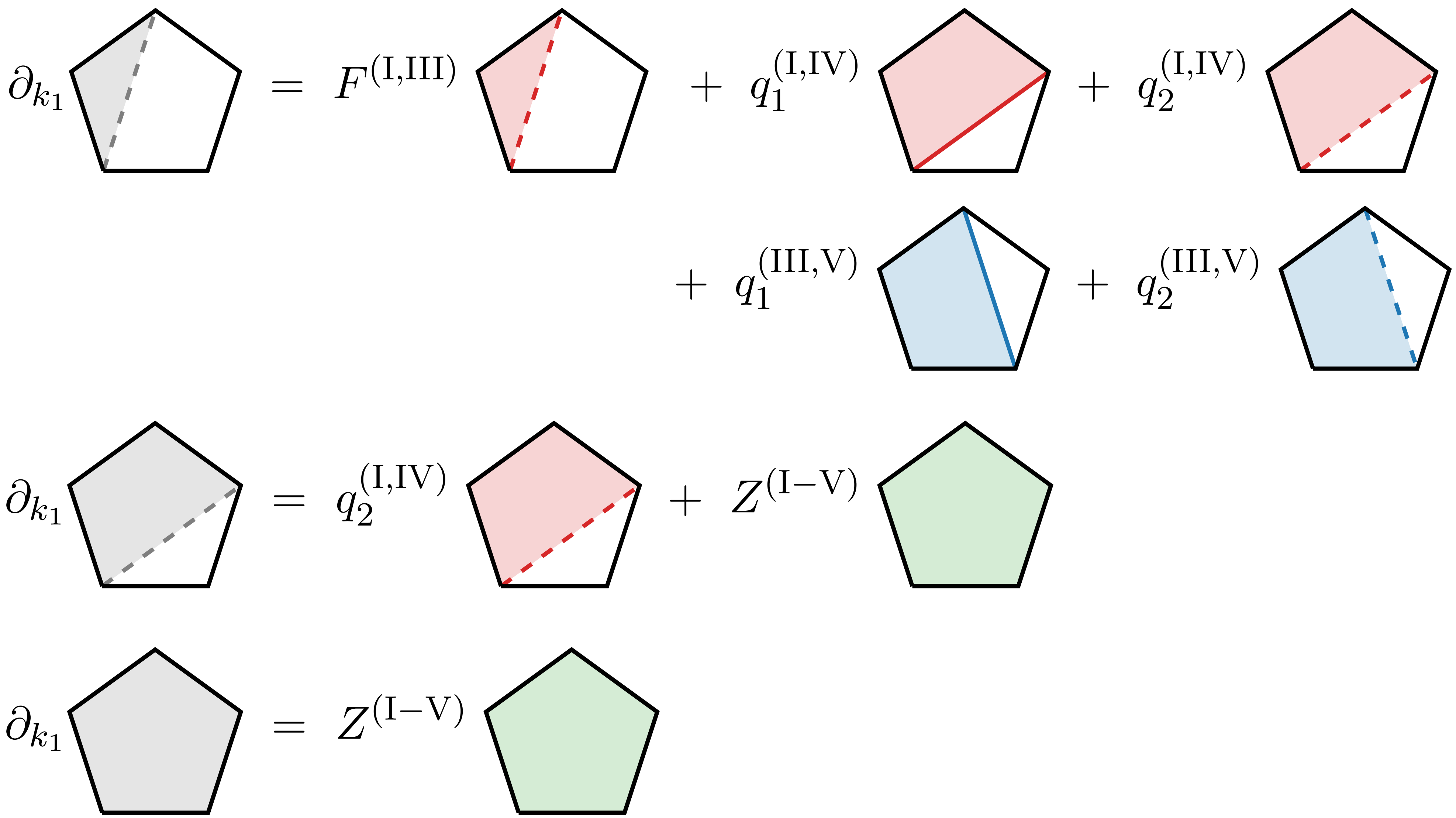
Beyond Single Graphs

For example, taking the k_1 derivative of the 5-point function gives



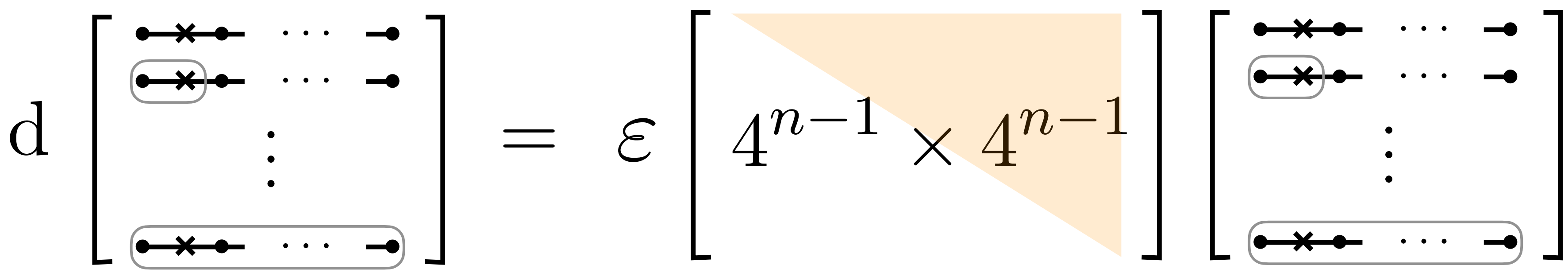
Beyond Single Graphs

The system of equations closes when the subpolygon is fully grown.



Outlook

We have found a completely systematic, graphical way of deriving the differential equations for cosmological correlators at tree level.



Many important questions still need to be addressed:

- loops?
- spin?
- massive particles?