# Progress in gravitational self-force theory: advances in modelling asymmetric binaries

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*From Amplitudes to Gravitational Waves*

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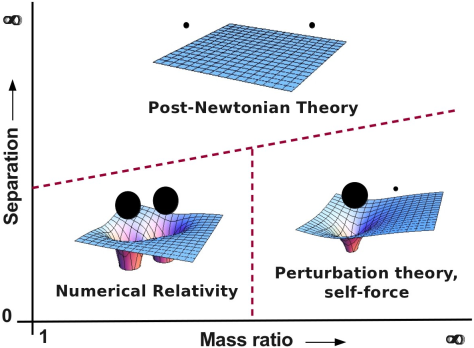
# Gravitational waves and the two-body problem

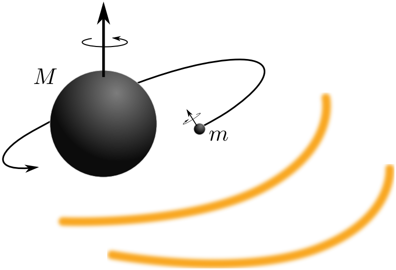
- next-gen detectors will see a much wider variety of binaries, with greater precision
- already detecting mass ratios ≈ 1:26 (GW191219_163120)
- we need new and more accurate models

Binary parameter space



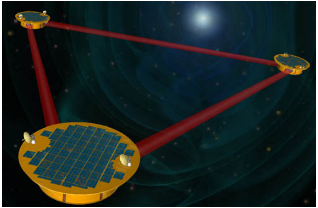

[Image credit: Leor Barack]

# Extreme-mass-ratio inspirals (EMRIs)




- LISA will observe inspirals of stellar-mass BHs or neutron stars into massive BHs

- stellar object spends $\sim M/m \sim 10^5$ orbits near BH $\Rightarrow$ unparalleled probe of strong-field region around BH



# Gravitational self-force theory—not just EMRIs!

- small body perturbs a spacetime:

$$\mathrm{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \dots$$

  where $\epsilon \propto m$

- this deformation of the geometry affects $m$'s motion
  $\Rightarrow$ exerts a *self-force*

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)} + \epsilon^2 f^\mu_{(2)} + \dots$$

# Outline

1. Self-force theory: the fundamentals
2. Self-force theory and asymmetric binaries
3. Results at second order: post-adiabatic waveforms
4. Merger and ringdown

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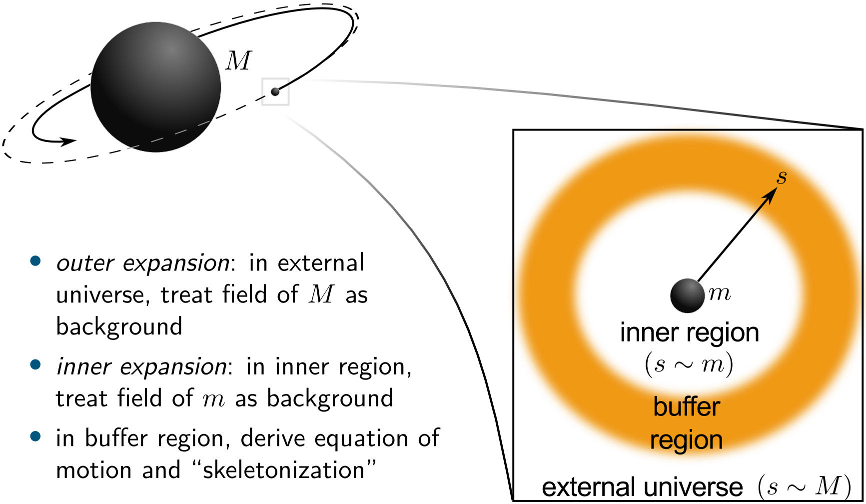
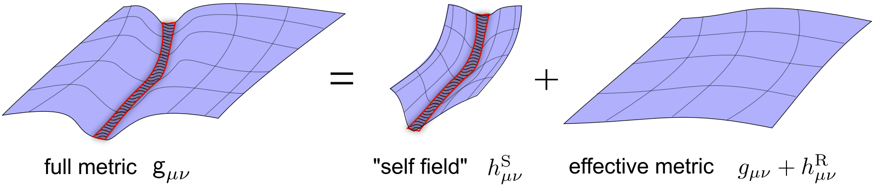
# Matched asymptotic expansions




- *outer expansion*: in external universe, treat field of $M$ as background
- *inner expansion*: in inner region, treat field of $m$ as background
- in buffer region, derive equation of motion and "skeletonization"

# Self-field and effective field [Detweiler & Whiting; Harte; AP]

- local solution to EFE in buffer region splits into a "self-field" and an effective metric




- $h^{\rm S}_{\mu\nu}$ directly determined by object's multipole moments
- $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\rm R}_{\mu\nu}$ is a *smooth vacuum metric* determined by global boundary conditions

# Equations of motion and skeletonization

[MiSaTa; QuWa; Gralla & Wald; Harte; AP]

EFE in buffer region determines the following:

1. equation of motion for object's effective center of mass [AP 2012]:

$$\frac{\tilde{D}^2 z^\mu}{d\tilde{\tau}^2} = O(\epsilon^3)$$

   (geodesic motion in $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$)

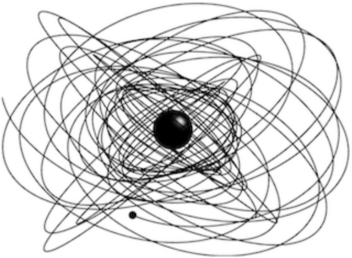
2. extended object can be replaced by a point particle [D'Eath; Gralla & Wald; Upton & AP 2021]

$$T^{\mu\nu} = m \int \tilde{u}^\mu \tilde{u}^\nu \frac{\delta^4(x-z)}{\sqrt{-\tilde{g}}} d\tilde{\tau} + O(\epsilon^3)$$

   ("Detweiler stress-energy": point mass in $\tilde{g}_{\mu\nu}$)

# Outline

# Zeroth order: test mass on a geodesic in Kerr



[image courtesy of Steve Drasco]

- geodesic characterized by three constants $J_A = (E, L_z, Q)$:
  1. energy $E$
  2. angular momentum $L_z$
  3. Carter constant $Q$, related to orbital inclination

- phases $\varphi_A = (\varphi_r, \varphi_\theta, \varphi_\phi)$ with frequencies $\frac{d\varphi_A}{dt} = \Omega_A(J_B)$

# Hierarchy of self-force models [Hinderer & Flanagan]

- self-force causes $\{E, L_z, Q\}$ to slowly evolve
  $\Rightarrow$ *two time scales*: orbital time $\sim 2\pi/\Omega$ and radiation-reaction time $\sim 2\pi/(\epsilon\Omega)$
- on radiation-reaction time, the orbital phases have an expansion

$$\varphi_A = \epsilon^{-1}\varphi_A^{(0)}(\epsilon t) + \epsilon^0\varphi_A^{(1)}(\epsilon t) + O(\epsilon)$$

- a model that gets $\varphi_A^{(0)}$ and $\varphi_A^{(1)}$ right should be enough for precise parameter extraction

# Hierarchy of self-force models [Hinderer & Flanagan]

**Adiabatic order**

determined by

- dissipative piece of $f_1^\mu$

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# Hierarchy of self-force models [Hinderer & Flanagan]

**Adiabatic order**

determined by

- dissipative piece of $f_1^\mu$

**First post-adiabatic order**

determined by

- dissipative piece of $f_2^\mu$
- conservative piece of $f_1^\mu$

- self-force causes $\{E, L_z, Q\}$ to slowly evolve
  $\Rightarrow$ *two time scales*: orbital time $\sim 2\pi/\Omega$ and radiation-reaction time $\sim 2\pi/(\epsilon\Omega)$
- on radiation-reaction time, the orbital phases have an expansion

$$\varphi_A = \epsilon^{-1}\varphi_A^{(0)}(\epsilon t) + \epsilon^0\varphi_A^{(1)}(\epsilon t) + O(\epsilon)$$

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# Multiscale expansion [Miller & AP; van de Meent & Warburton; AP & Wardell; Flanagan, Hinderer, Moxon, AP]

- adopt "good" perturbed variables $(\tilde{\varphi}_A, \tilde{J}_A)$ for orbit. Full set of system parameters $\mathcal{J}_A \sim (\tilde{J}_A, M_{BH}, J_{BH})$

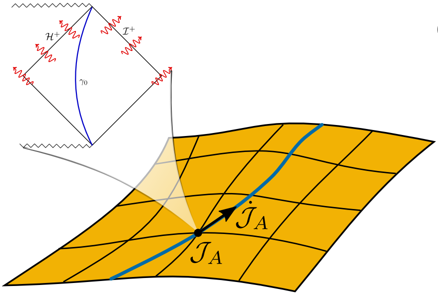
$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\mathcal{J}_B)$$
$$\frac{d\mathcal{J}_A}{dt} = \epsilon\left[\tilde{F}_A^{(0)}(\mathcal{J}_B) + \epsilon\tilde{F}_A^{(1)}(\mathcal{J}_B) + O(\epsilon^2)\right]$$

- treat $h_{\mu\nu}$ as function on extended manifold:

$$h_{\mu\nu}(t, x^i) \to \epsilon h^{(1)}_{\mu\nu}(\tilde{\varphi}_A, \mathcal{J}_A, x^i) + \epsilon^2 h^{(2)}_{\mu\nu}(\tilde{\varphi}_A, \mathcal{J}_A, x^i) + O(\epsilon^3)$$

- in Einstein equations, $\dfrac{\partial}{\partial t} = \Omega_A \dfrac{\partial}{\partial\tilde{\varphi}_A} + \dfrac{d\mathcal{J}_A}{dt}\dfrac{\partial}{\partial\mathcal{J}_A}$

# Rapid waveforms




**Offline step**

- Fourier series:

$$h^n_{\mu\nu} = \sum_{k^A} h^{n,\Omega_k}_{\mu\nu}(\mathcal{J}_A, x^i)e^{-ik^A\varphi_A}$$

$$\Omega_k := k^A\Omega_A$$

- solve field equations for amplitudes $h^{n,\Omega_k}_{\mu\nu}$ on grid of $\mathcal{J}_A$ values

**Online step**

- FastEMRIWaveforms package: rapidly evolve through parameter space [Katz, Chua, Speri, Warburton, Hughes]
  $\Rightarrow$ generate waveform in $\sim 10 - 100$ milliseconds

# Outline

# Complete 1PA calculations for quasicircular orbits

[AP, Warburton, Wardell 2013–]

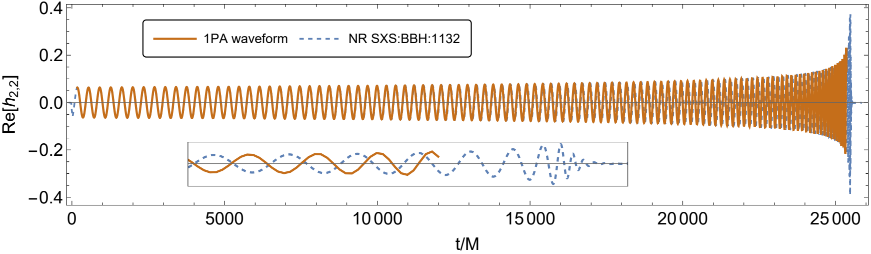
- parameters: $\mathcal{J}_A = (\Omega, M_{\rm BH}, J_{\rm BH})$, $J_{\rm BH} \sim \epsilon$
- evolution:

$$\frac{d\phi_p}{dt} = \Omega$$
$$\frac{d\Omega}{dt} = \epsilon\left[F_\Omega^{(0)}(\Omega) + \epsilon F_\Omega^{(1)}(\Omega) + O(\epsilon^2)\right]$$

- $h^{(n)}_{\mu\nu} = \sum_{i\ell m} h^{(n)}_{i\ell m}(\mathcal{J}^A, r)e^{-im\phi_p}Y^{i\ell m}_{\mu\nu}$
- solve field equations for amplitudes $h^{(n)}_{i\ell m}$

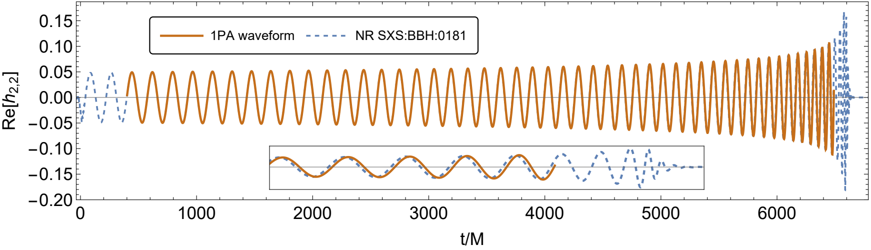
# 1PA waveforms [Wardell, AP, Warburton, Durkan, Miller, Le Tiec]

Mass ratio $\epsilon = 1$

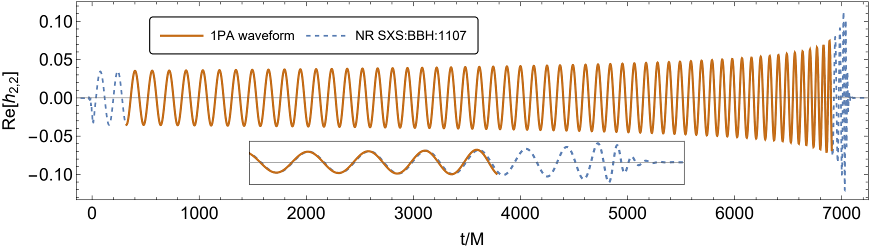


# 1PA waveforms [Wardell, AP, Warburton, Durkan, Miller, Le Tiec]

Mass ratio $\epsilon = 1/6$

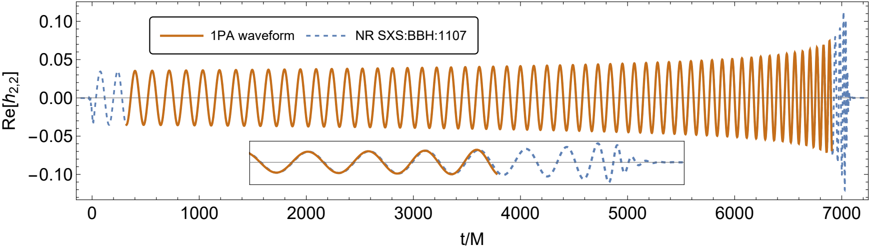


Mass ratio $\epsilon = 1/10$



# 1PA waveforms [Wardell, AP, Warburton, Durkan, Miller, Le Tiec]

Mass ratio $\epsilon = 1/10$




error estimate: $\sim 7.5\epsilon$ rad from $R = 20M$ to ISCO

# Outline

# Transition to plunge

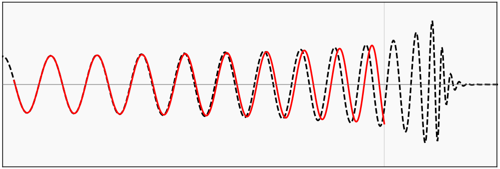
- evolution on timescale $\sim 1/(\epsilon^{1/5}\Omega)$ on frequency interval $(\Omega - \Omega_{\rm isco}) \sim \epsilon^{2/5}$
- parameters: $\mathcal{J}_A = \{\Delta\tilde{\Omega}, M_{BH}, J_{BH}\}$, $\Delta\tilde{\Omega} := \frac{\Omega - \Omega_{\rm isco}}{\epsilon^{2/5}}$
- evolution:

$$\frac{d\phi_p}{dt} = \Omega_{\rm isco} + \epsilon^{2/5}\Delta\tilde{\Omega}$$

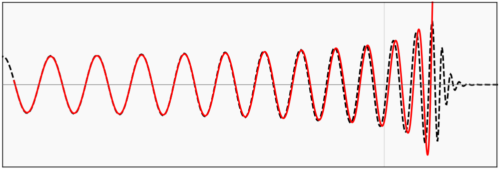
$$\frac{d\Delta\tilde{\Omega}}{dt} = \epsilon^{1/5}\left[F^{(0)}_{\Delta\tilde{\Omega}}(\Delta\tilde{\Omega}) + \epsilon^{1/5}F^{(1)}_{\Delta\tilde{\Omega}}(\Delta\tilde{\Omega}) + O(\epsilon^{2/5})\right]$$

- $h_{\mu\nu} = \epsilon \sum_{n\geq 0}\sum_{ilm} \epsilon^{n/5} j^{(n)}_{ilm}(\mathcal{J}_A, r) e^{-im\phi_p} Y^{ilm}_{\mu\nu}$

# Transition to plunge [Compere, Durkan, Kuchler, AP]



$$\frac{d\Omega}{dt} = F_{\Omega}^{(0)}$$
$$h_{lm} = \epsilon h_{lm}^{(1)}(\Omega) e^{-im\phi_p}$$



$$\frac{d\Delta\tilde{\Omega}}{dt} = \epsilon^{1/5}\left(F_{\Delta\tilde{\Omega}}^{(0)} + \epsilon^{2/5} F_{\Delta\tilde{\Omega}}^{(2)}\right)$$
$$h_{lm} = \epsilon\Big(j_{lm}^{(0)} + \epsilon^{2/5} j_{lm}^{(2)} + \epsilon^{3/5} j_{lm}^{(3)}\Big) e^{-im\phi_p}$$

# Plunge

- evolution on short time scale $\sim 1/(\epsilon^0\Omega)$
- parameters: $\mathcal{J}_A = \{\Omega, M_{BH}, J_{BH}\}$
- evolution:

$$\frac{d\phi_p}{dt} = \Omega$$
$$\frac{d\Omega}{dt} = F_\Omega^{(0)}(\Omega) + \epsilon F_\Omega^{(1)}(\Omega) + O(\epsilon^2)$$

- $h_{\mu\nu} = \sum_{i\ell m}\left[\epsilon h^{(1)}_{i\ell m}(\mathcal{J}^A, r) + \epsilon^2 h^{(2)}_{i\ell m}(\mathcal{J}^A, r) + O(\epsilon^3)\right] e^{-im\phi_p} Y^{i\ell m}_{\mu\nu}$

# Conclusion

**Self-force theory**

- high accuracy for IMRIs and EMRIs
- unique tool for merger and ringdown
- native waveform generation fast enough for data analysis

**Status**

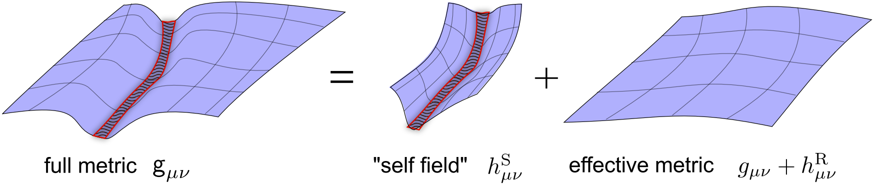
- 0PA: generic inspirals in Kerr are "available"
- 1PA: quasicircular inspiral into slowly spinning primary

**Challenges / Opportunities**

- need coverage of vast parameter space
- offline steps very expensive and complicated
  $\Rightarrow$ room for improved methods
- we don't need 1PA terms very accurately
  $\Rightarrow$ could extract 1PA terms from PN or PM

# Point particles and punctures

- replace "self-field" with "singular field"




- replace object with a point mass [D'Eath; Gralla & Wald; Upton & AP 2021]

$$T^{\mu\nu} := \frac{1}{8\pi}\left\{\epsilon\delta G^{\mu\nu}[h^{(1)}] + \epsilon^2\left(\delta G^{\mu\nu}[h^{(2)}] + \delta^2 G^{\mu\nu}[h^{(1)}]\right)\right\}$$
$$= m\int \tilde{u}^\mu \tilde{u}^\nu \frac{\delta^4(x-z)}{\sqrt{-\tilde{g}}} d\tilde{\tau} + O(\epsilon^3)$$

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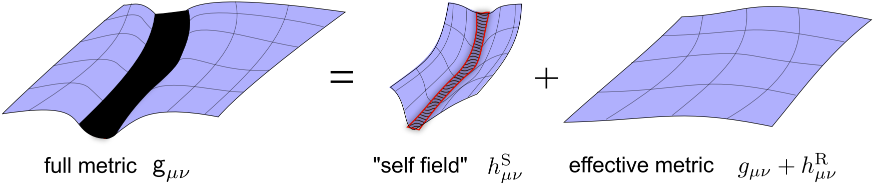


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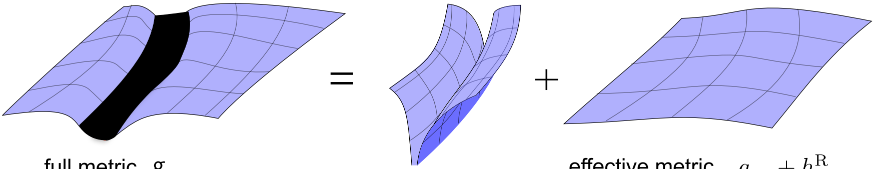


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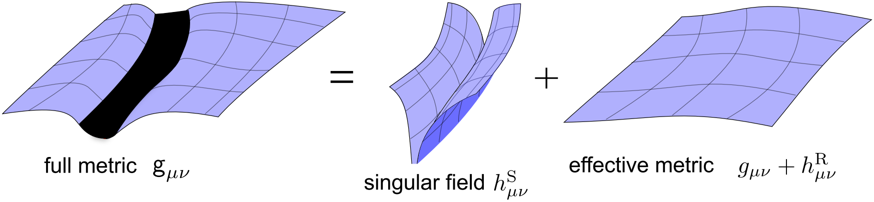


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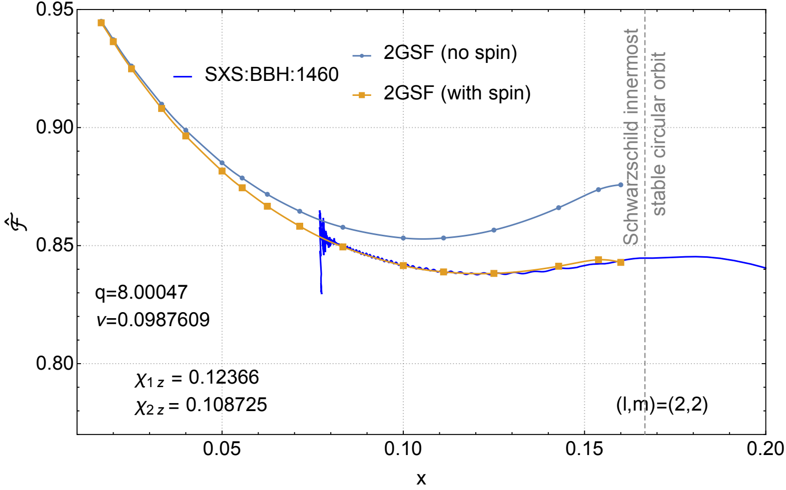
# Spinning bodies



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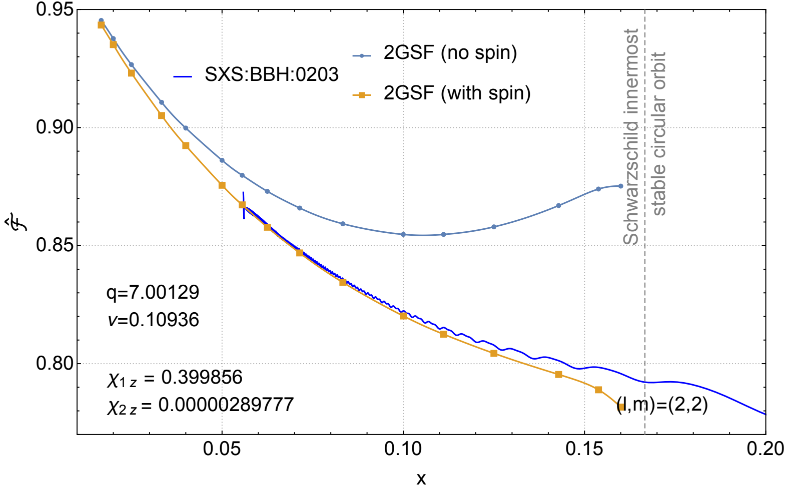

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