Towards gravitational scattering at fifth order in G

UCLA Mani L. Bhaumik Institute for Theoretical Physics

Michael Ruf
From Amplitudes to Gravitational Waves, Nordita, July 24 2023

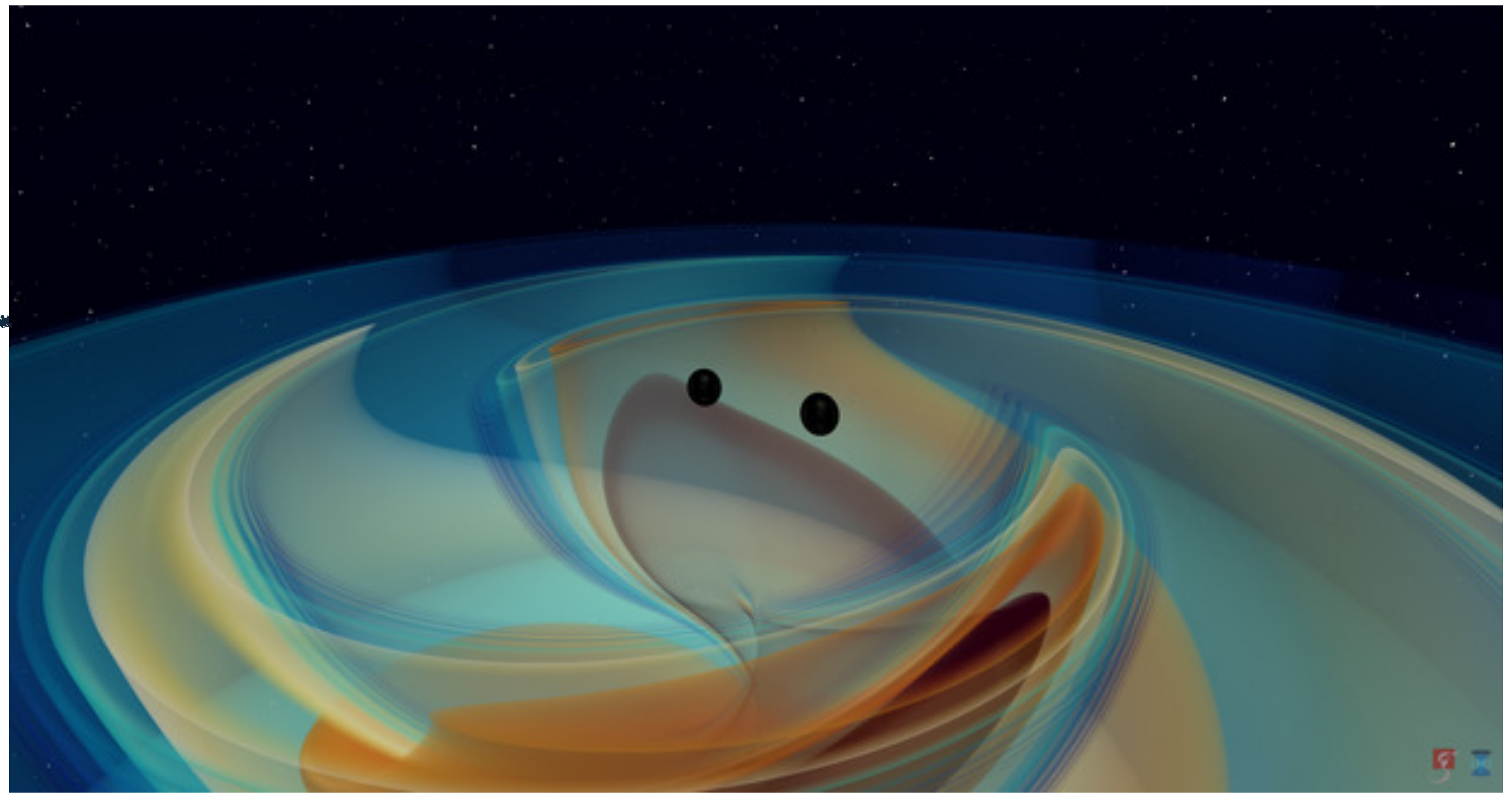
Based on work in collaboration w/ [Bern, Herrmann, Parra-Martinez, Roiban, A. Smirnov, V. Smirnov, Solon, Shen, Zeng]

Outline

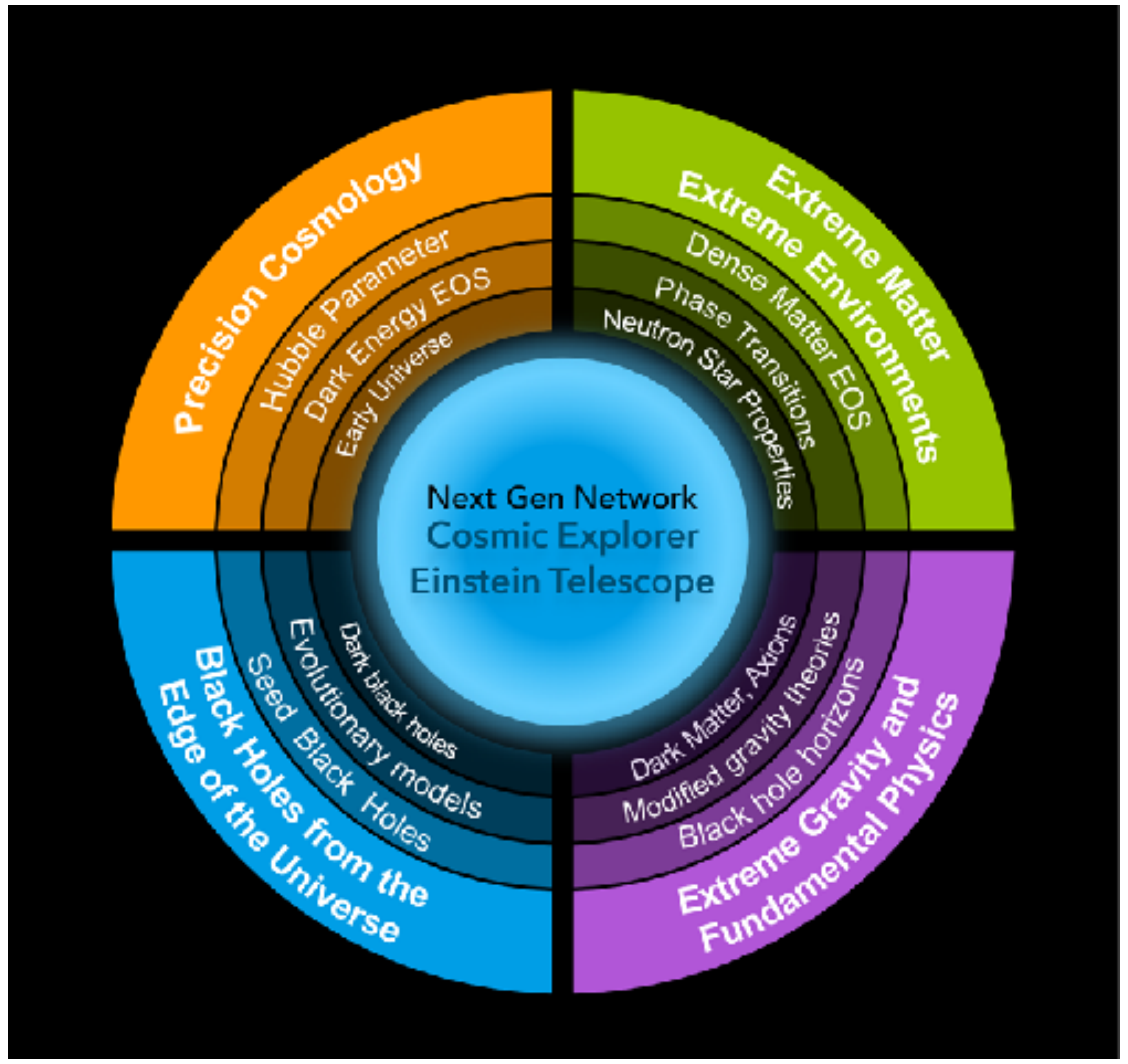
- Part 1: Recap scattering at order G^4
- Part 2: Towards scattering at order G^5

Motivation

- GW abundant, important source: compact binary systems
- Physics goals:
  - Strong-field tests of GR, new physics
  - BH properties, abundance etc.
  - Ultra-dense matter (neutron star equation of state)
  - Multi-messenger astronomy
  - …

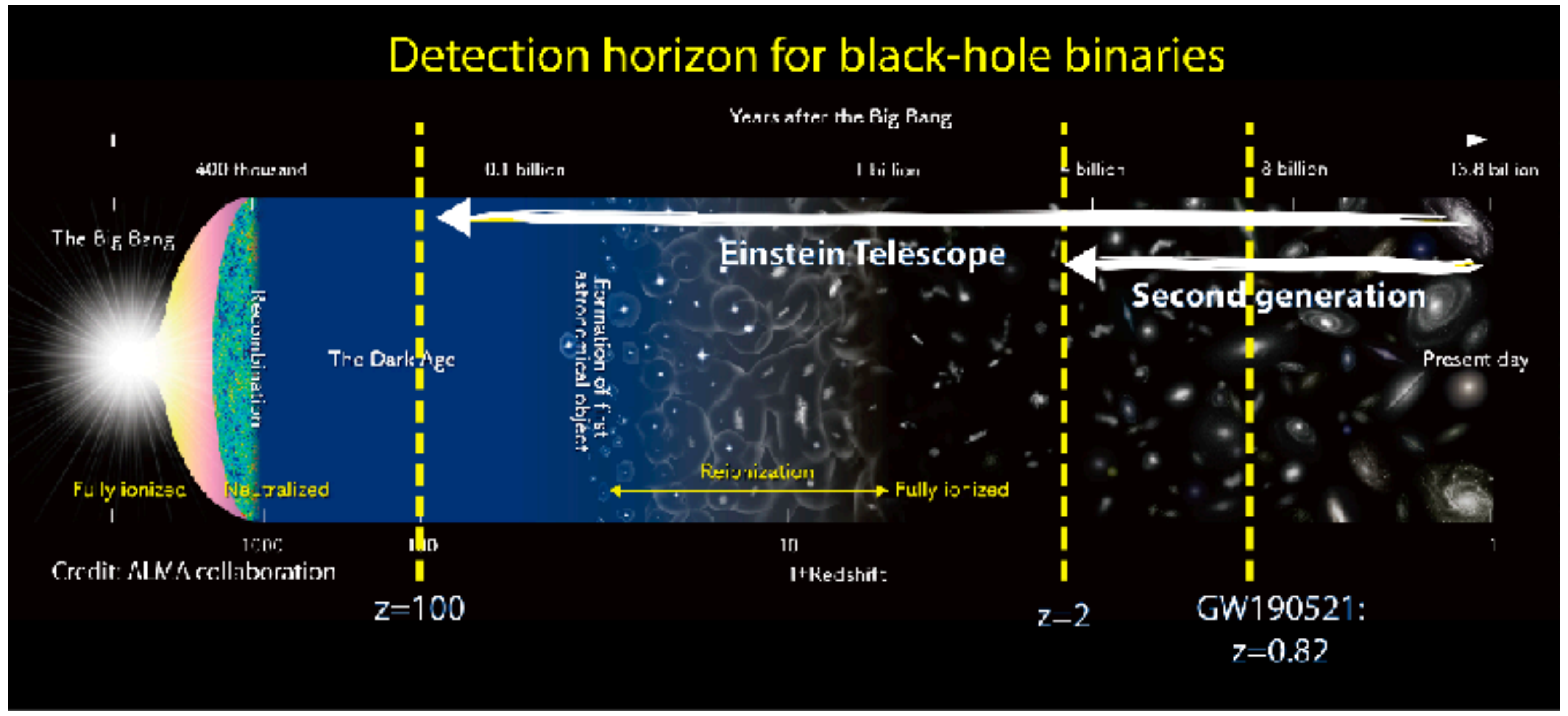


[GW190521, LIGO]

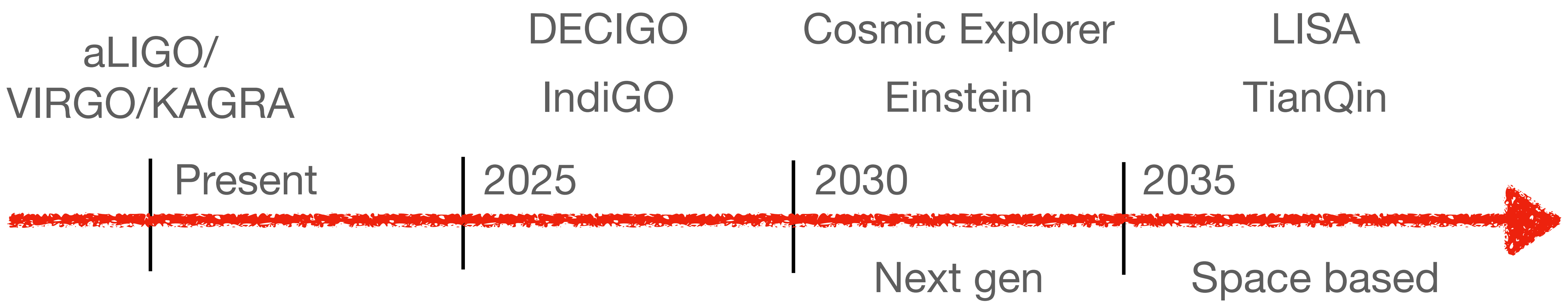


Motivation

- Present detectors (LIGO/VIRGO/KAGRA):
  - O(100) events. O4: 1 event/2.5 days



- Next-gen. experiments (ca. 2035):
  - More precision 10-100x improved S/N, wider frequency band
  - More data (bigger reach + sensitivity)
  - Extreme corners of parameter space, e.g. EMRI

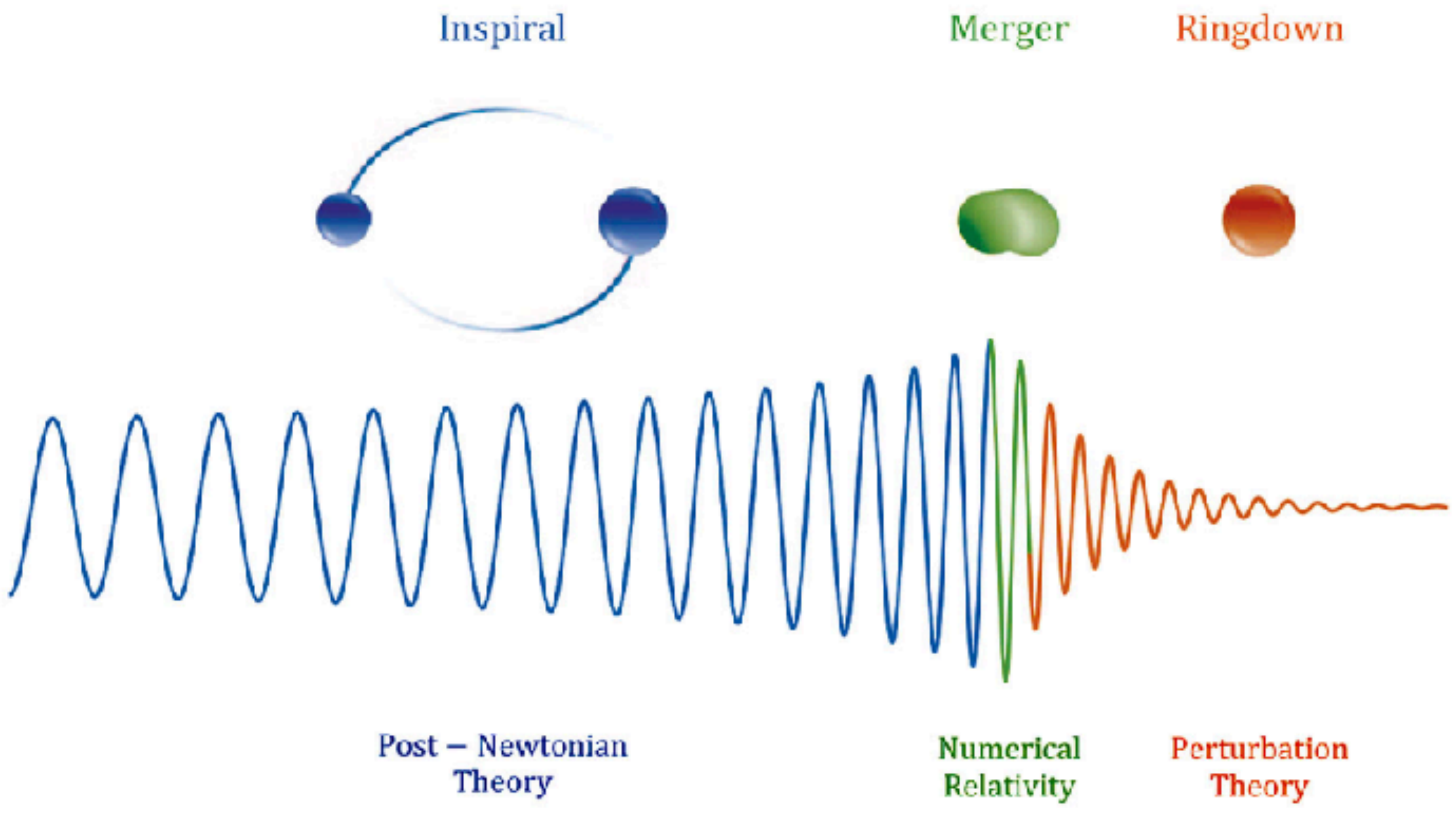


Need high-precision wave-form modelling!

Buonanno's talk

Motivation

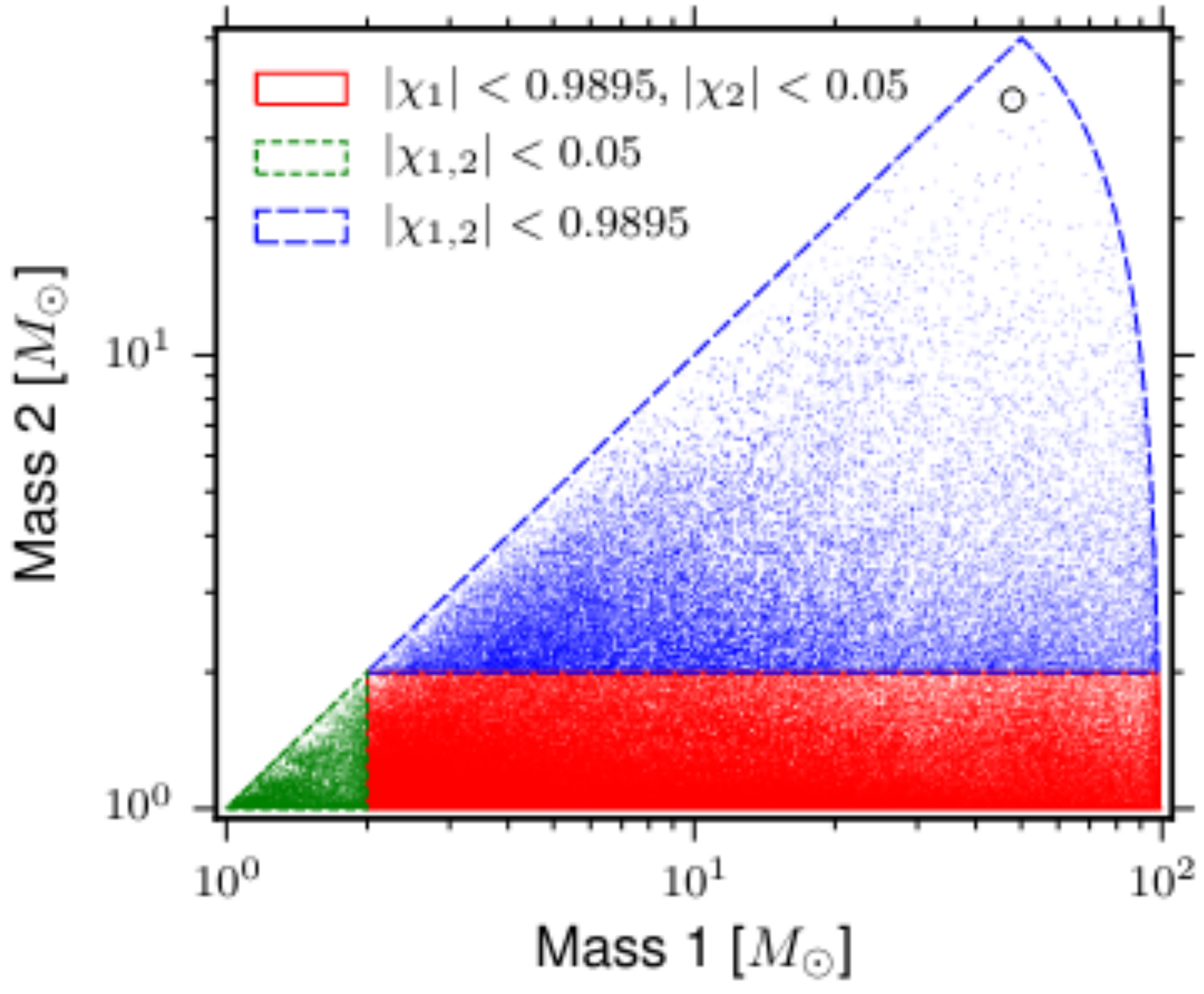
- Numerical waveform from Einstein eqn $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$



[1610.03567]

- Significant resource requirements:
  - $O(10^5)$ CPU h/ NR template
  - GW150914: 250k templates
  - Challenging in PS-corners: $m_1 \ll m_2, v \to c, |\vec{L}|/m \to 1$
- Solution: analytic and hybrid models (GW150914: post-Newtonian + effective-one-body)
- Corrections to Newton's potential to high orders

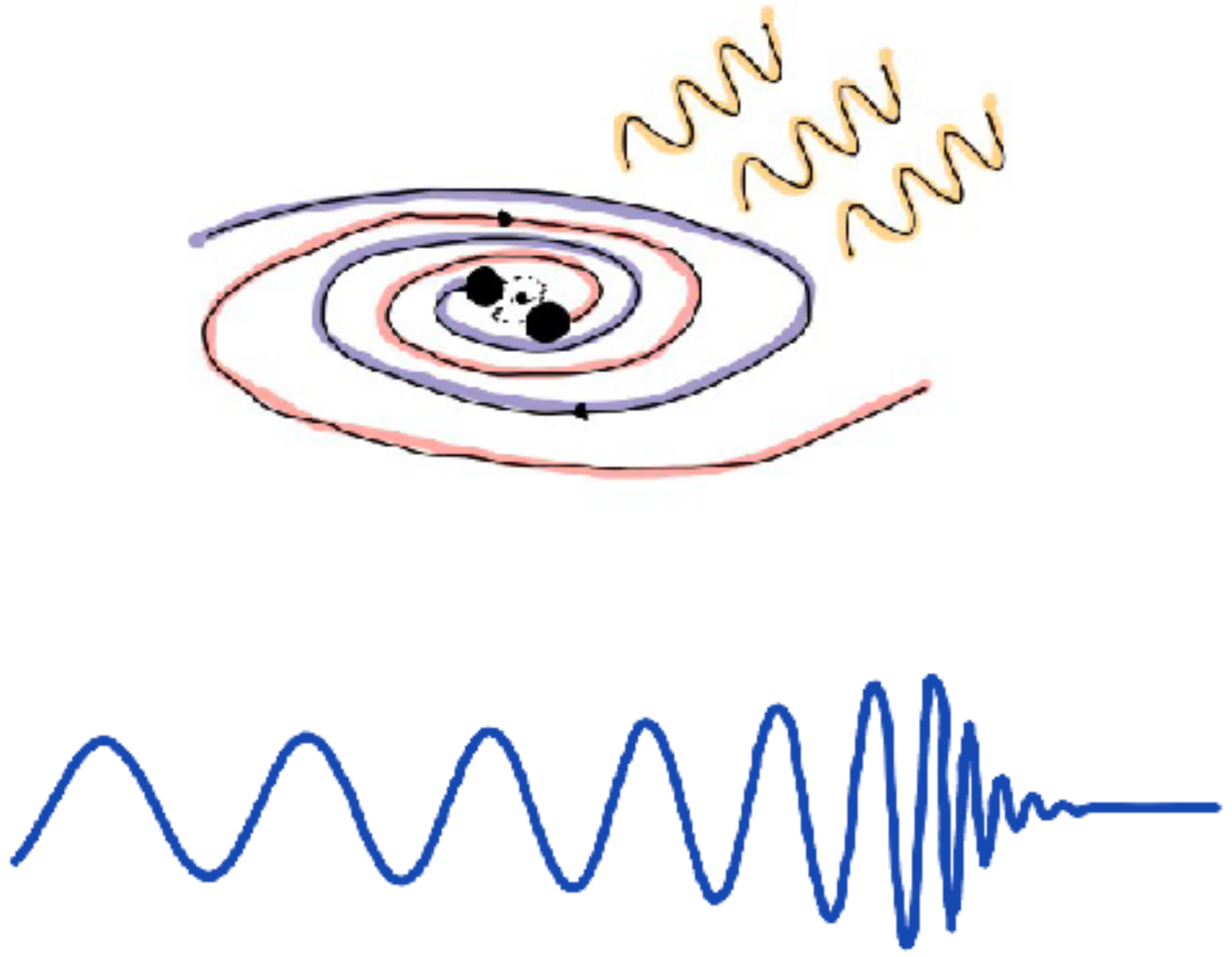
$$V(r) = -\frac{G\mu M}{r} + \frac{1}{c^2}\left[-\frac{3G\mu M \boldsymbol{v}^2}{2r} + \frac{G^2M^2}{r^2}\right] + \dots$$



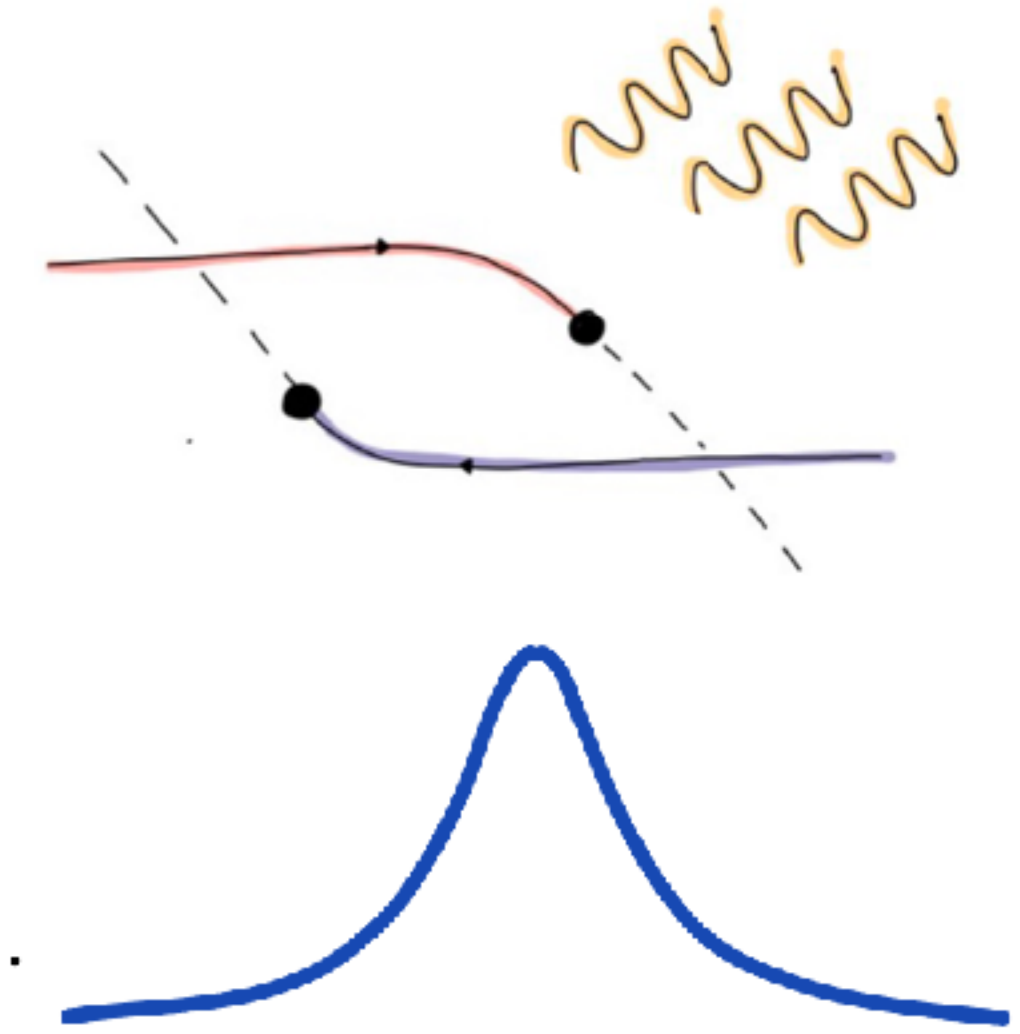
[GW150914, LIGO]

Gravitational Scattering

Process of interest: scattering of compact massive objects

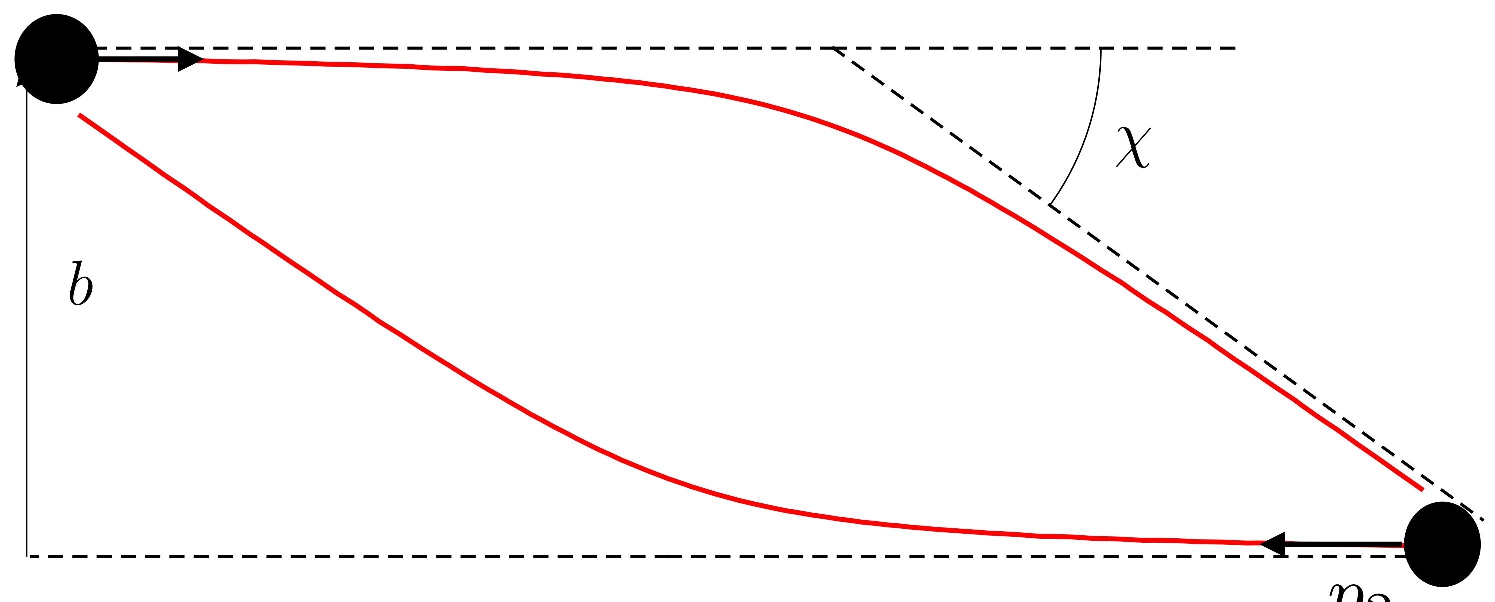


vs



Hard to observe in GW observatiories. Why bother?

Gravitational Scattering



Why bother?

1. Arguable simplest process, determined by initial data $\{p_1, p_2, b\}$
   - Gauge/coordinate invariant approach
   - Large separations: perturbative, no merger
   - Benefits for numerical and analytic approach

2. Connection to the bound problem
   - Key subtleties (e.g. hereditary effects) are present
   - Universal information (e.g. instantaneous potentials)

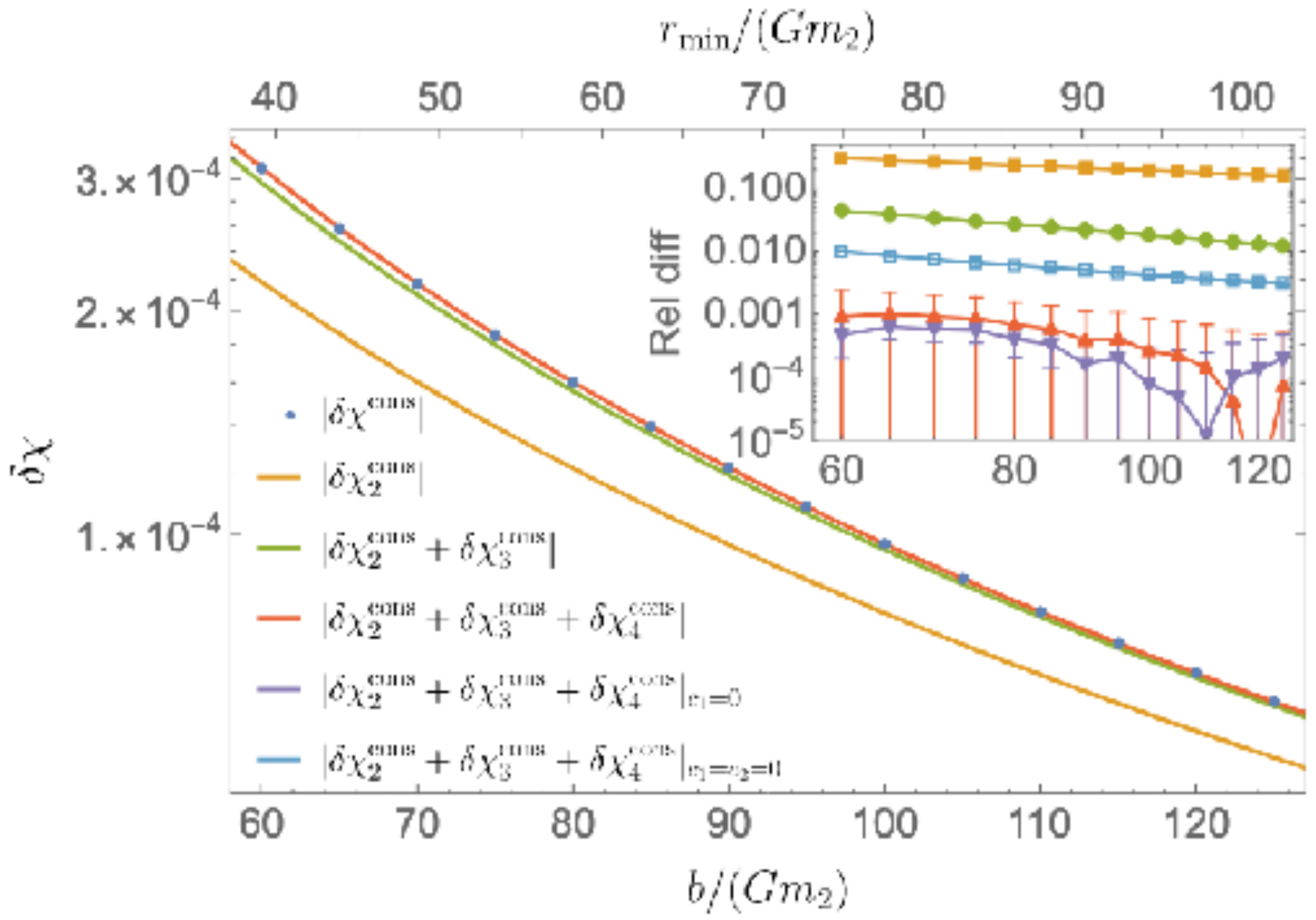
$$V(r) = -\frac{G\mu M}{r} + \frac{1}{c^2}\left[-\frac{3G\mu M\boldsymbol{v}^2}{2r} + \frac{G^2M^2}{r^2}\right] + \dots$$

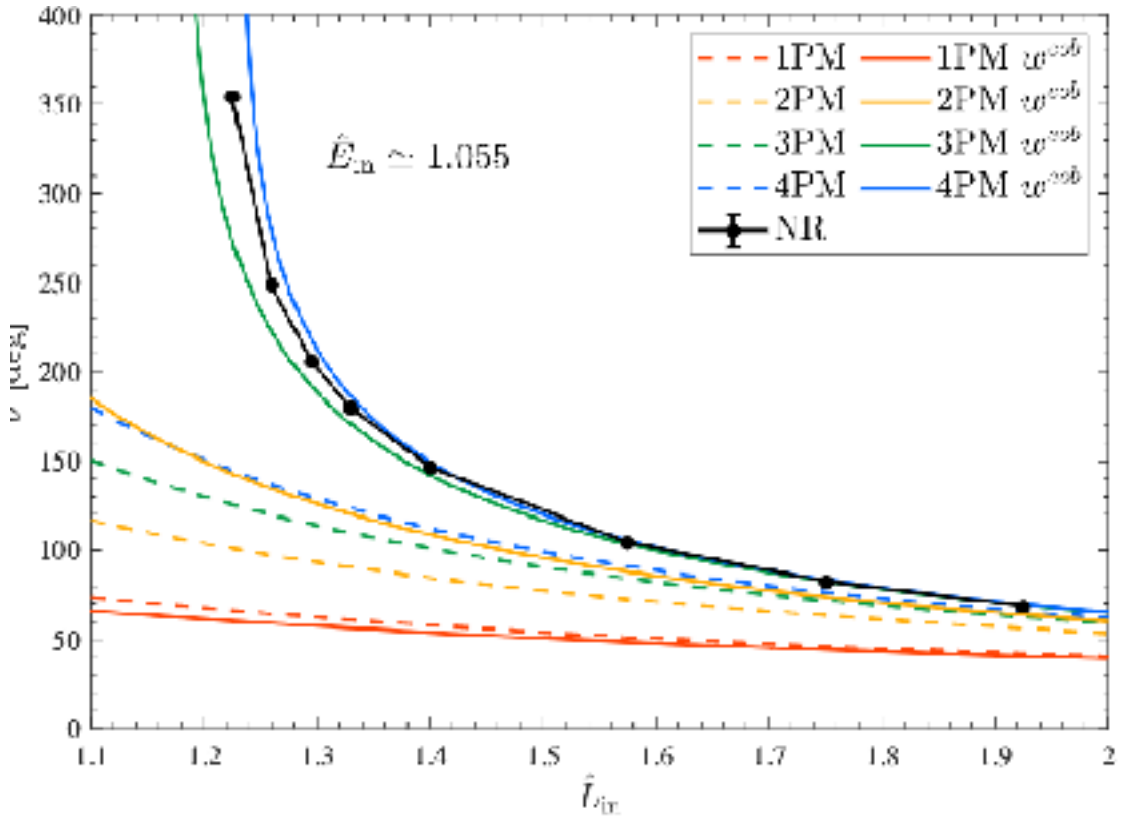
3. Meshes well with the amplitudes program

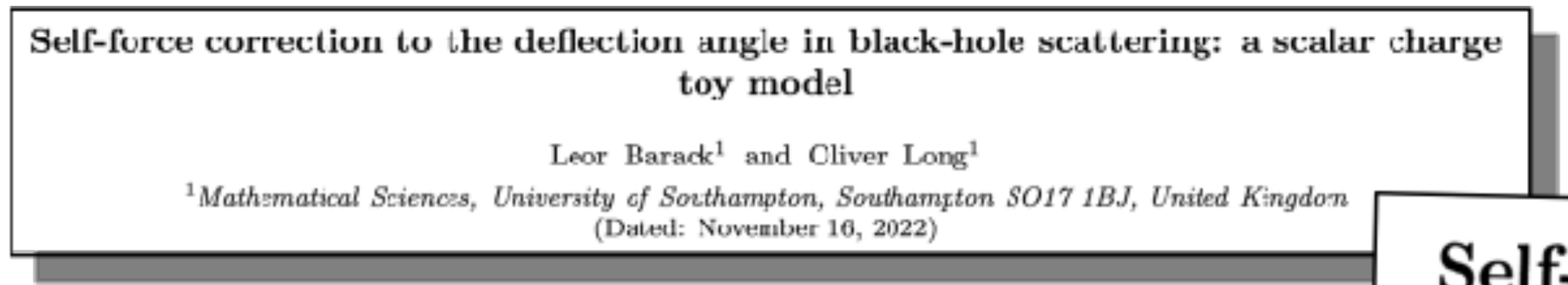
Relativistic treatment exposes additional structures, e.g. mass polynomiality [Damour]

Gravitational Scattering

- The classical GR community is very interested in scattering
  [Barack, Berti, Bini, Buonanno, Cardoso, Damour, East, Geralico, Gralla, Guercilena, Hinder, Hinderer, Hopper, Khalil, Lobo, Long, van de Meent, Nagar, Pfeiffer, Pratten Pretorious, Pretorius, Rettegno, Rezzolla, Schmidt, Sperhake, Steinhoff, Thomas, Vines, Whittall, Yunes,. . .]
- See e.g. Barack's workshop talk for collaborative effort
- Comparisons with percent-level agreement!
- Complementary approaches:
  - PM in weak field
  - SF/NR in strong field
  - ultimate goal: new hybrid models (e.g. SF-PM)

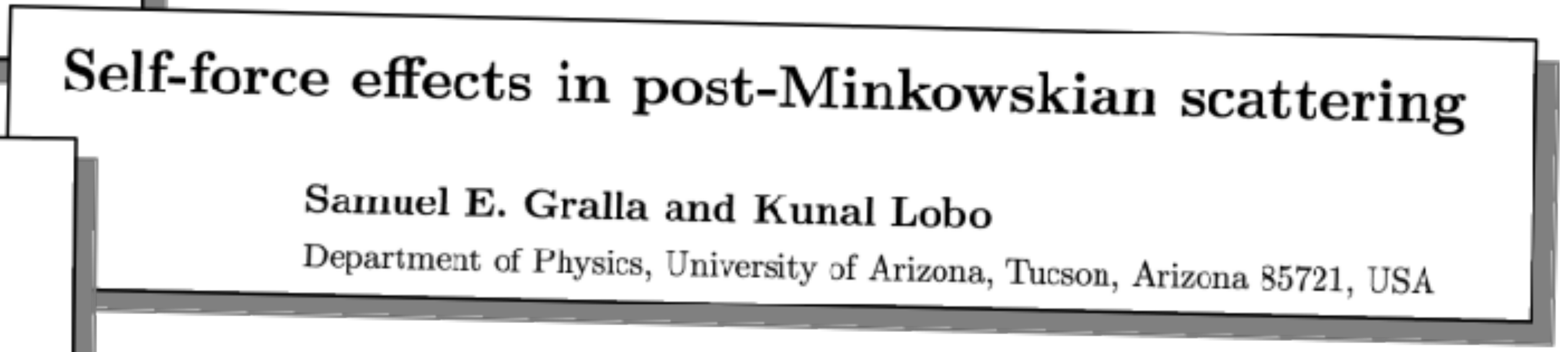





Self-force correction to the deflection angle in black-hole scattering: a scalar charge toy model

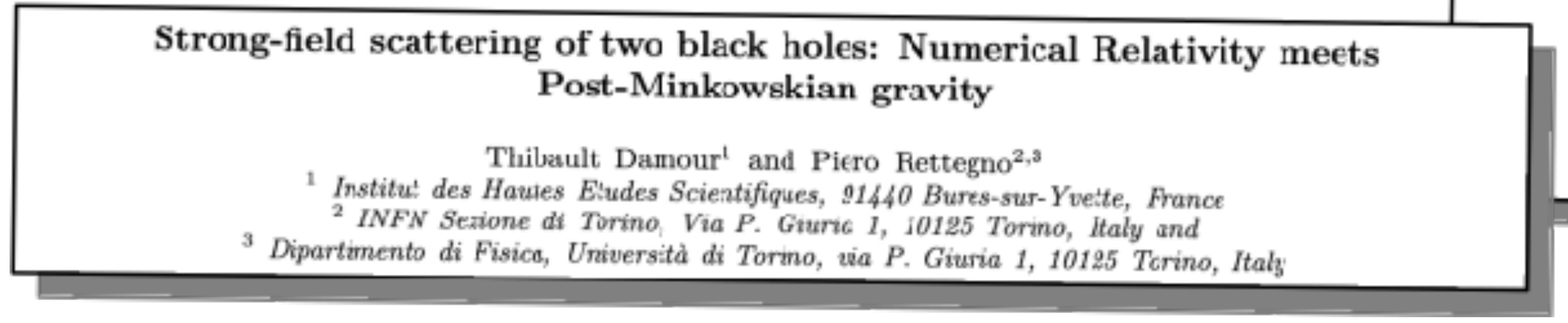
Leor Barack[1] and Oliver Long[1]

[1]Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom
(Dated: November 16, 2022)


Self-force effects in post-Minkowskian scattering

Samuel E. Gralla and Kunal Lobo

Department of Physics, University of Arizona, Tucson, Arizona 85721, USA


Strong-field scattering of two black holes: Numerical Relativity meets Post-Minkowskian gravity

Thibault Damour[1] and Piero Rettegno[2,3]

[1] Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France
[2] INFN Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy and
[3] Dipartimento di Fisica, Università di Torino, via P. Giuria 1, 10125 Torino, Italy

Gravitational Scattering

- Weak field perturbation theory $Gm/b \ll 1$ (PM-expansion) for large impact parameter scattering
- Two flavours:
  - Worldline based (classical or QFT [Mogull, Plefka, Steinhoff]) ←Talks by Kälin and Mogull + Workshop talk by Plefka
  - Amplitudes based
- Field is very mature, large number of new results in the past 4 years
  - High orders in perturbation theory
  - Spin ←Talk by Vines & tidal effects
  - Waveforms ←Talk by Travaglini
  - Raditative quantities ←Talk by Heissenberg
  - Self-force←Talk by Cristofoli's
- And better understanding
  - High energy limit
  - Exponentation (Eikonal, HEFT, Radial action,…)
  - Spin, angular momentum,…

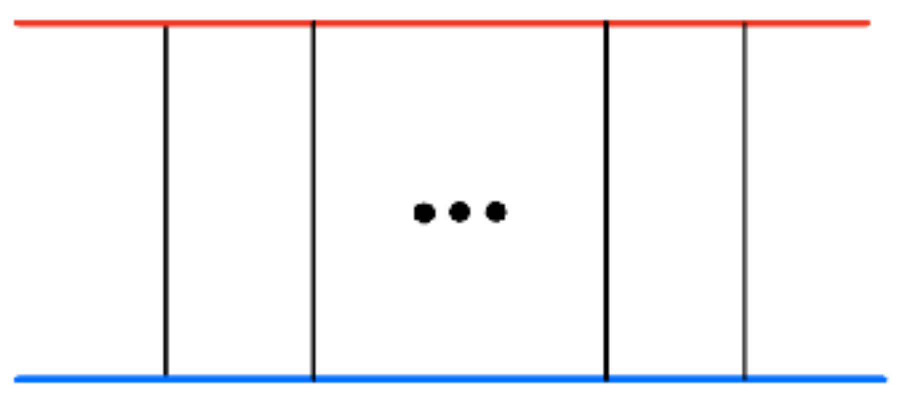
Classical physics from Ampltudes

- Basic idea:
  - Can evaluate scattering amplitudes very efficiently
  - S-Matrix has all information on classical scattering
- Old idea, dating back to ’60 [Corinaldesi, Iwasaki, Feynman, Barker, Gupta, Kaskas]
- Revived by Damour’s 2017 paper
- Modern program: state of the art results to enter GW template pipeline

Quantum viewpoint for classical physics extremely healthy!

Amplitudes to observables

- Amplitude not observable: $\mathcal{M}_{L\text{-loop}} \sim \frac{1}{\hbar^{L+1}}, \quad \mathcal{M}_{L\text{-loop}} \sim \frac{1}{\epsilon^L}$



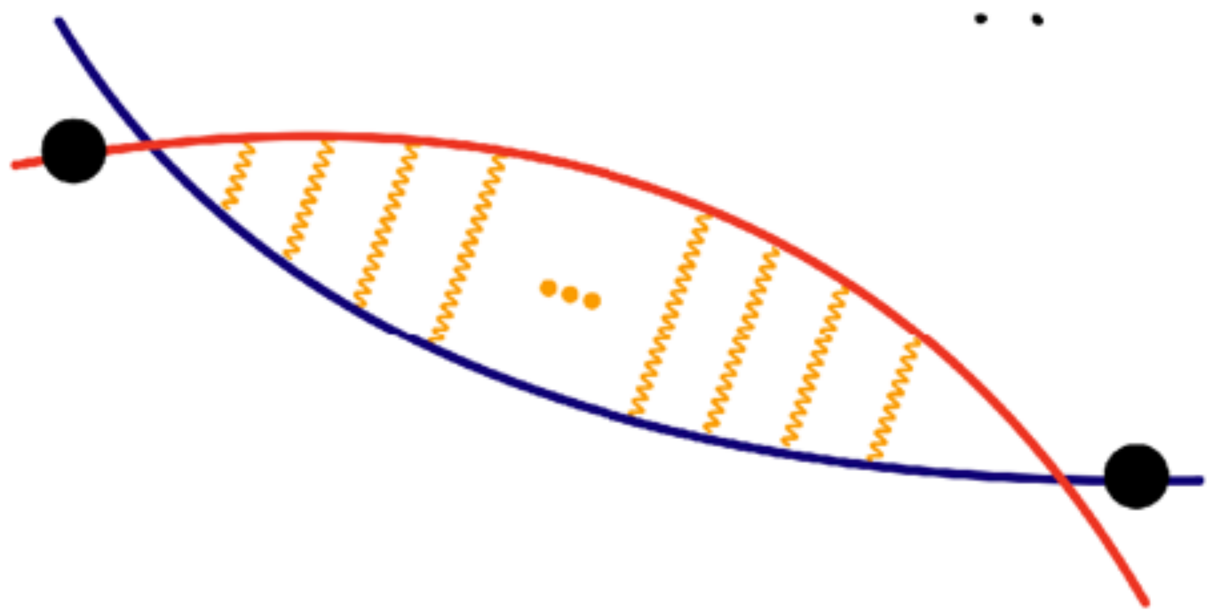
- Observables through
  - Direct computation [Kosower, Maybee, O'Connell; Damgaard, Hansen, Planté, Vanhove (4PM)]
  - Hamiltonian (Schrödinger eqn. or EFT matching [Rothstein,Neill; Cheung, Solon, Rothstein])
  - Stationary phase/generating functionals (eikonal, partial waves, heavy particle phase,…)
- Amplitude ↔ radial action [Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng]

$$\mathcal{M} = \mathrm{i}\int_J (e^{\mathrm{i}I_r(J)/\hbar} - 1), \quad I_r(J,E) = \int_{\text{trajectory}} p_r(J,E)\mathrm{d}r \qquad \chi(J,E) = -\partial_J I_r(J,E)$$

Gauge invariants

Very efficient extraction that meshes with relativistic integration

Classical Limit



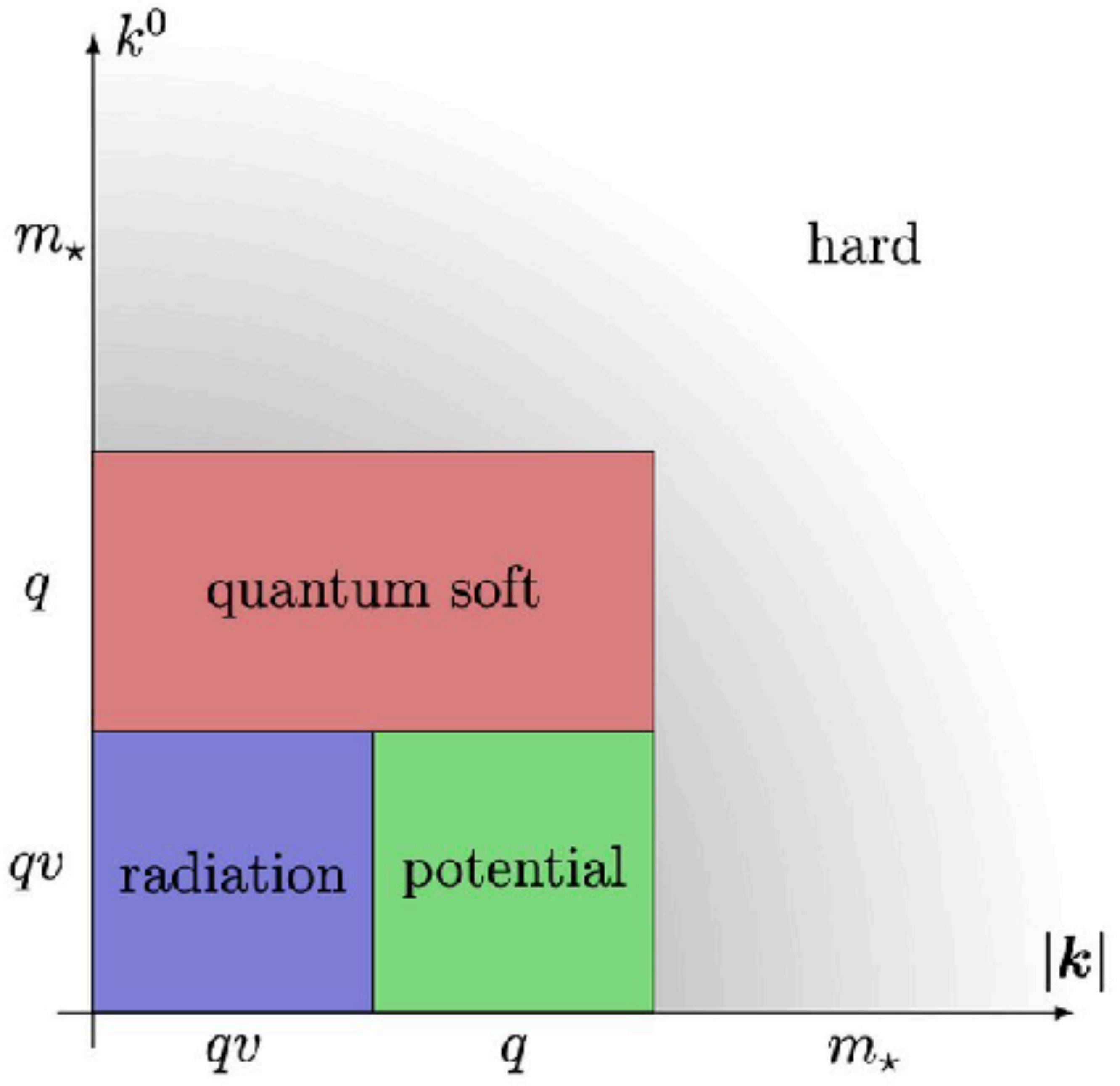
- Classical physics: Large number of soft exchanges $q = \hbar \bar{q}$

$$1 \ll J^2 \sim \frac{s}{q^2} \sim \frac{m_i^2}{q^2} \to q^2 \ll m_i^2 \sim s$$

$$l_{\text{compton}} \sim \frac{1}{M} \ll R_S \sim GM \ll b$$

- Relativistic regions: [Benecke, Smirnov]
  - Hard (h): $\ell \sim m \leftarrow$ UV, quantum $\lambda_{\text{compton}} \sim b$
  - Soft (s): $\ell \sim q \leftarrow$ long range $\lambda_{\text{compton}} \ll b$
- Threshold expansion: $v \sim |\vec{p}_{\text{COM}}|/\sqrt{s}$
  - Potential (p): $(\omega, \vec{\ell}) \sim (|q|v, |q|) \leftarrow$ instantaneous
  - Radiation (r): $(\omega, \vec{\ell}) \sim (|q|v, |q|v)$
- Classical physics (p)+(r), not well-defined separately
- Formally $v \ll 1$, resumption to $v \sim \mathcal{O}(1)$

$$v + \frac{v^3}{3} + \frac{v^5}{5} + \ldots = \text{arctanh}(v)$$



Amplitude computation

- Amplitude computation in parts
  - Constructing the integrand
  - Integral reduction
  - Evaluation of "master" integrals

$$\mathscr{M}(p_1, p_2, p_3, p_4) = \int \mathrm{d}^d \ell M(\ell) = f(\sigma)\log(\sigma) + \ldots$$

- Separate problems each requiring some attention
- Integration identical in different approaches [Parra-Martinez, **MSR**, Zeng '20]
  - Amplitudes-based (traditional [Bern et. al.], heavy particle EFT [Damgaard, Haddad, Helset; Brandhuber et. al], velocity cuts [Bjerrum-Bohr, Plante, Vanhove], …)
  - Worldline-based (traditional [Porto, Kalin], WQFT [Mogull, Plefka,Steinhoff])
- Improvements in integration beneficial for community
- More prescision → more loops. 4PM = G^4 = 3 loops, 5PM = G^5 = 4 loops, …

Integrand — Generalized unitarity $M(\ell)$

- Integrands from different methods
  - Feynman rules from Einstein-Hilbert action → straightforward
  - Generalized unitarity [Bern, Dixon, Dunbar, Kosower] → clean, more efficient
- Building blocks: on-shell tree amplitudes
  - Recursion from simpler trees (BCFW, Berends-Giele)
  - Double-copy → QCD trees
  - Sometimes even closed-from expressions!
- Recurring theme: on-shell amplitudes have structure (symmetries, analytic properties) — good idea to use this

Surprisingly, gravity interactions complicated but no bottleneck!

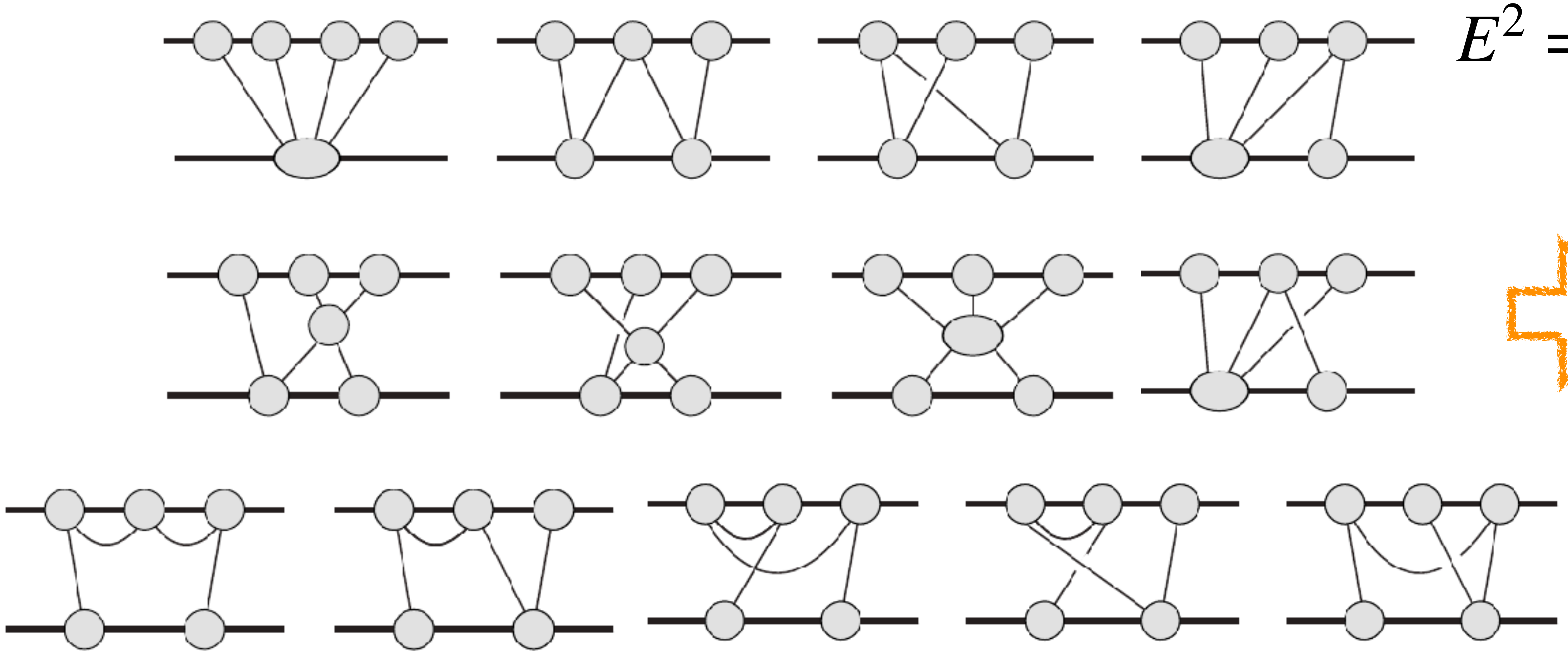
Integrand — Generalized unitarity

- Construct integrand from unitary cuts, at $\mathcal{O}(G^4)$:

Blobs = tree amplitudes
Lines = on-shell states
$E^2 = \vec{p}^2 + m^2$

Potential/
Instantaneous

Tail

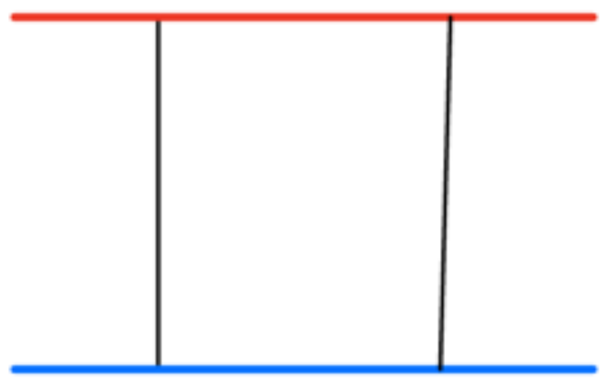


$\Rightarrow M(\ell)$

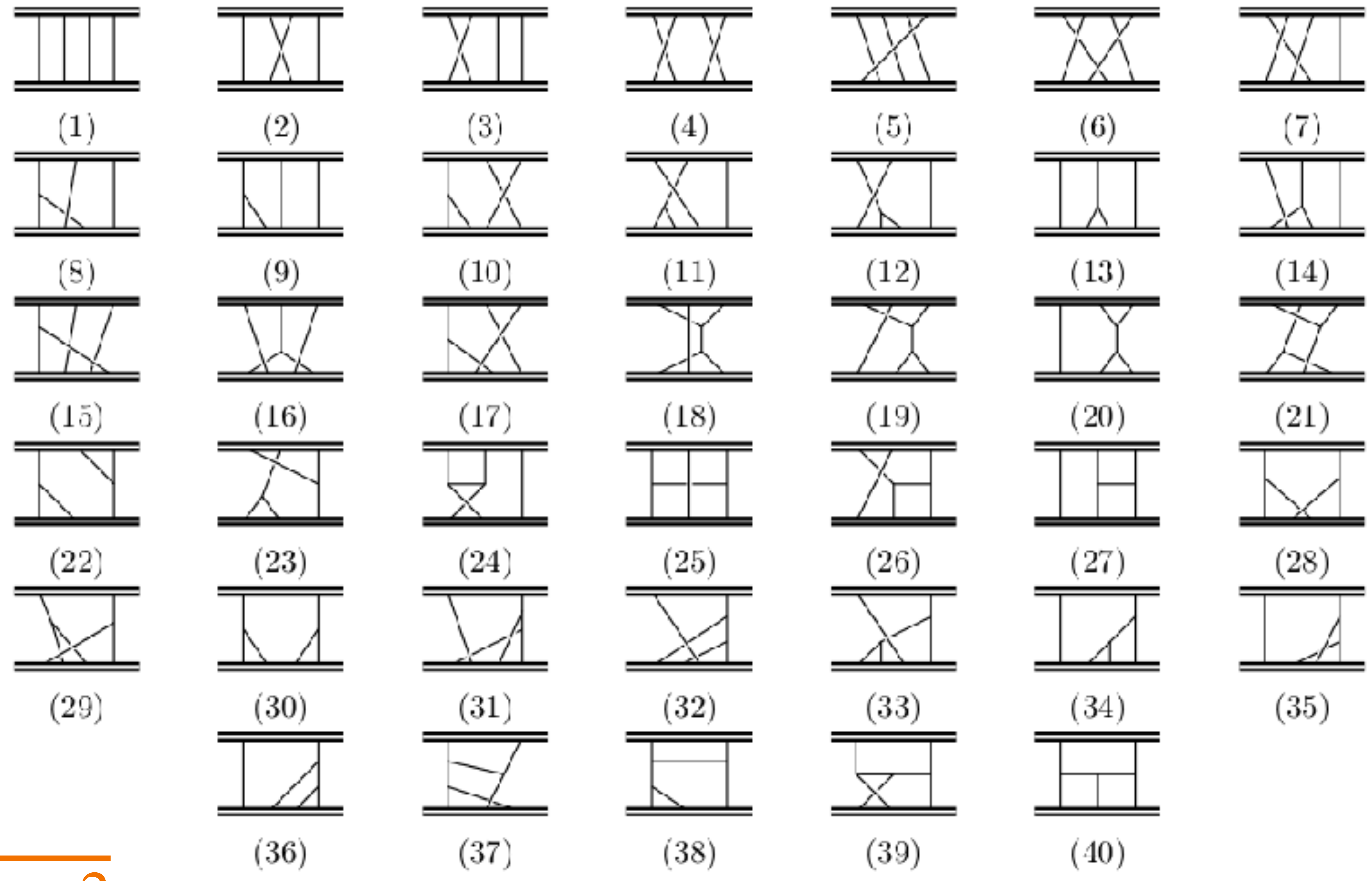
- Drastically simplified in classical limit
  - No graviton loops, self energies, matter contacts
  - 1 matter line per loop

Integration

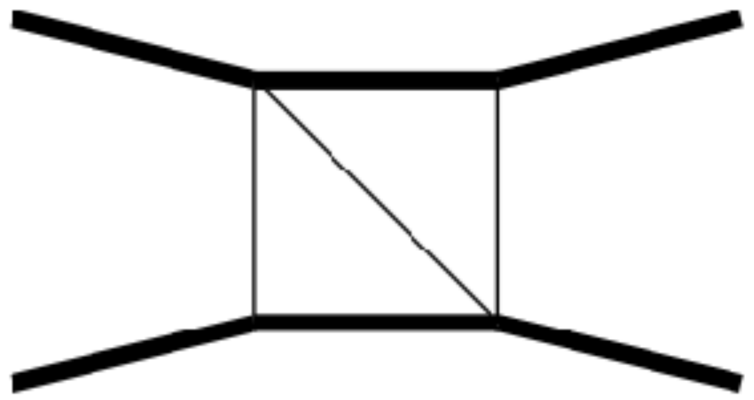
- Amplitude organized in 40 integral families
- Integrals as in heavy quark EFT



$$= \int \mathrm{d}^D\ell \, \frac{1}{\ell^2} \frac{1}{(\ell-q)^2} \frac{1}{2u_1\cdot\ell} \frac{1}{2u_2\cdot\ell}$$



- Classical limit crucial:
  - Fewer independent integrals
  - Single-variable problem! $y = u_1 \cdot u_2 = 1/\sqrt{1-v^2}$
  - Simpler functions



$\sim$

$$\begin{aligned}
&= 16\log\frac{-t}{m^2}\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ 0 & 1+1/y & 1\end{smallmatrix};\bar{x},\vec{a}\big) - \mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ 0 & 1+y & 1\end{smallmatrix};\bar{x},\vec{a}\big) \qquad (5.8)\\
&+ \mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ \infty & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) + \mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1\\ 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) + \zeta_2\,\mathcal{E}_4\big(\begin{smallmatrix}-1\\ \infty\end{smallmatrix};\bar{x},\vec{a}\big) + \zeta_2\,\mathcal{E}_4\big(\begin{smallmatrix}-1\\ 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]\\
&- 8\,(8\zeta_2 + 4\mathrm{Li}_2(y) + \log^2 y)\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 0 & 1+1/y\end{smallmatrix};\bar{x},\vec{a}\big) + \mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 0 & 1+y\end{smallmatrix};\bar{x},\vec{a}\big) - \mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 0 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]\\
&- 32\,\zeta_2\,\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ \infty & 1\end{smallmatrix};\bar{x},\vec{a}\big) - \mathcal{E}_4\big(\begin{smallmatrix}-1 & 1\\ 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big] + 16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+1/y & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&- 32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+1/y & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big) - 16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+y & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) + 32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 0 & 1+y & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+ 16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) - 24\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) - 32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 1 & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+ 16\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) + 40\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\big) - 32\,\mathcal{E}_4\big(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 1 & 2 & 1\end{smallmatrix};\bar{x},\vec{a}\big)\\
&+ \frac{4}{3}(12\mathrm{Li}_3(y) + 24\zeta_2\log y + \log^3 y)\Big[\mathcal{E}_4\big(\begin{smallmatrix}-1\\ \infty\end{smallmatrix};\bar{x},\vec{a}\big) + \mathcal{E}_4\big(\begin{smallmatrix}-1\\ 1\end{smallmatrix};\bar{x},\vec{a}\big)\Big]\\
&+ 64\zeta_4 - 32\zeta_2\mathrm{Li}_2(y) + 16\mathrm{Li}_4(y) + 8\zeta_2\log^2 y + \frac{1}{3}\log^4 y\,.
\end{aligned}$$

→ $\mathrm{arccosh}(y)\log(-t)$

[Broedel, Dulat, Duhr, Penante, Tancredi]

Integration

- Thousands of integrals

$$I_{1-\text{loop}}[a_1, a_2, a_3, a_4] = \int \mathrm{d}^D\ell \, \frac{1}{[\ell^2]^{a_1}} \frac{1}{[(\ell-q)^2]^{a_2}} \frac{1}{[2u_1\cdot\ell]^{a_3}} \frac{1}{[2u_2\cdot\ell]^{a_4}}\,, \quad I_{3-\text{loop}}[a_1, \ldots, a_{15}]$$

- Integrals satisfy integration-by-parts (IBP) relations [Tkachov, Chetyrkin 1981]

$$0 = \int \mathrm{d}^D\ell \, u_2 \cdot \frac{\partial}{\partial\ell} \frac{1}{[\ell^2]^{a_1}} \frac{1}{[(\ell-q)^2]^{a_2}} \frac{1}{[u_1\cdot\ell]^{a_3}} \frac{1}{[u_2\cdot\ell]^{a_4}} = -a_1 I[a_1+1, a_2, a_3, a_4-1] - a_2 I[a_1, a_2+1, a_3, a_4-1] + \ldots$$

- Linear algebra problem using Laporta's algorithm [Laporta '01] [FIRE, Kira,Reduze…]

$$0 = -I[2,1,1,0] - I[1,2,1,0] + \ldots$$
$$0 = -I[2,1,1,1] - I[1,2,0,1] + \ldots$$
$$\vdots$$

- Scales with number and range of a_i

Integration [Parra-Martinez, MSR, Zeng '20]

- Integral reduction: 137 irreducible "master" integrals
- Master integrals satisfy Fuchsian differential equations (DE) [Kotikov '91]

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{I} = \sum_k \frac{\mathrm{d}\log(w_k)}{\mathrm{d}x} A_k(\epsilon)\vec{I}\,, \quad w_k \in \{x, 1 \pm x, 1 + x^2\}\,, \quad |\vec{I}| = 137\,, \quad A_k(\epsilon) \in M_{137}(\mathbb{Q})[\epsilon]$$

- Change of basis $\vec{I} \to \vec{J} = T\vec{I}$ to canonical form [Henn '13]

$$\sigma = \frac{1 + x^2}{2x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{J} = \epsilon \sum_k \frac{\mathrm{d}\log(w_k)}{\mathrm{d}x} B_k \vec{J}$$

$$1 < \sigma \leftrightarrow x \in (0,1)$$

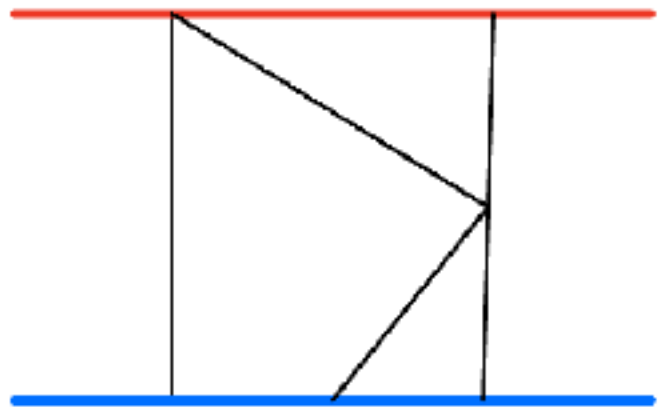
- Order-by-order solution in terms of (generalized) polylogarithms

$$\vec{J} = \sum_n \epsilon^n \vec{J}_n \quad \vec{J}_{n+1} = \sum_k A_k \int_0^x \mathrm{d}z \left[\frac{\mathrm{d}}{\mathrm{d}z}\log(w_k(z))\right] \vec{J}_n$$

- Boundary conditions: Regularity/scaling fixes most. Rest computed in the static limit $\sigma \to 1$

Integration — Elliptic sector

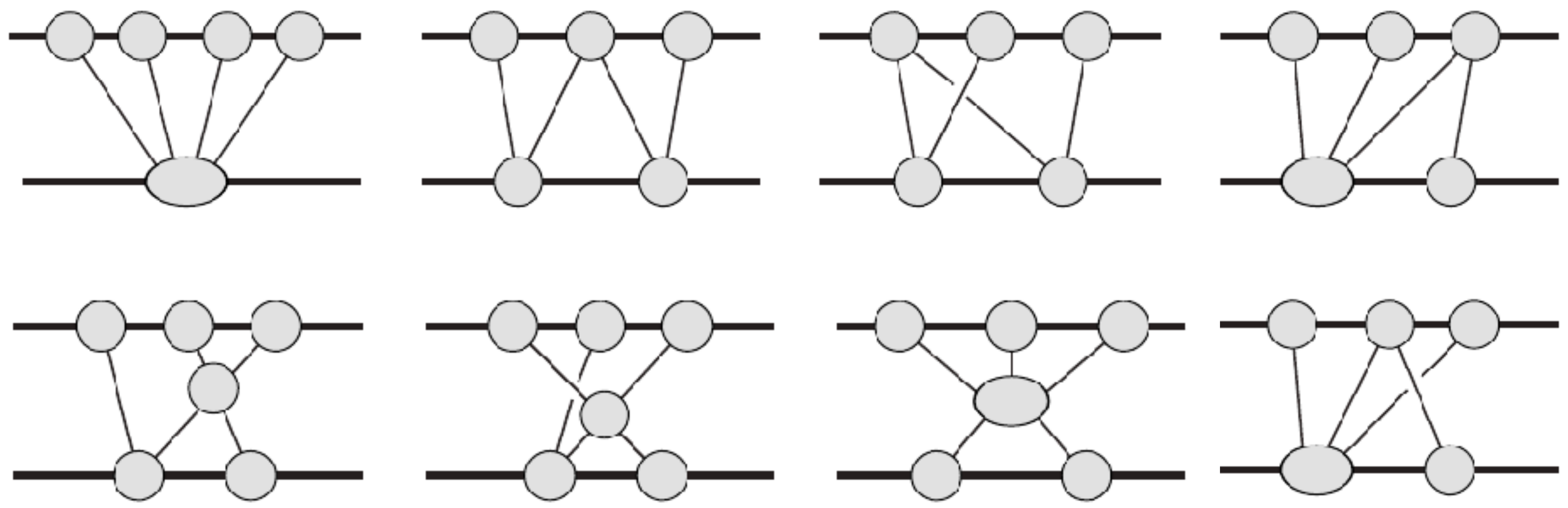
- Some cases: no canonical form



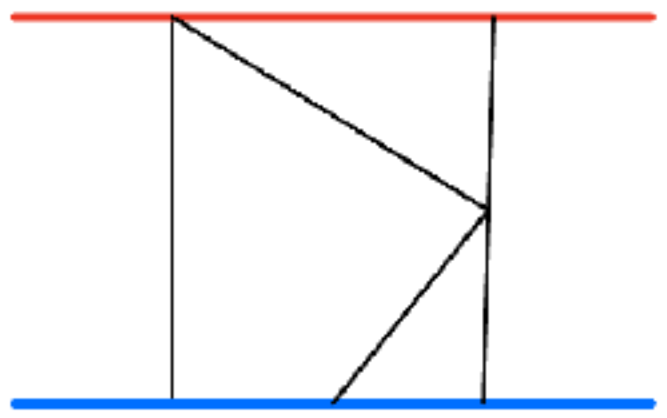
$$= \frac{1}{\epsilon^2}\frac{8}{\sigma+1}K^2\left(\frac{1-\sigma}{1+\sigma}\right) + \mathcal{O}(\epsilon^{-1})$$

- Elliptics only in the potential region → hardest for integration
- Strategy 1:
  - Split amplitude $\mathcal{M} = \mathcal{M}_{\text{poly}} + \mathcal{M}_{\text{elliptic}}$
  - Solve $\mathcal{M}_{\text{poly}}$ through DE
  - For $\mathcal{M}_{\text{elliptic}}$ compute series to high ordersand match to ansatz
- Strategy 2: Generalized kernels instead of $\frac{\mathrm{d}\log(w_k)}{\mathrm{d}x}$, [Dlapa et al.]

Integration — Elliptic sector



- Ansatz from contact integrals integrals



$$\sim \left\{ K^2\left(\frac{1-\sigma}{1+\sigma}\right), E\left(\frac{1-\sigma}{1+\sigma}\right) K\left(\frac{1-\sigma}{1+\sigma}\right), E^2\left(\frac{1-\sigma}{1+\sigma}\right) \right\}$$

Not allowed to collapse any line!

- Expand integrals/amplitude in v using DE and match

$$\mathscr{M}_{4,\text{elliptic}} \sim -\pi^2\left(\frac{41}{16}+\frac{33601v^2}{3072}+\ldots\#v^{400}\right) = r_4\pi^2 + r_5K\left(\frac{1-\sigma}{1+\sigma}\right)^2 + r_6E\left(\frac{1-\sigma}{1+\sigma}\right)K\left(\frac{1-\sigma}{1+\sigma}\right) + r_7E\left(\frac{1-\sigma}{1+\sigma}\right)^2$$

- 60 orders to fix r_i, 400 to check
- Avoids complicated integrals (elliptic polylogs) in intermediate steps
- (Pre-)Canonical form useful to make series expansion efficient

Classical scattering at $O(G^4)$ —Results

$$\mathcal{M}_4^{\text{cons}} = G^4 M^7 \nu^2 |\vec{q}| \pi^2 \left[\mathcal{M}_4^{\text{p}} + \nu \left(4\mathcal{M}_4^{\text{t}} \log\left(\frac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \text{iterations}$$

$$\mathcal{M}_4^{\text{p}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}$$

$$\mathcal{M}_4^{\text{t}} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6 K^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7 E^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

$$\begin{aligned} \mathcal{M}_4^{\text{rem}} = {} & r_8 + r_9 \log\left(\frac{\sigma+1}{2}\right) + r_{10} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma) \\ & + r_{13} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\frac{\sigma+1}{2}\right) + r_{14} \frac{\text{arccosh}^2(\sigma)}{\sigma^2 - 1} \\ & + r_{15} \operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16} \operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2-1}} \left[\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) \right] \end{aligned}$$

Classical scattering at $O(G^4)$ —Results

$$\mathcal{M}_4^{\text{cons}} = G^4 M^7 \nu^2 |\vec{q}| \pi^2 \left[\mathcal{M}_4^{\text{p}} + \nu \left(4\mathcal{M}_4^{\text{t}} \log\left(\frac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \text{iterations}$$

$$\mathcal{M}_4^{\text{p}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}$$

$$\mathcal{M}_4^{\text{t}} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6 K^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7 E^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

$$\mathcal{M}_4^{\text{rem}} = r_8 + r_9 \log\left(\frac{\sigma+1}{2}\right) + r_{10} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma)$$

$$+ r_{13} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\frac{\sigma+1}{2}\right) + r_{14} \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1}$$

$$+ r_{15} \operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16} \operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2-1}} \left[\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) \right]$$

$$\int_{\vec{\ell}} \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_{\vec{\ell}} \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\vec{\ell}} \frac{\tilde{I}_{r,1} \tilde{I}_{r,3}}{Z_1} + \int_{\vec{\ell}} \frac{\tilde{I}_{r,2}^2}{Z_1}$$

As predicted by amplitude-action relation
Highly non-trivial check!

Classical scattering at $O(G^4)$ —Results

Radial action at order G^4 $I_r^4(\vec{q}) \sim \chi^4$

$$\mathcal{M}_4^{\rm cons} = G^4 M^7 \nu^2 |\vec{q}| \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu \left(4 \mathcal{M}_4^{\rm t} \log\left(\frac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem} \right) \right]$$ + iterations

$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}$$

$$\mathcal{M}_4^{\rm t} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6 K^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7 E^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

$$\begin{aligned}\mathcal{M}_4^{\rm rem} = {} & r_8 + r_9 \log\left(\frac{\sigma+1}{2}\right) + r_{10} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma) \\ & + r_{13} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\frac{\sigma+1}{2}\right) + r_{14} \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} \\ & + r_{15} \operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16} \operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2-1}} \left[\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) \right]\end{aligned}$$

Classical scattering at $O(G^4)$ —Results

$$\mathcal{M}_4^{\rm cons} = G^4 M^7 \nu^2 |\vec{q}| \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu \left(4\mathcal{M}_4^{\rm t} \log\left(\frac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem} \right) \right] + \text{iterations}$$

$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}$$

$$\mathcal{M}_4^{\rm t} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6 K^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7 E^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

Elliptics, from ansatz

$$\mathcal{M}_4^{\rm rem} = r_8 + r_9 \log\left(\frac{\sigma+1}{2}\right) + r_{10} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma)$$
$$+ r_{13} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\frac{\sigma+1}{2}\right) + r_{14} \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2 - 1}$$
$$+ r_{15} \operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16} \operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2-1}} \left[\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) \right]$$

Classical scattering at $O(G^4)$ —Results

$$\mathcal{M}_4^{\rm cons} = G^4 M^7 \nu^2 |\vec{q}\,| \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu \left(4\mathcal{M}_4^{\rm t} \log\left(\frac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem} \right) \right] + \text{iterations}$$

$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}$$

$$\mathcal{M}_4^{\rm t} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}$$

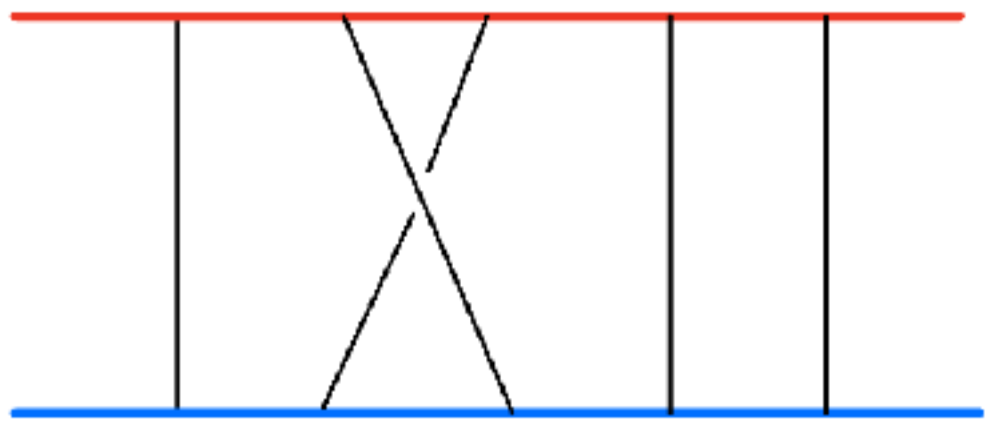
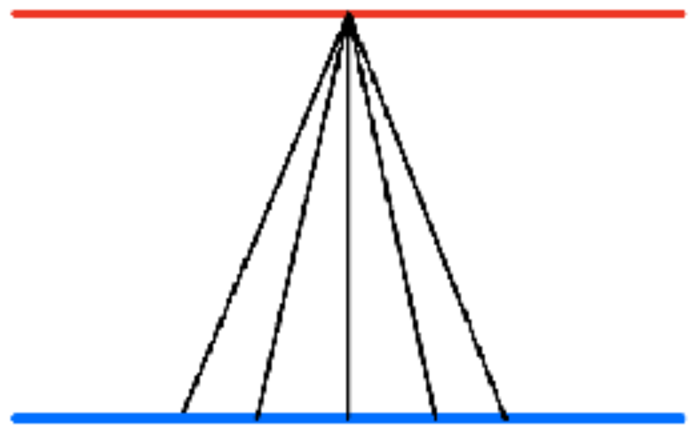
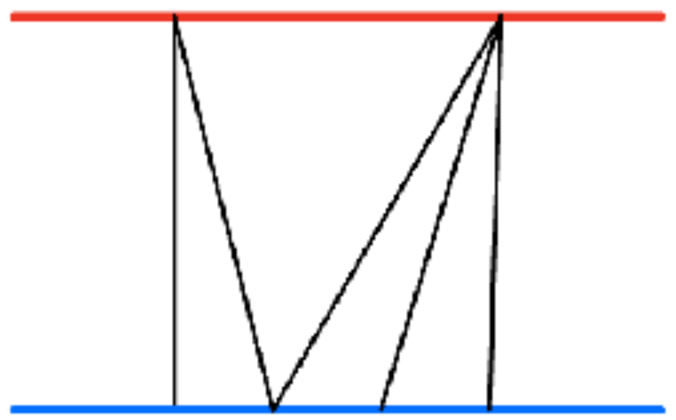
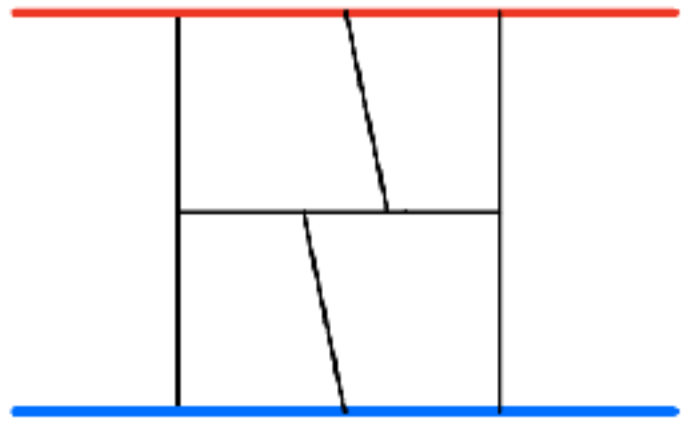
From explicit solution of DE

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6 K^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7 E^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

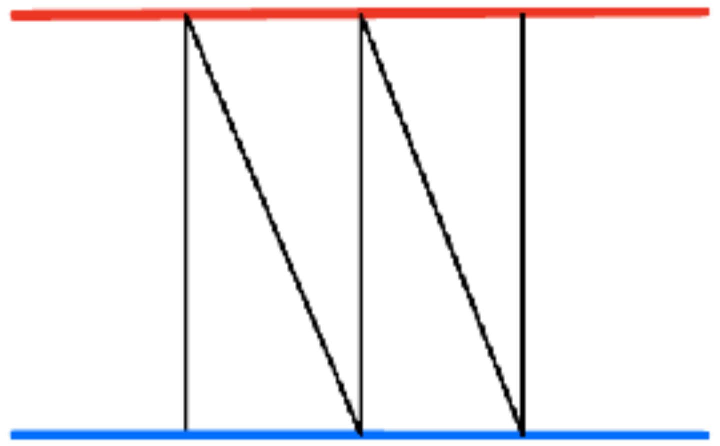
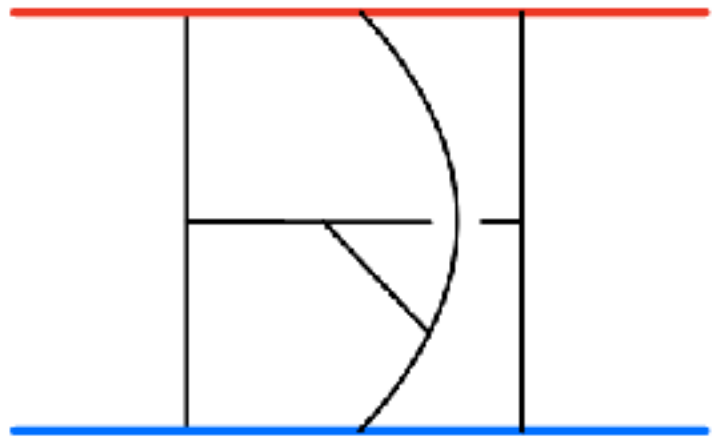
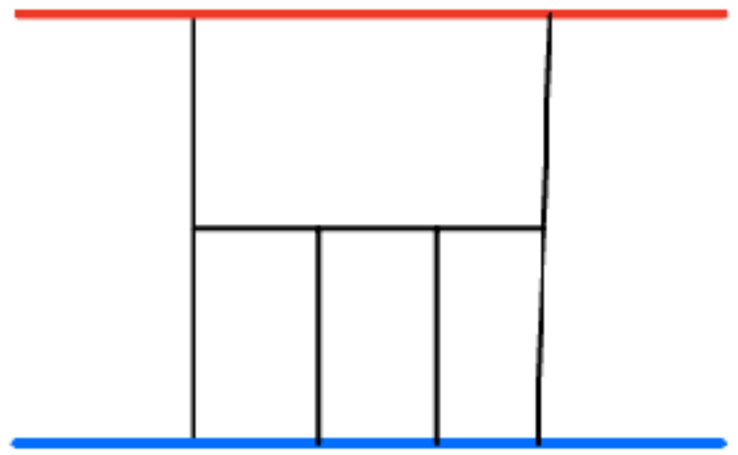
$$\begin{aligned}\mathcal{M}_4^{\rm rem} = {} & r_8 + r_9 \log\left(\frac{\sigma+1}{2}\right) + r_{10} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma) \\ & + r_{13} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\frac{\sigma+1}{2}\right) + r_{14} \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} \\ & + r_{15} \operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16} \operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2-1}} \left[\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) \right]\end{aligned}$$

Classical scattering at $O(G^4)$ — Summary

- Classical scattering at G^4 from amplitudes approach very efficient
- Main improvement from importing integration techniques from collider physics (IBP, DE, Generalised unitarity)
- Results confirmed and extended to include radiation and spin [Dlapa et al., Jacobsen et al., Damgaard et. al.]
- Main bottleneck integration
  - Integral reduction
  - Evaluation of master integrals

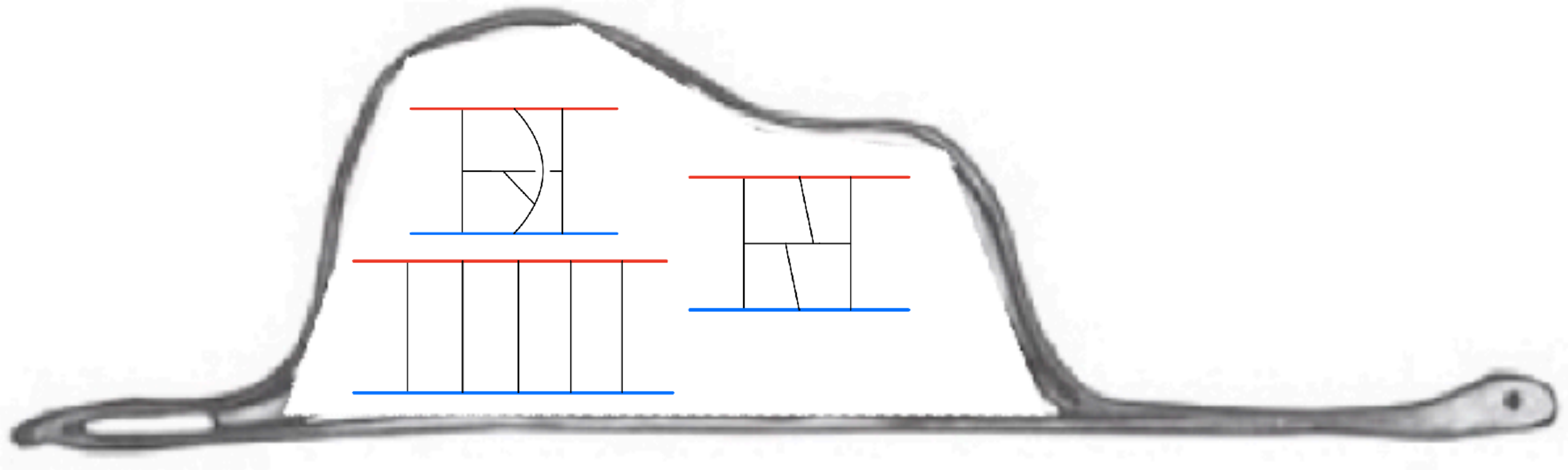





Towards Gravitational scattering at G^5

Amplitude computation

- Amplitude computation comes in parts
  - Constructing the integrand ✓ 51 cuts for 5PM, 6PM straightforward
  - Integral reduction ?
  - Evaluation of master integrals ?
- Integral reduction:
  - linear algebra problem, exponential growth in # eqns.
  - Intractable without major improvements
- Evaluating integrals:
  - DE’s for almost all families
  - Series soon, analytic results harder



The 5PM problem is hard, don’t try to swallow it whole!

Slicing the 5PM problem

- Can we eat our cake one piece at a time? Expansions?

$$\mathcal{M}_{5\mathrm{PM}}(v, m_1, m_2)$$

- PN expansion $v \to 0$
  - Complicated topologies suppressed
  - Important for phenomenology
- High-energy expansion $v \to 1$
  - Less well understood, important conceptual questions
- Expanding looses information on functional structure

Slicing the 5PM problem

- Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}_{5\mathrm{PM}}(v, m_1, m_2)$$

- Hierarchical limit (SF) $m_1 \ll m_2$

$$\nu = \frac{\mu}{M} = m_1 m_2/(m_1 + m_2)^2 \leq 1/4$$

  - Organization into gauge-invariant objects

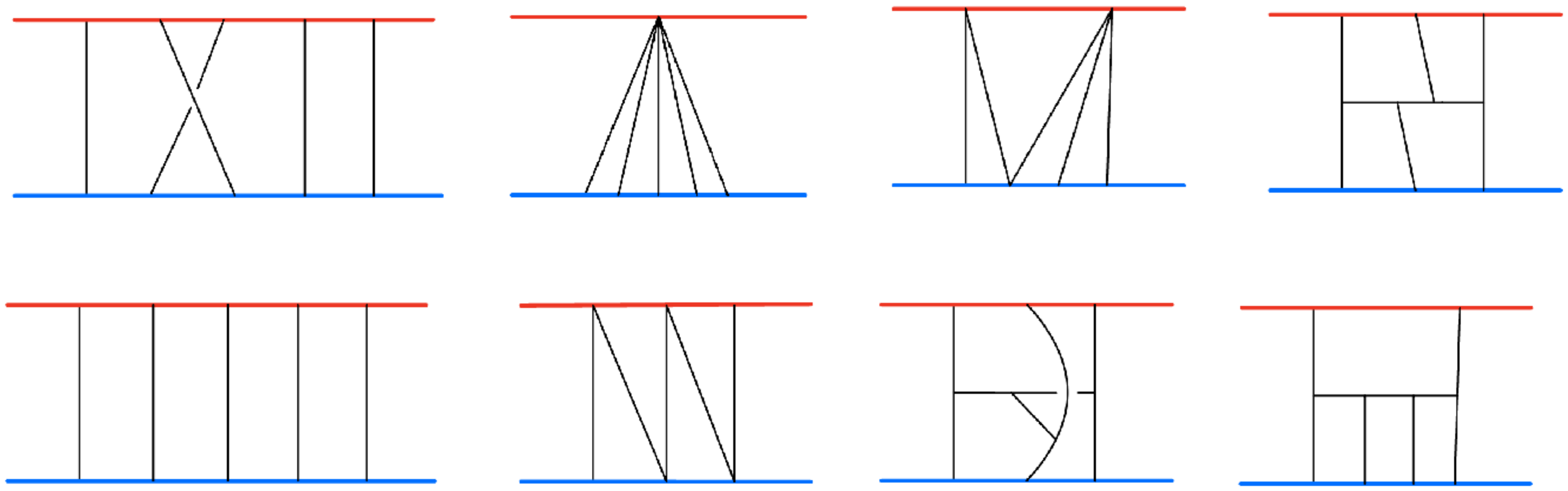
$$\mathscr{M}_{5\mathrm{PM}} = \mathscr{M}^{0\mathrm{SF}}_{5\mathrm{PM}} + \nu \mathscr{M}^{1\mathrm{SF}}_{5\mathrm{PM}} + \nu^2 \mathscr{M}^{2\mathrm{SF}}_{5\mathrm{PM}} \leftarrow \text{trivial from amplitudes}$$

  - Useful expansion for equal-mass case! Similar to QCD $\frac{1}{N_c} = \frac{1}{3}$
  - Complicated integrals suppressed

Slicing the 5PM problem

- Can we eat our cake one piece at a time? Expansions?

$\mathscr{M}_{5PM} =$

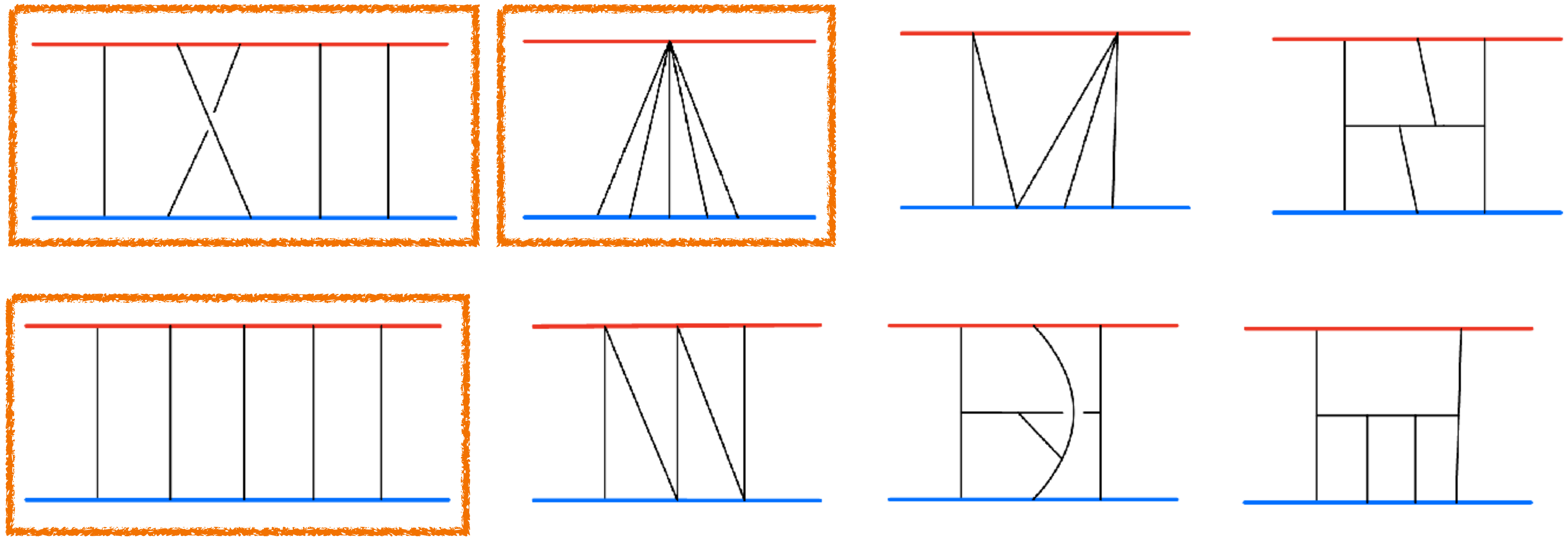


+ $\mathcal{O}(10^4)$ more

Slicing the 5PM problem

- Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}^{0\mathrm{SF}}_{5\mathrm{PM}} =$$

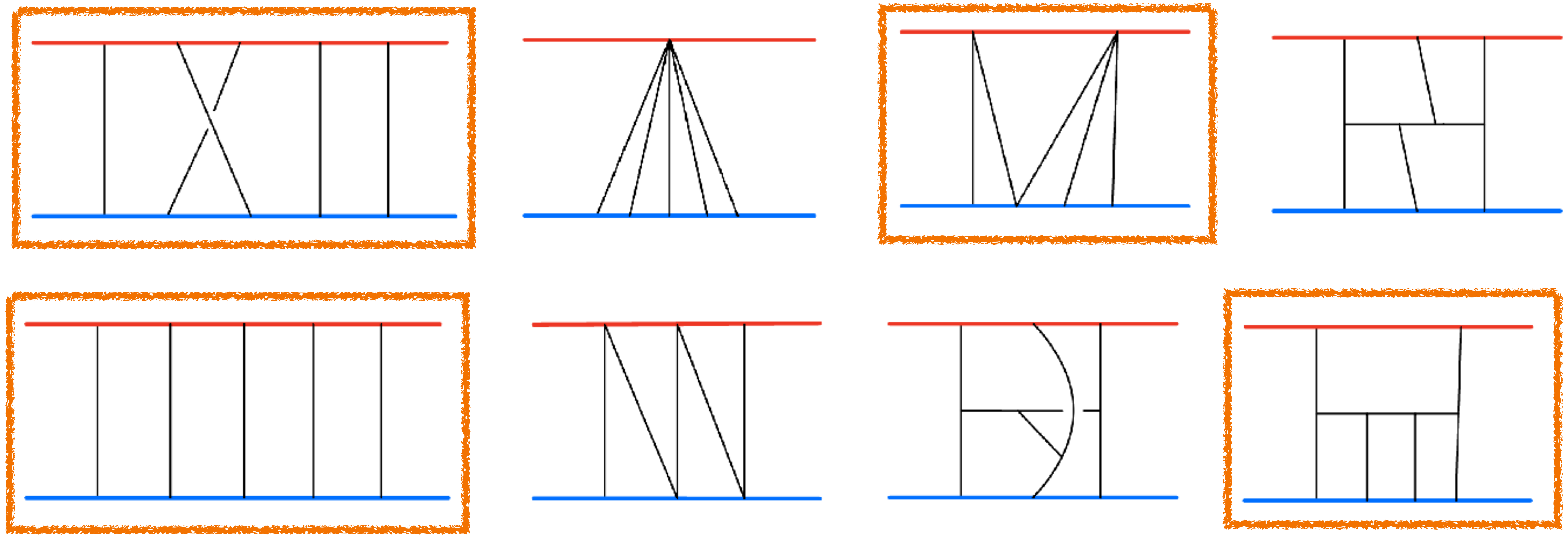


+ $\mathscr{O}(10^4)$ more

Slicing the 5PM problem

- Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}^{\mathrm{1SF}}_{\mathrm{5PM}} =$$

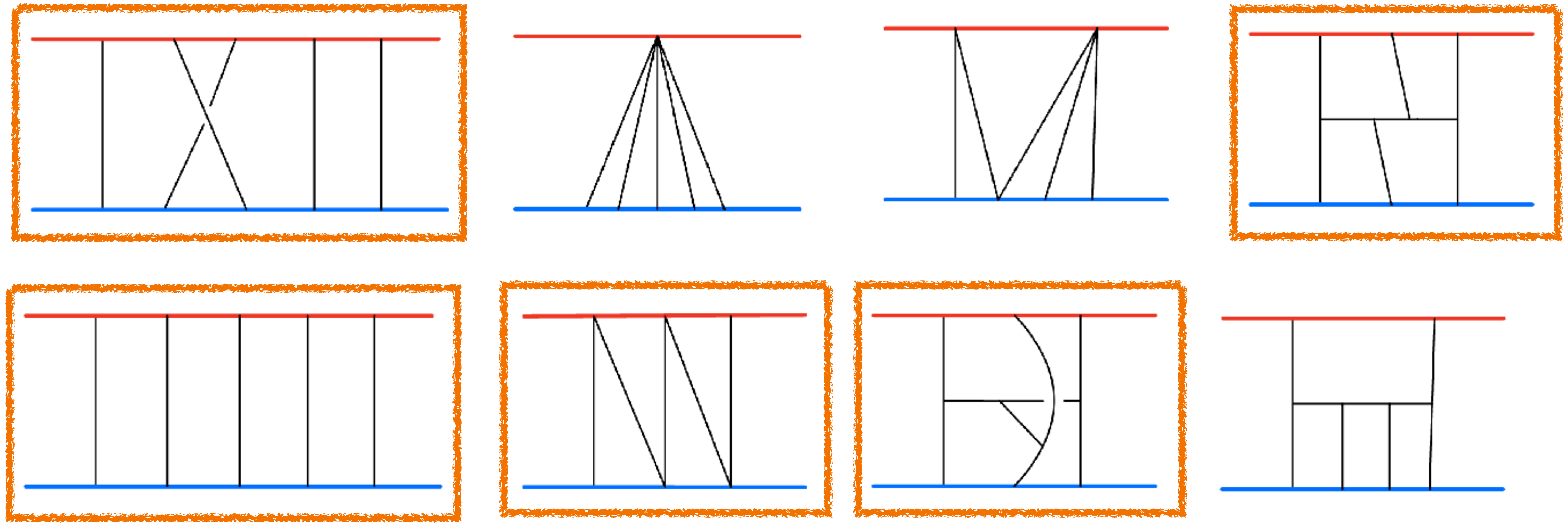


$+\ \mathcal{O}(10^4)$ more

Slicing the 5PM problem

- Can we eat our cake one piece at a time? Expansions?

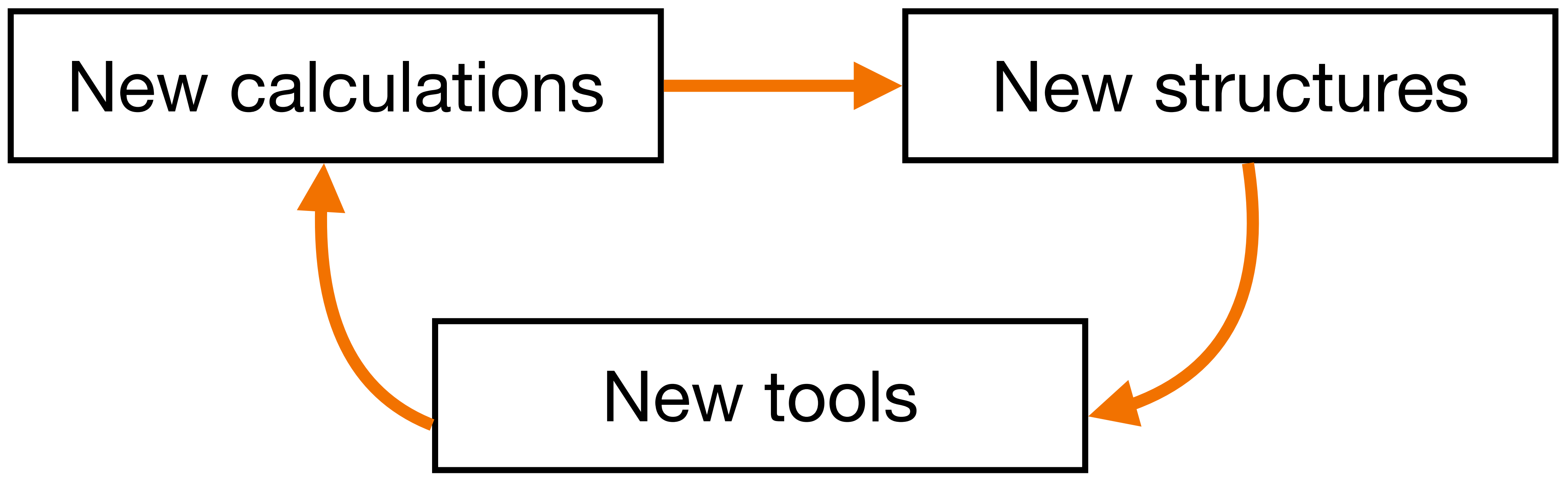
$$\mathscr{M}^{\rm 2SF}_{\rm 5PM} =$$



+ $\mathcal{O}(10^4)$ more

Slicing the 5PM problem

- Work on 1SF in progress
- Use a simpler model without approximations:
  - Maximal SUSY, scalar toy models,…
  - Electrodynamics
- Main criteria:
  - Sizeable overlap with GR
  - Significantly more complicated than 4PM
  - Real world system, applications to phenomenology

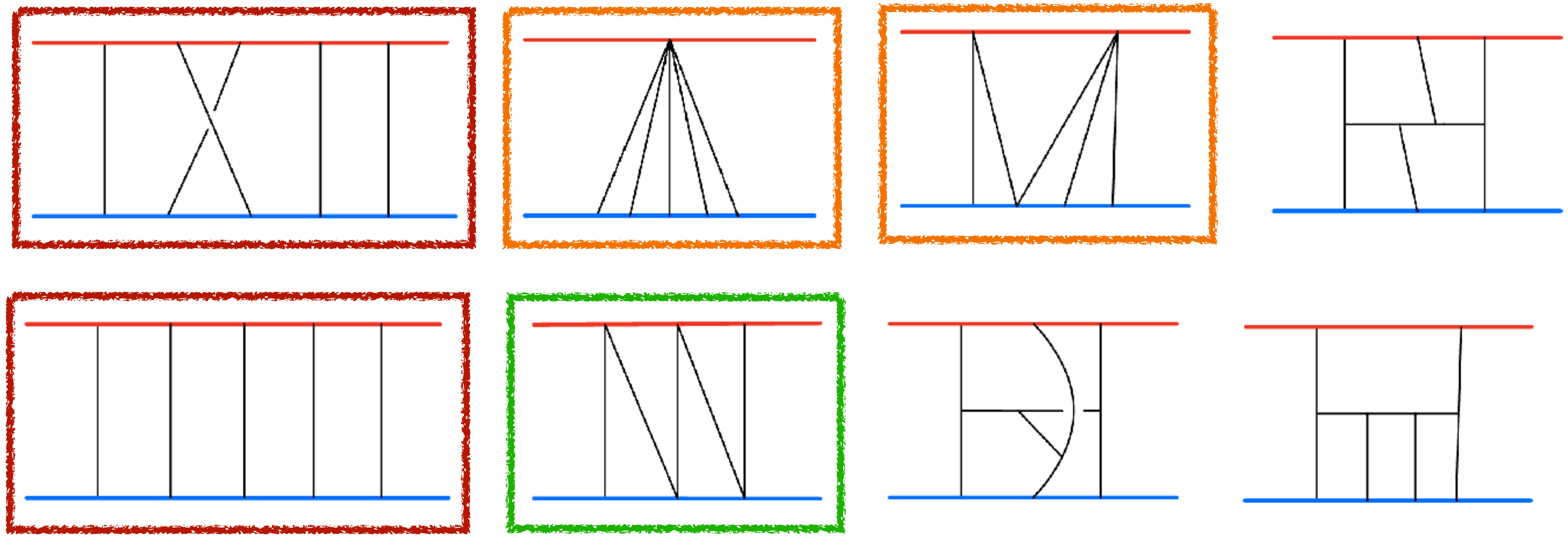


Electrodynamics checks all the boxes

Slicing the 5PM problem

- Can we eat our cake one piece at a time?

$$\mathscr{M}^{\text{QED}}_{\text{5PL}} =$$



As complicated as in GR!

2SF graphs

$+\ \mathcal{O}(10^4)$ more

Classical scattering at $O(\alpha^5)$

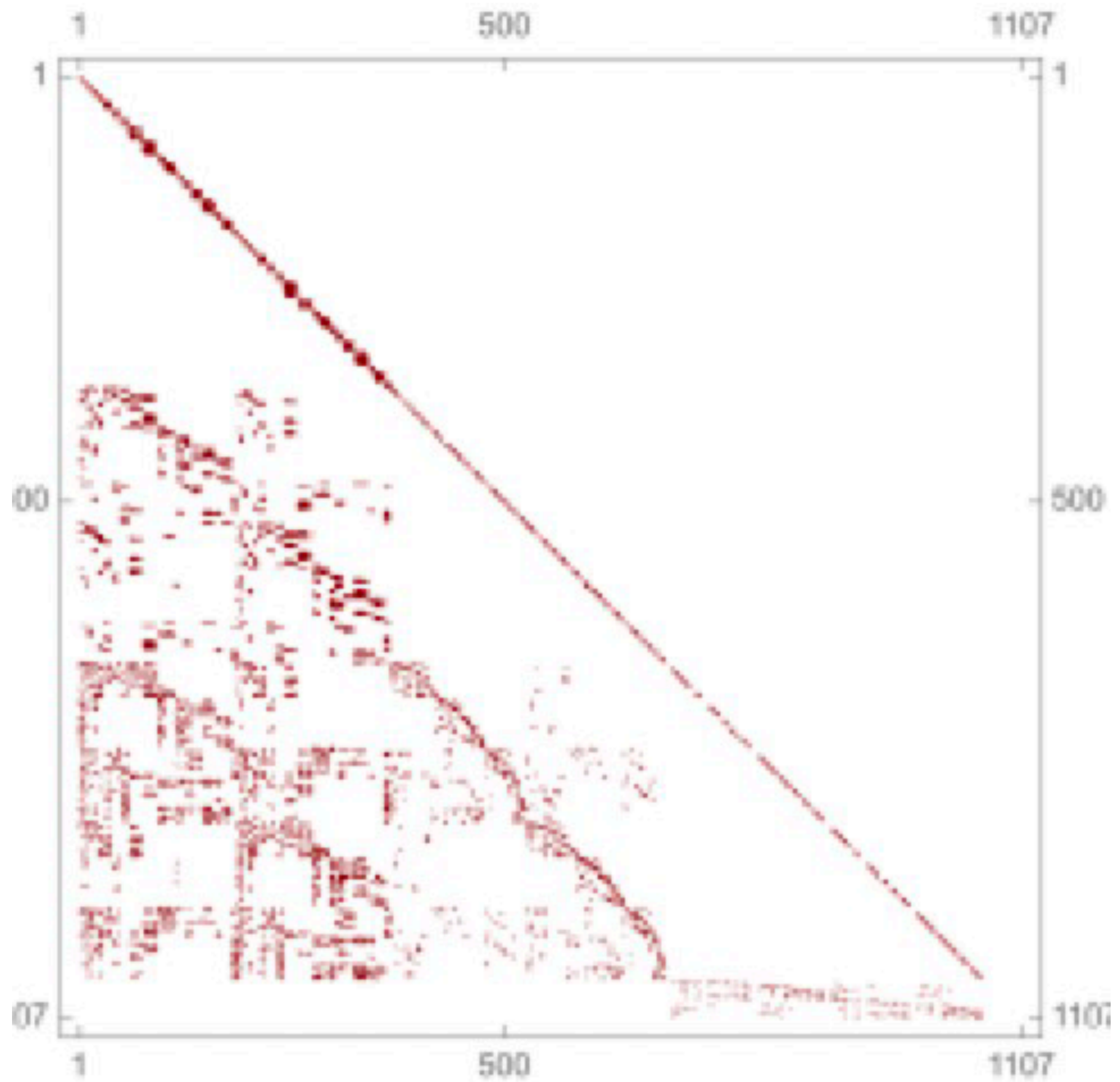
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \sum_{i=1}^{2}\left[|D_\mu \phi_i|^2 - m_i^2|\phi_i|^2\right]$$

- QED integrand trivial: ~1000 Feynman diagrams
- Deep expansion in the classical limit $\mathcal{M}_4 \sim \frac{1}{\hbar^5} + \ldots + \frac{1}{\hbar}$
- Integral reduction: 10^6 integrals → 1107 masters, 23 families
- For the integration integral reduction use FIRE+LiteRed:
  - Choosing better basis of master integrals [Smirnov; Usovitch]
  - Removing redundant equations

Classical scattering at $O(\alpha^5)$ — Integration

- No elliptic integrals for QED
- Differential equations in canonical form

$$\partial_x \vec{I} = \epsilon \sum_{k,n} f_k^n(x) A_{k,n} \vec{I}, \quad r = |\vec{I}| = 1107,$$



- In terms of cyclotomic kernels

$$f_n^k(x) = \frac{x^k}{\Phi_n(x)}, \quad \Phi_{1,2} = x \pm 1, \Phi_4 = 1 + x^2, \Phi_{3,6} = 1 \pm x + x^2$$

Classical scattering at $O(\alpha^5)$

- Cyclotomic harmonic polylogs introduced in [Ablinger, Blumelein, Schneider 2011]
  - Manifestly real $C_4^0(x) = \int \frac{\mathrm{d}x}{1+x^2} = \frac{\mathrm{i}}{2}\int \mathrm{d}x \left[\frac{1}{x+\mathrm{i}} - \frac{1}{x-\mathrm{i}}\right] = \frac{\mathrm{i}}{2}(\log(x+\mathrm{i}) - \log(x-\mathrm{i})) = \arctan(x)$
  - Shuffle algebra (minimal basis)
  - Integration and differential rules
  - Series expansion & numerics

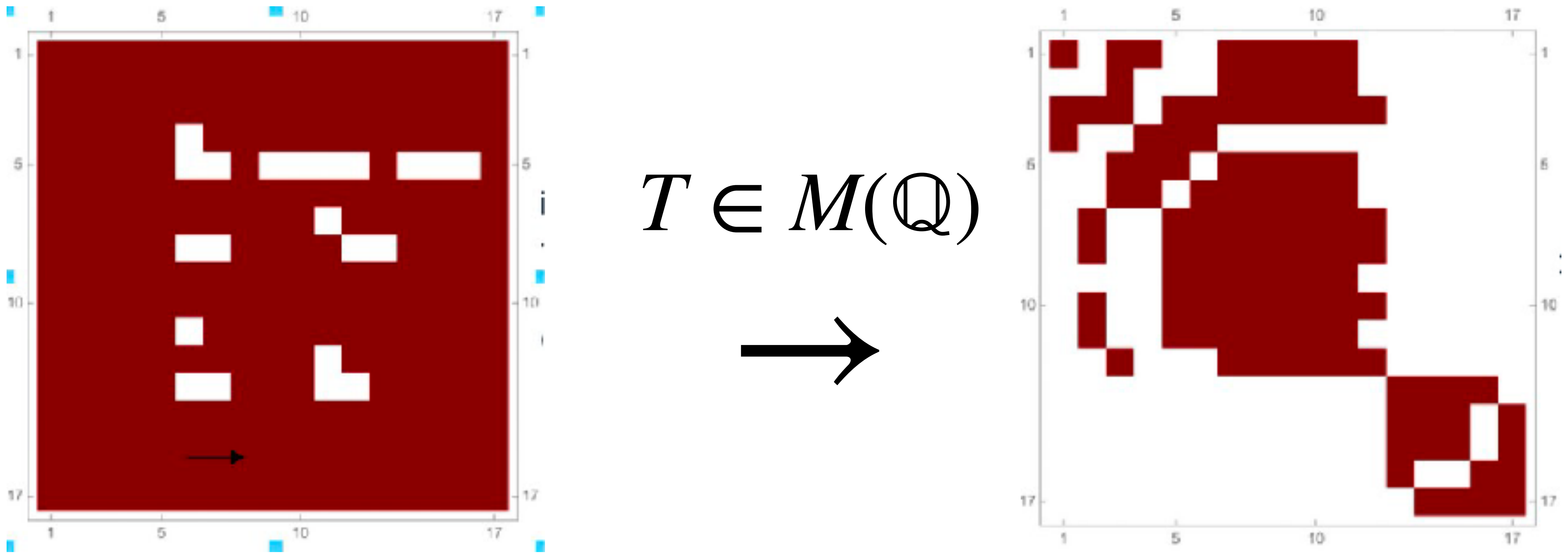
- Special combinations (amplitude) in terms of real Li's

$$C_{400}^{100}(x) - C_{000}^{200}(x) = -\frac{\mathrm{Li}_3\left(x^6\right)}{18} + \frac{1}{6}\mathrm{Li}_2\left(x^6\right)\log(x) + \frac{4\,\mathrm{Li}_3\left(x^3\right)}{9} - \frac{2}{3}\mathrm{Li}_2\left(x^3\right)\log(x) + \frac{\mathrm{Li}_3\left(x^2\right)}{2}$$
$$-\frac{1}{2}\mathrm{Li}_2\left(1-x^2\right)\log(x) - 4\,\mathrm{Li}_3(x) - 2\,\mathrm{Li}_2(-x)\log(x) - \log\left(1-x^2\right)\log^2(x) + \frac{1}{4}\pi^2\log(x) + \frac{707}{32}\zeta_3$$

Classical scattering at $O(\alpha^5)$ — Integration

- DE has additional structure: sparse, top sectors don't talk to bottom
- Canonical DE invariant under rational transformation: factorize

$$\color{red}{\epsilon}\sum_{k,n} f_k^n(x) A_{k,n} \to \color{red}{\epsilon}\sum_{k,n} f_k^n(x) A'_{k,n}$$



- Bonus relations between integrals, linked to special structure of eikonal integrals

Classical scattering at $O(\alpha^5)$ —Results

$$\chi_{\text{pot}}^{5\text{PL}} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 (\sigma^2 - 1)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Classical scattering at $O(\alpha^5)$ —Results

Fifth post-Lorentzian order

$$\chi_{\rm pot}^{\rm 5PL} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 \left(\sigma^2 - 1\right)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Classical scattering at $O(\alpha^5)$ —Results

$$\chi^{\rm 5PL}_{\rm pot} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 \left(\sigma^2 - 1\right)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Contribution from potential modes well-defined! No tail!

Classical scattering at $O(\alpha^5)$ —Results

$$\chi_{\text{pot}}^{5\text{PL}} = \frac{\alpha^5(m_1+m_2)^4}{30J^5E^4\,(\sigma^2-1)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Mass polynomiality

Second order E&M self-force!

Classical scattering at $O(\alpha^5)$ —Results

$$\chi^{5PL}_{pot} = \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

$$
\begin{aligned}
r_9^{(1)} &= r_{12}^{(1)} = 240(\sigma^2-1)^2\,, \\
r_{11}^{(1)} &= 120(\sigma^2-1)(\sigma^2+2\sigma-1)\,, \\
r_6^{(1)} &= r_7^{(1)} = r_{10}^{(1)} = 0\,, \\
r_1^{(2)} &= \frac{405\sigma\left(15-44\sigma^2\right)}{16\left(1-4\sigma^2\right)^2} - \frac{15\left(10\sigma^2+2\sigma-3\right)}{\sigma^3} \\
&\quad + \frac{-2048\sigma^7+6656\sigma^6+17872\sigma^5+20000\sigma^4}{16} \\
&\quad + \frac{-7740\sigma^3-22560\sigma^2-6635\sigma-2080}{16}\,, \\
r_2^{(2)} &= \sqrt{\sigma^2-1}\left[\frac{45\left(1232\sigma^4-1168\sigma^2+287\right)}{16\left(4\sigma^2-1\right)^3} \right. \\
&\quad + \frac{30\left(20\sigma^3-9\sigma^2-4\sigma+3\right)}{\sigma^4} \\
&\quad \left. + \frac{5}{16}\left(1776\sigma^4+8192\sigma^3+10820\sigma^2+11776\sigma+3223\right)\right]\,, \\
r_3^{(2)} &= -\frac{30\left(16\sigma^4+36\sigma^3-11\sigma^2-6\sigma+3\right)}{\sigma^5} \\
&\quad + 20\left(212\sigma^3+350\sigma^2+328\sigma+319\right)\,, \\
r_4^{(2)} &= \frac{2880(\sigma+1)(3\sigma+1)}{\sqrt{\sigma^2-1}}\,, \\
r_6^{(2)} &= 480\left(\sigma^2-1\right)^{3/2}\left(2\sigma^2-1\right)\,, \\
r_7^{(2)} &= 45\sigma\left(\sigma^2-1\right)^{5/2}\,, \\
r_9^{(2)} &= -480\left(\sigma^2-1\right)\left(\sigma^2-\sigma-1\right)\,, \\
r_{10}^{(2)} &= -135\left(\sigma^2-1\right)^2\,, \\
r_{12}^{(2)} &= -480\left(\sigma^2-1\right)\left(\sigma^2-2\sigma-1\right)\,, \\
r_5^{(2)} &= r_8^{(2)} = r_{11}^{(2)} = 0\,. \qquad (14)
\end{aligned}
$$

Poles at $\sigma = 0, \pm 1/2, \pm 1$

Outside of the scattering region $1 < \sigma$

Implications for bound state?

Rational coefficients

Classical scattering at $O(\alpha^5)$ —Results

Transcendental functions

$$\chi^{5PL}_{pot} = \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

$$\begin{aligned}
&f_1 = 1\,,\ f_2 = C_0^0(x)\,,\ f_3 = C_{0,0}^{0,0}(x)\,,\ f_4 = C_{0,0,0}^{0,0,0}(x)\,,\\
&f_5 = -C_{1,0}^{0,0}(x) + C_{2,0}^{0,0}(x) + \frac{\pi^2}{4}\,,\\
&f_6 = -C_{2,0}^{0,0}(x) + C_{4,0}^{1,0}(x) - \frac{\pi^2}{16}\,,\\
&f_7 = C_{3,0}^{0,0}(x) + 2C_{3,0}^{1,0}(x) + C_{6,0}^{0,0}(x) - 2C_{6,0}^{1,0}(x) + \frac{\pi^2}{6}\,,\\
&f_8 = -C_{0,1,0}^{0,0,0}(x) + C_{0,2,0}^{0,0,0}(x) + \frac{\pi^2}{4}C_0^0(x) + \frac{7\zeta_3}{2}\,,\\
&f_9 = -C_{0,2,0}^{0,0,0}(x) + C_{0,4,0}^{0,1,0}(x) - \frac{\pi^2}{16}C_0^0(x) - \frac{21\zeta_3}{16}\,,\\
&f_{10} = C_{0,3,0}^{0,0,0}(x) + 2C_{0,3,0}^{0,1,0}(x) + C_{0,6,0}^{0,0,0}(x) - 2C_{0,6,0}^{0,1,0}(x)\\
&\qquad + \frac{1}{6}\pi^2 C_0^0(x) + \frac{28\zeta_3}{9}\,,\\
&f_{11} = -C_{1,0,0}^{0,0,0}(x) + C_{2,0,0}^{0,0,0}(x) - \frac{7\zeta_3}{4}\,,\\
&f_{12} = -C_{2,0,0}^{0,0,0}(x) + C_{4,0,0}^{1,0,0}(x) + \frac{21\zeta_3}{32}\,. \qquad (15)
\end{aligned}$$

Functions are special:

No ζ-values $f_k = \sum_{n,r} \log^r(1-x) a_n^r (1-x)^n\,, \quad a_n^r \in \mathbb{Q}$

Only specific contributions of indices (symbology)

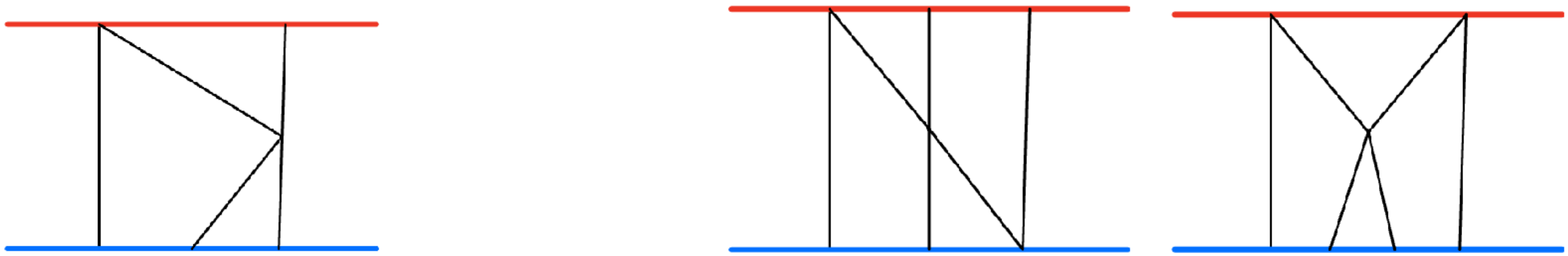
Alternative form: polylogs with real arguments

Classical scattering at $O(\alpha^5)$ —Lessons

- Computations at fifth order in perturbation theory are possible!
- E&M scattering at large impact parameter — heavy ion scattering
- First glimpse at the function space, organization in terms of CHPL useful
- Understand which parts of the computation are hardest → 2SF graphs
- QED is an useful playground for bootstrap ideas
  - High post-Coulombian orders from Fokker action
  - Simple space of transcendental function
  - Additional relation to energy loss
- Performed lots of checks of the result

Classical scattering at $O(G^5)$ —Status

- QED was surprisingly simple — only CHPL's
- For GR likely functions beyond polylogs and E, K
- Integrals more complicated than amplitude → series + ansatz



$$\sim \left\{ K^2\left(\frac{1-\sigma}{1+\sigma}\right), E\left(\frac{1-\sigma}{1+\sigma}\right) K\left(\frac{1-\sigma}{1+\sigma}\right), E^2\left(\frac{1-\sigma}{1+\sigma}\right) \right\}$$

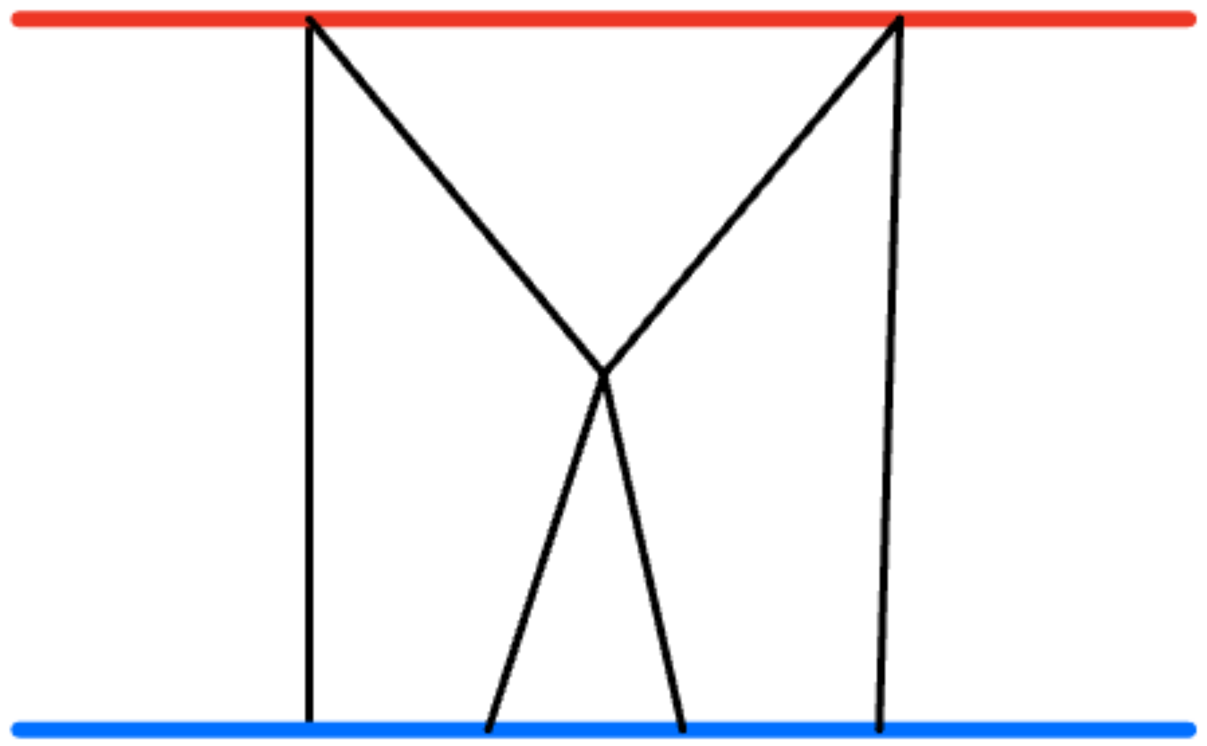
$$\sim \{\, ? \,\}$$

Classical scattering at $O(G^5)$ —Status

Example 1 (1SF)

$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_5 \end{pmatrix} = \epsilon \sum_k \mathrm{d}\log(w_k) A_k \begin{pmatrix} f_1 \\ \vdots \\ f_5 \end{pmatrix}, \quad w_k \in \{x, 1 \pm x, 1 + x^2\}$$

Same as at 3 loop



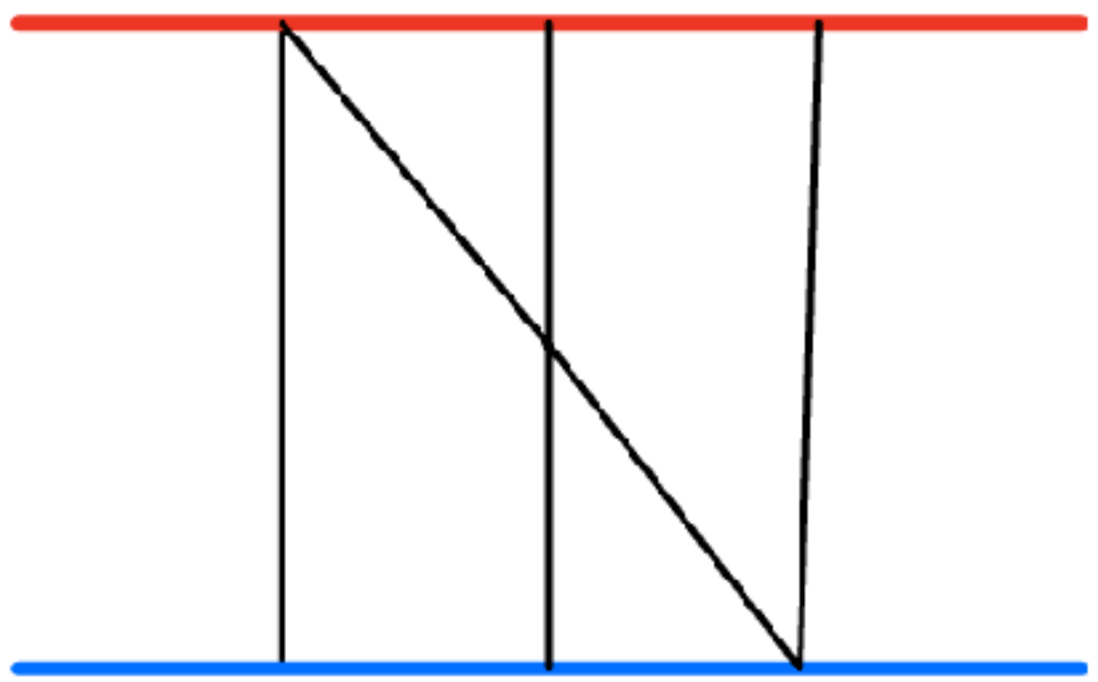
evaluates to CHPL's !

Classical scattering at $O(G^5)$ —Status

Example 2 (2SF)

Beyond cyclotomic alphabet

$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix} = \sum_k \mathrm{d}\log(w_k) A_k \begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix}, \quad w_k = \{x, 1 \pm x, 1 \pm x + x^2, 1 \pm 6x + x^2, 1 + 6x^2 + x^4\}$$



$$x = 3 - 2\sqrt{2} = 0.171... \in [0,1]$$

$$x = -\mathrm{i}\sqrt{3 - 2\sqrt{2}} \to \gamma = \mathrm{i}$$

Inside the scattering region $x \in [0,1]$!

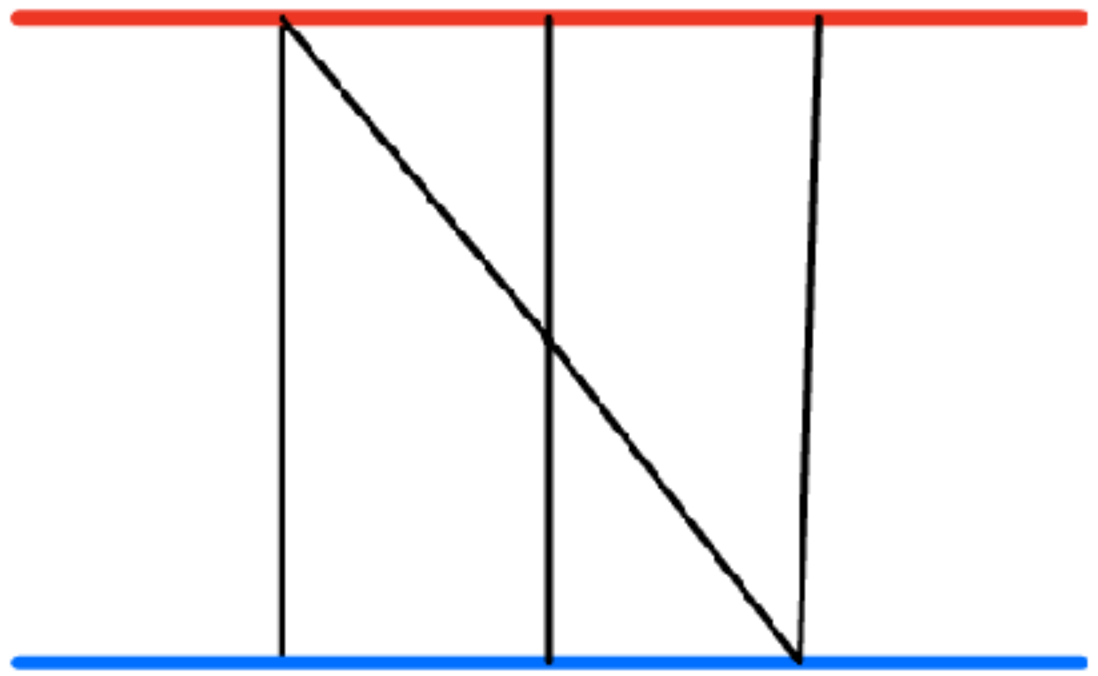
A_k eigenvalues $1/2 + a\epsilon \, , a \in \mathbb{Z}$ suggest elliptic (or more complicated) integrals

Classical scattering at $O(G^5)$ —Todo

- Equivalently solve 9th order ODE — hard

$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix} = \epsilon \sum_k \mathrm{d}\log(w_k) A_k \begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix}, \quad 0 = \sum_{k=0}^{9} p_k(x,\epsilon) \frac{\mathrm{d}^k}{\mathrm{d}^k x} f_1$$

- Needs detailed study of the geometry involved
- Might need to resort to series expansion



Conclusion

- Amplitudes-based program to extract GR observables to high orders in perturbation theory
- Higher precision, more loops
- Scattering in QED at 4 loops important case study
  - Proof of principle computation
  - Large overlap with GR computation ~25% of the master integrals, including 2SF integrals
  - Identified bottlenecks and improved setup in integral reduction
- Progress towards scattering at G^5
  - Integrand constructed ✓
  - Differential equations for all but a few families (hopefully completed soon) ✓
  - Integral reduction challenging **?** New ideas from collider physics could help
- Eventually reconnect with the bound problem

Optimistic on near-term progress — 5PM hard but doable!