

# Radiative Observables from the Eikonal Operator

Carlo Heissenberg

Uppsala University and Nordita

From Amplitudes to Gravitational Waves, July 24th, 2023

Based on

P. Di Vecchia, CH, R. Russo, G. Veneziano [2210.12118]

CH [2210.15689]

A. Georgoudis, CH, I. Vazquez–Holm [2303.07006]

P. Di Vecchia, CH, R. Russo, G. Veneziano [2306.16488] (review)



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# Outline

- 1 Introduction: the Elastic Eikonal
- 2 Eikonal Operator
- 3 Angular Momentum
- 4 Waveforms

# Outline

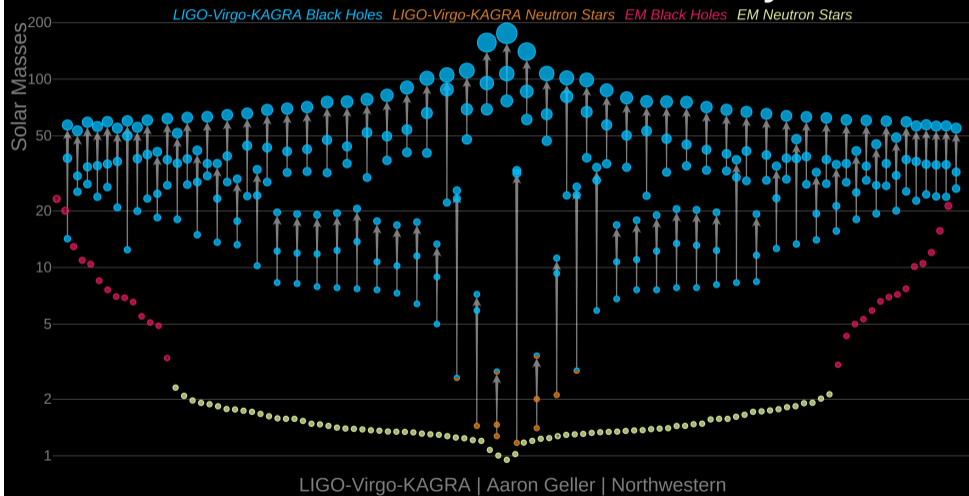
1 Introduction: the Elastic Eikonal

2 Eikonal Operator

3 Angular Momentum

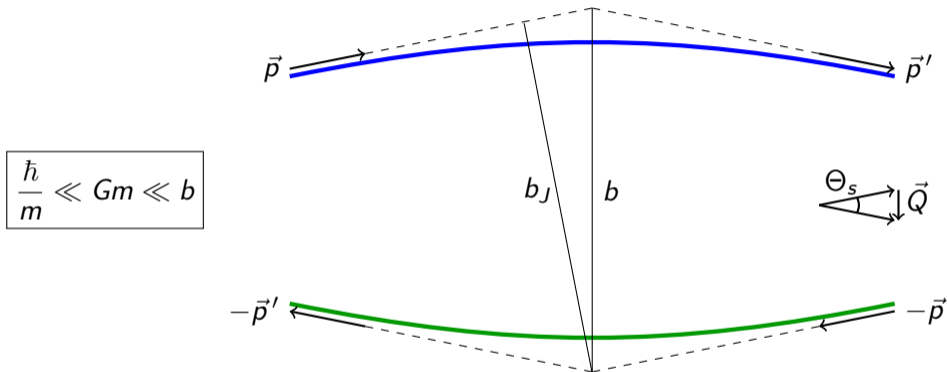
4 Waveforms

## Masses in the Stellar Graveyard



# Post-Minkowskian (PM) Scattering

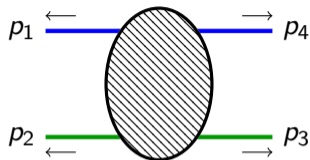
[see e.g. Kosower, Maybee, O'Connell '18; Bern et al. '19; Di Vecchia, CH, Russo, Veneziano '21; Bellazzini, Isabella, Riva '22]



$$\frac{\hbar}{m} \ll Gm \ll b$$

$$\frac{Gm^2}{\hbar} \gg 1, \quad \frac{Gm}{b} \ll 1, \quad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic).}$$

# The Elastic Eikonal



$$\begin{aligned}
 s &= -(p_1 + p_2)^2 = E^2 \\
 &= m_1^2 + 2m_1 m_2 \sigma + m_2^2, \\
 t &= -(p_1 + p_4)^2 = -q^2.
 \end{aligned}$$

- From  $q$  to  $b$ : Fourier transform [ $q \sim \mathcal{O}(\frac{\hbar}{b})$ ]

$$\tilde{\mathcal{A}}(b) = \text{FT}(b) = \int \frac{d^D q}{(2\pi)^D} 2\pi\delta(2p_1 \cdot q) 2\pi\delta(2p_2 \cdot q) e^{ib \cdot q} \mathcal{A}(s, q), \quad \boxed{1 + i\tilde{\mathcal{A}}(b) = e^{2i\delta(b)}}$$

with  $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \dots \sim \frac{Gm^2}{\hbar} \left( \log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \dots \right)$

- From  $b$  to  $Q$ : stationary-phase approximation [ $Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})$ ]

$$\int d^{D-2} b e^{-ib \cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial \text{Re } 2\delta}{\partial b^\mu}$$

# The 3PM Eikonal in General Relativity

[Di Vecchia, CH, Russo, Veneziano '20, '21]

[Related work at 3PM: Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Damour '20; Herrmann, Parra-Martinez, Ruf, Zeng '21, Bjerrum-Bohr, Damgaard, Planté, Vanhove '21; Brandhuber, Chen, Travaglini, Wen '21]

- Eikonal phase:

$$\text{Re } 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2} \left[ \frac{s(12\sigma^4 - 10\sigma^2 + 1)}{2m_1 m_2 (\sigma^2 - 1)^{\frac{3}{2}}} - \frac{\sigma(14\sigma^2 + 25)}{3\sqrt{\sigma^2 - 1}} - \frac{4\sigma^4 - 12\sigma^2 - 3}{\sigma^2 - 1} \text{arccosh } \sigma \right] + \text{Re } 2\delta_2^{\text{RR}}$$

with

$$\text{Re } 2\delta_2^{\text{RR}} = \frac{G}{2} Q_{\text{1PM}}^2 \mathcal{I}(\sigma), \quad \mathcal{I}(\sigma) \equiv \frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{\sigma(2\sigma^2 - 3)}{(\sigma^2 - 1)^{3/2}} \text{arccosh } \sigma.$$

- Infrared divergent exponential suppression:

$$\text{Im } 2\delta_2 = \frac{1}{\pi} \left[ -\frac{1}{\epsilon} + \log(\sigma^2 - 1) \right] \text{Re } 2\delta_2^{\text{RR}} + \dots$$

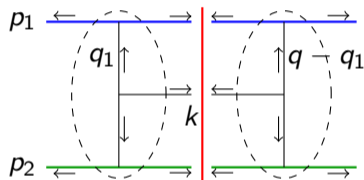
# Reverse Unitarity

[Herrmann, Parra-Martinez, Ruf, Zeng '21] [Di Vecchia, CH, Russo, Veneziano '21]

By unitarity,  $2 \operatorname{Im} 2\delta_2 = \int_k \tilde{\mathcal{A}}_{\mu\nu}^{(5)}(k) (\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}) \tilde{\mathcal{A}}_{\rho\sigma}^{(5)*}(k) \equiv \int_k \tilde{\mathcal{A}}^{(5)} \tilde{\mathcal{A}}^{(5)*}$

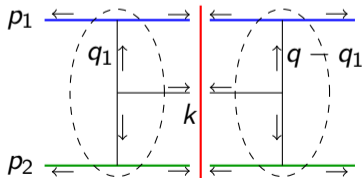
- Conveniently recast as the FT of a cut in momentum-space [Di Vecchia, CH, Russo, Veneziano '21]

$$2 \operatorname{Im} 2\delta_2 = \text{FT} \int d(\text{LIPS})$$



- Similarly to [Herrmann, Parra-Martinez, Ruf, Zeng '21]

$$P^\alpha = \int_k k^\alpha \tilde{\mathcal{A}}^{(5)} \tilde{\mathcal{A}}^{(5)*} = \text{FT} \int d(\text{LIPS}) k^\alpha$$





# Elastic Final State

- Final state (schematically):

$$|\text{out}\rangle = e^{2i\delta(b)}|\text{in}\rangle$$

- Impulse:

$$Q_\mu = \left( -i \langle \text{out} | \frac{\overleftrightarrow{\partial}}{\partial b^\mu} | \text{out} \rangle \right) / \langle \text{out} | \text{out} \rangle = \frac{\partial \text{Re } 2\delta}{\partial b^\mu}.$$

## Questions:

- 1 How do we restore (nonperturbative) **unitarity**?

$$\langle \text{out} | \text{out} \rangle = e^{-2 \text{Im } 2\delta} \langle \text{in} | \text{in} \rangle \rightarrow 0 \quad \text{as } D \rightarrow 4$$

- 2 How do we calculate the **radiative observables**?
  - Angular momentum loss(es)
  - Scattering waveform

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# Operator Exponential

- **Option 1:** Look for an exponential structure in the full  $S$ -matrix

[Damgaard, Planté, Vanhove '21; Damgaard, Hansen, Planté, Vanhove '23]

$$S = 1 + iT = e^{iN} \quad \boxed{|\text{out}\rangle = e^{iN}|\text{in}\rangle}$$

The “ $N$ -matrix” elements are better behaved in the classical limit.

- **Option 2:** Look for an exponential structure in the classical limit,

$$\boxed{|\text{out}\rangle \simeq e^{2i\hat{\delta}}|\text{in}\rangle}$$

with  $2\hat{\delta}$  a suitable **eikonal operator**

[Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White '21] [Di Vecchia, CH, Russo, Veneziano '22].

# $N$ -Operator, $T$ -Operator and Unitarity

- **$N$ -operator:**  $S = e^{iN}$

$$N = -i \log(1 + iT) = T - \frac{i}{2} T^2 + \dots$$

up to one loop.

- **Unitarity:**  $S^\dagger S = 1$ ,

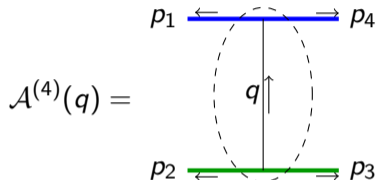
$$\frac{1}{2}(T - T^\dagger) = +\frac{i}{2} T^\dagger T$$

- Let us denote by  $\mathcal{B}$  the  $N$ -matrix elements, just like  $\mathcal{A}$  denotes the the usual amplitudes ( $T$ -matrix elements). Then, by unitarity,

$$\mathcal{B}_0 = \mathcal{A}_0, \quad \mathcal{B}_1 = \text{Re } \mathcal{A}_1.$$

# The $2 \rightarrow 2$ Amplitude

[Parra-Martinez, Ruf, Zeng '20]



$$p_1^\mu = -\bar{m}_1 u_1^\mu + q^\mu / 2$$

$$p_4^\mu = +\bar{m}_1 u_1^\mu + q^\mu / 2$$

$$p_2^\mu = -\bar{m}_2 u_2^\mu - q^\mu / 2$$

$$p_3^\mu = +\bar{m}_2 u_2^\mu - q^\mu / 2$$

$$q \sim \hbar / b$$

- The “average velocities”  $u_{1,2}^\mu$  obey

$$u_1^2 = -1 = u_2^2 \quad \& \quad u_1 \cdot q = 0 = u_2 \cdot q$$

- We consider the physical variables

$$y = -u_1 \cdot u_2 \geq 1, \quad q^2.$$

Note that  $y = \sigma + \mathcal{O}(\hbar^2)$  and  $\bar{m}_{1,2} = m_{1,2} + \mathcal{O}(\hbar^2)$ .

## Tree-Level $2 \rightarrow 2$ Amplitude

- $2 \rightarrow 2$  amplitude in momentum space

$$\mathcal{A}_0^{(4)} = \mathcal{B}_0^{(4)} = \text{Diagram} = \frac{32\pi G \bar{m}_1^2 \bar{m}_2^2 \left( y^2 - \frac{1}{2(1-\epsilon)} \right)}{q^2} + \left[ \frac{4\pi G (\bar{m}_1^2 + \bar{m}_2^2)}{1-\epsilon} + \frac{\pi G (3-2\epsilon)}{\epsilon-1} q^2 \right]$$

analytic in  $q^2$

- In impact-parameter space  $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{2i\delta_0}}$

$$\tilde{\mathcal{A}}_0^{(4)} = \tilde{\mathcal{B}}_0^{(4)} = 2\delta_0 = \frac{4Gm_1m_2 \left( \sigma^2 - \frac{1}{2(1-\epsilon)} \right)}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b)^{-\epsilon}}$$

+ short-range

# One-Loop $2 \rightarrow 2$ Amplitude

[Cheung, Rothstein, Solon '18; Collado, Di Vecchia, Russo '19]

Elastic subtraction:

$$\mathcal{A}_1^{(4)} = \text{[Diagram: One-loop bubble with two external lines]} = \frac{i}{2} \text{[Diagram: Two tree-level diagrams]} + \mathcal{B}_1^{(4)}$$

with

$$\mathcal{B}_1^{(4)} = \frac{6\pi^2 G^2 \bar{m}_1^2 \bar{m}_2^2 (5y^2 - 1) (\bar{m}_1 + \bar{m}_2)}{\sqrt{q^2}}$$

in  $D = 4$

- The subtracted term  $\frac{i}{2}$  [Diagram: Two tree-level diagrams] is **imaginary** and **infrared divergent**
- It is also “superclassical”  $\mathcal{O}(\frac{1}{\hbar^2})$
- $\mathcal{B}_1^{(4)}$  is **real**, **finite** and **classical**  $\mathcal{O}(\frac{1}{\hbar})$  [and **transcendental weight 2**].

# Elastic Exponentiation at One Loop

[Collado, Di Vecchia, Russo '19]

Inelastic subtraction in  $b$ -space:

$$\tilde{\mathcal{A}}_1^{(4)} = \text{FT} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad = \frac{i}{2} (2\delta_0)^2 + \tilde{\mathcal{B}}_1^{(4)}$$

- This matches the exponential  $1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{i(2\delta_0+2\delta_1)}$

- And it identifies

$$\tilde{\mathcal{B}}_1^{(4)} = 2\delta_1 = \frac{3\pi G^2 m_1 m_2 (5\sigma^2 - 1) (m_1 + m_2)}{4b\sqrt{\sigma^2 - 1}}$$

$D = 4$ , which determines 2PM correction to the deflection angle.



# The $2 \rightarrow 3$ Amplitude

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

$$\mathcal{A}^{(5)\mu\nu}(q_1, q_2, k) =$$

$$p_1^\mu = -\bar{m}_1 u_1^\mu + q_1^\mu/2$$

$$p_4^\mu = +\bar{m}_1 u_1^\mu + q_1^\mu/2$$

$$p_2^\mu = -\bar{m}_2 u_2^\mu + q_2^\mu/2$$

$$p_3^\mu = +\bar{m}_2 u_2^\mu + q_2^\mu/2$$

$$q_1 \sim q_2 \sim k \sim \hbar/b$$

- The “average velocities”  $u_{1,2}^\mu$  obey

$$u_1^2 = -1 = u_2^2 \quad \& \quad u_1 \cdot q_1 = 0 = u_2 \cdot q_2$$

- We consider the physical variables

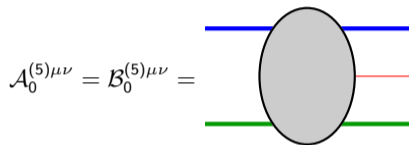
$$y = -u_1 \cdot u_2 \geq 1, \quad \omega_{1,2} = -u_{1,2} \cdot k \geq 0, \quad q_{1,2}^2.$$

Note that  $y = \sigma + \mathcal{O}(\hbar)$  and  $\bar{m}_{1,2} = m_{1,2} + \mathcal{O}(\hbar^2)$ .

# Tree-Level 2 $\rightarrow$ 3 Amplitude

[2  $\rightarrow$  3 amplitude: Goldberger, Ridgway '17; Luna, Nicholson, O'Connell, White '17] [Di Vecchia, CH, Russo, Veneziano, '21]

Gauge-invariant amplitude  $k_\mu \mathcal{A}_0^{(5)\mu\nu} = 0$ ,



$$\begin{aligned}
 &= (8\pi G)^{\frac{3}{2}} \left\{ 4m_1^2 m_2^2 (\sigma^2 - \frac{1}{D-2}) \left[ -\frac{p_1^\mu p_1^\nu (k \cdot q_1)}{(p_1 \cdot k)^2 q_2^2} - \frac{p_2^\mu p_2^\nu (k \cdot q_2)}{(p_2 \cdot k)^2 q_1^2} + \frac{p_1^{(\mu} (q_1 - q_2)^{\nu)}}{2(p_1 \cdot k) q_2^2} - \frac{p_2^{(\mu} (q_1 - q_2)^{\nu)}}{2(p_2 \cdot k) q_1^2} \right. \right. \\
 &+ \left. \frac{(q_1 - q_2)^\mu (q_1 - q_2)^\nu}{2q_1^2 q_2^2} \right] + 8 \frac{((p_1 \cdot k) p_2^\mu - (p_2 \cdot k) p_1^\mu) ((p_1 \cdot k) p_2^\nu - (p_2 \cdot k) p_1^\nu)}{q_1^2 q_2^2} \\
 &+ (2p_1 \cdot p_2) \left( \frac{4p_1^\mu p_1^\nu \frac{k \cdot p_2}{k \cdot p_1} - 2p_1^{(\mu} p_2^{\nu)}}{q_2^2} + \frac{4p_2^\mu p_2^\nu \frac{k \cdot p_1}{k \cdot p_2} - 2p_1^{(\mu} p_2^{\nu)}}{q_1^2} + \frac{(q_1 - q_2)^{\mu} (-2(p_1 \cdot k) p_2 + 2(p_2 \cdot k) p_1)^{\nu}}{q_1^2 q_2^2} \right) \left. \right\}.
 \end{aligned}$$

# One-Loop 2 $\rightarrow$ 3 Amplitude

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

Inelastic subtraction:

$$\mathcal{A}_1^{(5)} = \frac{i}{2} \left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] + \frac{i}{2} \left[ \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] + \mathcal{B}_1^{(5)}$$

with  $\mathcal{B}_1^{(5)}$  a rational function of ( $\sqrt{\text{of}}$ )  $y, \omega_1, \omega_2, q_1^2, q_2^2$ .

- The subtracted terms are all **imaginary** and **infrared divergent**
- $\frac{i}{2} \left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] + \frac{i}{2} \left[ \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right]$  are also “superclassical”  $\mathcal{O}(\frac{1}{\hbar^2})$
- The leftover  $\mathcal{B}_1^{(5)}$  is **real**, **finite** and **classical**  $\mathcal{O}(\frac{1}{\hbar})$  [and **transcendental weight 2**].

# Inelastic Exponentiation at One Loop

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

Inelastic subtraction in  $b$ -space:

$$\tilde{\mathcal{A}}_1^{(5)} = 2i\delta_0 \tilde{\mathcal{B}}_0^{(5)}$$

$$+ \frac{i}{2} \text{FT} \left[ \text{Diagram 1} \right] + \frac{i}{2} \text{FT} \left[ \text{Diagram 2} \right] + \tilde{\mathcal{B}}_1^{(5)}$$

- This matches the **exponential** (here  $\int_k \simeq \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)$ )

$$i\tilde{\mathcal{A}}_1^{(5)} = \langle \text{in} | a(k) e^{i(2\delta_0+2\delta_1)} e^{i \int_k [\tilde{\mathcal{V}}(k)a^\dagger(k) + \tilde{\mathcal{V}}^*(k)a(k)]} | \text{in} \rangle$$

- And it identifies

$$\mathcal{W}_0 = \mathcal{B}_0^{(5)}, \quad \mathcal{W}_1 = \mathcal{B}_1^{(5)} + \frac{i}{2} \left[ \text{Diagram 1} \right] + \frac{i}{2} \left[ \text{Diagram 2} \right]$$

# Calculating the one-loop $2 \rightarrow 3$ amplitude

[We start from the integrand given in: Carrasco, Vazquez-Holm '21]

[Georgoudis, CH, Vazquez-Holm '23]

$\mathcal{G}_1, P', s_1 = \frac{1}{2}$	$\mathcal{G}_2, P, s_2 = \frac{1}{2}$	$\mathcal{G}_3, P, s_3 = 1$	$\mathcal{G}_4, M', s_4 = \frac{1}{4}$
$\mathcal{G}_5, P', s_5 = \frac{1}{4}$	$\mathcal{G}_6, M, s_6 = \frac{1}{2}$	$\mathcal{G}_7, P, s_7 = \frac{1}{2}$	$\mathcal{G}_8, M, s_8 = \frac{1}{2}$
$\mathcal{G}_9, P, s_9 = \frac{1}{2}$	$\mathcal{G}_{10}, P, s_{10} = \frac{1}{2}$	$\mathcal{G}_{11}, M, s_{11} = \frac{1}{2}$	$\mathcal{G}_{12}, M, s_{12} = \frac{1}{2}$
$\mathcal{G}_{13}, P, s_{13} = \frac{1}{4}$	$\mathcal{G}_{14}, M, s_{14} = \frac{1}{2}$	$\mathcal{G}_{15}, P, s_{15} = \frac{1}{2}$	$\mathcal{G}_{16}, P, s_{16} = \frac{1}{8}$
$\mathcal{G}_{20}, P, s_{20} = \frac{1}{4} \times \frac{1}{2}$	$\mathcal{G}_{23}, P, s_{23} = \frac{1}{2} \times \frac{1}{2}$	$\mathcal{G}_{24}, P, s_{24} = \frac{1}{2}$	$\mathcal{G}_{17,18,19,21,22}$

$I_{0,0,1,0,1}$	$I_{0,0,1,1,0}$	$I_{0,1,0,0,1}$	$I_{1,0,0,1,0}$
$I_{0,1,1,1,0}$	$I_{1,0,1,0,1}$	$I_{0,1,1,0,1}$	$I_{1,0,1,1,0}$
$I_{1,1,0,1,0}$	$I_{1,1,0,0,1}$	$I_{0,1,1,1,1}$	$I_{1,0,1,1,1}$
$I_{1,1,0,1,1}$	$I_{1,1,1,0,1}$	$I_{1,1,1,1,0}$	$I_{1,1,1,1,1}$

# Properties of $\mathcal{B}_0^{(5)\mu\nu}$ , $\mathcal{B}_1^{(5)\mu\nu}$

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

- Under “time reversal”

$$\omega_{1,2} \mapsto -\omega_{1,2}$$

- The leading contribution  $\mathcal{B}_0^{(5)\mu\nu}$  is *even*,
- The subleading contribution has both *odd* and *even* parts

$$\mathcal{B}_1 = \mathcal{B}_{1E} + \mathcal{B}_{1O}.$$

- The **odd** one is equal to

$$\mathcal{B}_{1O} = \left( 1 - \frac{y(y^2 - \frac{3}{2})}{(y^2 - 1)^{\frac{3}{2}}} \right) \lim_{\epsilon \rightarrow 0} [-\pi\epsilon \operatorname{Im} \mathcal{W}_1]$$

Trademark of radiation-reaction [Di Vecchia, CH, Russo, Veneziano '21]

# Inelastic Final State

[Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White '21]

[Di Vecchia, CH, Russo, Veneziano '22]

## Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$e^{2i\hat{\delta}(b_1, b_2)} = e^{i \operatorname{Re} 2\delta(b)} e^{i \int_k [\tilde{\mathcal{W}}(k) a^\dagger(k) + \tilde{\mathcal{W}}^*(k) a(k)]}.$$

- Final state (again, schematically):

$$|\text{out}\rangle = e^{2i\hat{\delta}(b_1, b_2)} |\text{in}\rangle$$

- **Unitarity:**

$$\langle \text{out} | \text{out} \rangle = \langle \text{in} | \text{in} \rangle = 1$$

- Consistency with the elastic exponentiation: by the BCH formula,

$$\langle \text{in} | \text{out} \rangle = e^{i \operatorname{Re} 2\delta(b)} e^{-\operatorname{Im} 2\delta(b)} = e^{2i\delta(b)}$$

because, by unitarity,  $\operatorname{Im} 2\delta_2 = \frac{1}{2} \int_k \tilde{\mathcal{A}}_0^{(5)} \tilde{\mathcal{A}}_0^{(5)*}$ .

# Static Modes as a Soft Dressing [Di Vecchia, CH, Russo, Veneziano '22]

[Soft dressing: Bloch, Nordsieck '37; Thirring, Tauschek '51; Weinberg '65; Mirbabayi, Porrati '16; Choi, Akhouri '17; Arkani-Hamed et al.'20.

Operator exponentiation: Damgaard, Planté, Vanhove '21; Cristofoli et al.'12. Classical soft theorems: Laddha, Sen '18; Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo, Sen '21.]

## Operator dressing of the elastic eikonal in $b$ space

$$\hat{S}_{s.r.} = e^{\int_k^\Lambda [F_j(k) \hat{a}_j^\dagger(k) - F_j^*(k) \hat{a}_j(k)]}.$$

- $F_j(k) = \varepsilon_{j\mu\nu}^*(k) F^{\mu\nu}(k)$

$$F^{\mu\nu}(k) = \sum_n \frac{\sqrt{8\pi G} p_n^\mu p_n^\nu}{p_n \cdot k - i0},$$

and  $\int_k^\Lambda = \int \frac{d^D k}{(2\pi)^D} 2\pi \delta(k^2) \theta(k^0) \theta(\Lambda - k^0)$ , with  $\Lambda$  a cutoff.

- Effectively

$$e^{2i\hat{\delta}(b_1, b_2)} \mapsto \hat{S}_{s.r.} e^{2i\hat{\delta}(b_1, b_2)}, \quad \text{Re } 2\delta(b) \mapsto 2\delta_c(b).$$



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# Radiated Angular Momentum

- $\langle \text{out} | \hat{\mathcal{J}}^{\alpha\beta} | \text{out} \rangle = \mathbf{J}^{\alpha\beta} + \mathcal{J}^{\alpha\beta}$  where [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo '22]

$$\mathbf{J}_{\alpha\beta} = \mathbf{J}_{\alpha\beta}^{(o)} + \mathbf{J}_{\alpha\beta}^{(s)}, \quad i\mathbf{J}_{\alpha\beta}^{(o)} = \int_k k_{[\alpha} \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial k^{\beta]}} \tilde{\mathcal{A}}^{(5)*}, \quad \mathbf{J}_{\alpha\beta}^{(s)} = 2i \int_k \tilde{\mathcal{A}}_{[\alpha}^{(5)\mu} \tilde{\mathcal{A}}_{\beta]\mu}^{(5)*} .$$

- **Reverse unitarity:**  $q_{\parallel 2} = -u_2 \cdot q$  [Di Vecchia, CH, Russo, Veneziano '22]

$$i\mathbf{J}_{\alpha\beta}^{(o)} = \text{FT} \int k_{[\alpha} \frac{\partial}{\partial k^{\beta]} \left[ d(\text{LIPS}) \begin{array}{c} p_1 \leftarrow \overrightarrow{\quad} \\ \uparrow \text{---} \text{---} \text{---} \text{---} \uparrow \\ | \quad | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ p_2 \leftarrow \overrightarrow{\quad} \end{array} \right] \begin{array}{c} \overrightarrow{\quad} \leftarrow \\ \uparrow \quad \downarrow \\ \overrightarrow{\quad} \leftarrow \end{array} - u_{2[\alpha} \text{FT} \frac{\partial}{\partial q_{\parallel 2]} \int d(\text{LIPS}) k_{\beta]} \begin{array}{c} p_1 \leftarrow \overrightarrow{\quad} \\ \uparrow \text{---} \text{---} \text{---} \text{---} \uparrow \\ | \quad | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ p_2 \leftarrow \overrightarrow{\quad} \end{array} \left| \begin{array}{c} \overrightarrow{\quad} \leftarrow \\ \uparrow \quad \downarrow \\ \overrightarrow{\quad} \leftarrow \end{array} \right.$$

- Inclusion of tidal effects [CH '22] by using the  $t_{E,B}^{\mu\nu}$  given in [Mougiakakos, Riva, Vernizzi '22].
- Inclusion of spin effects [CH (in progress...)] by using the  $t_{S_1, S_2}^{\mu\nu}$  given e.g. in [Riva, Vernizzi, Wong '22].

# Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[Di Vecchia, CH, Russo '22] [see also: Veneziano, Vilkovisky '22; Javadinezhad, Porrati '22; Riva, Vernizzi, Whong '23].

## Angular momentum/mass dipole loss due to static modes

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2} \sum_{n,m} \left[ \left( \sigma_{nm}^2 - \frac{1}{2} \right) \frac{\frac{\sigma_{nm} \operatorname{arccosh} \sigma_{nm} - 1}{\sqrt{\sigma_{nm}^2 - 1}}}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm} \operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}} \right] (\eta_n - \eta_m) p_n^{[\alpha} p_m^{\beta]}.$$

- Here  $-\eta_n \eta_m p_n \cdot p_m = m_n m_m \sigma_{nm}$  with  $\eta_n = +1$  ( $\eta_n = -1$ ) if  $n$  is outgoing (incoming).
- Matches [Damour '20; Manohar, Ridgway, Shen '22; Bini, Damour '22] up to  $\mathcal{O}(G^3)$  upon expanding

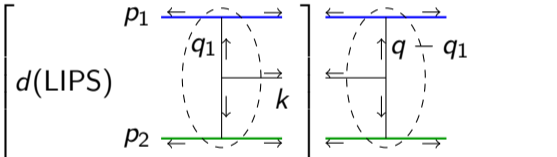
$$\mathcal{J}^{\alpha\beta} = -\frac{G}{2} (p_1 - p_2)^{[\alpha} Q^{\beta]} \mathcal{I}(\sigma) + \mathcal{O}(G^4), \quad Q^\mu = Q_{1\text{PM}}^\mu + Q_{2\text{PM}}^\mu + \mathcal{O}(G^3)$$

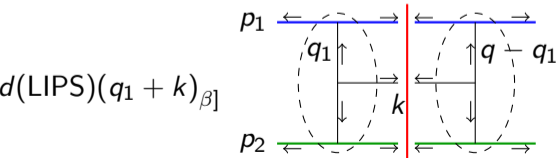
- Valid also in the case of tidally deformable [CH '22] or spinning objects [Alessio, Di Vecchia '22].

- $\langle \text{out} | \hat{L}_2^{\alpha\beta} | \text{out} \rangle - \langle \text{in} | \hat{L}_2^{\alpha\beta} | \text{in} \rangle = \Delta L_2^{\alpha\beta} + \Delta L_{2\text{cons}}^{\alpha\beta} + \Delta \mathcal{L}_2^{\alpha\beta}$  where

$$\Delta L_2^{\alpha\beta} = \text{Im } J_2^{\alpha\beta} + b_2^{[\alpha} \Delta P_2^{\beta]}, \quad J_{2\alpha\beta} = \int_k p_{2[\alpha} \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial p_2^{\beta]}} \tilde{\mathcal{A}}^{(5)*}.$$

- **Reverse unitarity:**

$$J_{2\alpha\beta} = \text{FT} \int u_{2[\alpha} \frac{\partial}{\partial u_2^{\beta]}} \left[ d(\text{LIPS}) \right]$$


$$+ u_{2[\alpha} \text{FT} \frac{\partial}{\partial q_{||2}} \int d(\text{LIPS}) (q_1 + k)_{\beta]}$$


# Angular Momentum Balance

- Convenient functions:  $C\sqrt{\sigma^2 - 1} = -\mathcal{E}_+ + \sigma\mathcal{E}_-$ ,  $\mathcal{F} = \pm\mathcal{E}_\pm \mp \frac{1}{2}\mathcal{E}$ .
- Radiated angular momentum [Manohar, Ridgway, Shen '21]

$$\mathbf{J}^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3} \mathcal{F} \left( b^{[\alpha} \check{u}_1^{\beta]} - b^{[\alpha} \check{u}_2^{\beta]} \right).$$

- Radiative changes of mechanical angular momentum [Di Vecchia, CH, Russo, Veneziano '22]

$$\Delta \mathbf{L}_1^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3} \left[ + \frac{\mathcal{E}_+ b^{[\alpha} u_1^{\beta]}}{\sigma - 1} - \frac{1}{2} \mathcal{E} b^{[\alpha} \check{u}_2^{\beta]} \right],$$
$$\Delta \mathbf{L}_2^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3} \left[ - \frac{\mathcal{E}_+ b^{[\alpha} u_2^{\beta]}}{\sigma - 1} + \frac{1}{2} \mathcal{E} b^{[\alpha} \check{u}_1^{\beta]} \right].$$

$$\boxed{\mathbf{J}^{\alpha\beta} + \Delta \mathbf{L}_1^{\alpha\beta} + \Delta \mathbf{L}_2^{\alpha\beta} = 0}$$

# Outline

- 1 Introduction: the Elastic Eikonal
- 2 Eikonal Operator
- 3 Angular Momentum
- 4 Waveforms**

# Metric Fluctuation in the Final State

- Free **graviton field** (recall  $\int_k = \int 2\pi\delta(k^2)\theta(k^0)\frac{d^D k}{(2\pi)^D}$ )

$$H_{\mu\nu}(x) = \int_k \left[ e^{ik\cdot x} a_{\mu\nu}(k) + e^{-ik\cdot x} a_{\mu\nu}^\dagger(k) \right].$$

- Expectation in the final state [Kosower, Maybee, O'Connell '18; Cristofoli, Gonzo, Kosower, O'Connell '21]

$$h_{\mu\nu}(x) = \langle \text{out} | H_{\mu\nu}(x) | \text{out} \rangle = \int_k e^{ik\cdot x} i\tilde{\mathcal{W}}_{\mu\nu}(k) + (\text{c.c.})$$

with

$$i\tilde{\mathcal{W}}_{\mu\nu}(k) = \langle \text{out} | a_{\mu\nu}(k) | \text{out} \rangle$$

- This is the canonically normalized **metric fluctuation**

$$g_{\mu\nu}(x) - \eta_{\mu\nu} = \sqrt{32\pi G} h_{\mu\nu}(x).$$

# The Asymptotic Waveform

- **Asymptotic limit:** We consider a detector with four-velocity  $t^\mu$  and retarded time  $u$  placed at a distance  $r \rightarrow \infty$  in a null direction  $n^\mu$ , i.e. near future null infinity  $\mathcal{I}^+$ ,

$$x^\mu = u t^\mu + r n^\mu, \quad n \cdot n = 0, \quad n \cdot t = -1.$$

- E.g. the detector's rest frame,

$$t^\mu = (1, 0, 0, 0), \quad n^\mu = (1, \hat{x}), \quad \hat{x} \cdot \hat{x} = 1.$$

- Then

$$h^{\mu\nu} \sim \frac{1}{4\pi r} \int_0^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} \tilde{\mathcal{V}}^{\mu\nu}(k = \omega n) + (\text{c.c.})$$

with  $\omega$  the frequency measured by the detector.



# L.O. Gravitational Waveform

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21]

[Di Vecchia, CH, Russo, Veneziano '23 (review)]

- Convenient variables:  $(x + y = 1)$

$$\Omega_j = \frac{b \omega_j}{\sqrt{\sigma^2 - 1}}, \quad b^\mu(x) = x b_1^\mu + y b_2^\mu, \quad \Omega_1(x) = \sqrt{\Omega_1^2 x^2 + 2\Omega_1 \Omega_2 xy + \Omega_2^2 y^2}$$

- The covariant waveform in frequency domain is  $\tilde{W}_0^{\mu\nu} = \tilde{W}_{0,12}^{\mu\nu} + \tilde{W}_{0,\text{irr}}^{\mu\nu}$

$$\frac{\tilde{W}_{0,12}^{\mu\nu}}{\sqrt{8\pi G}} = \frac{G m_1 m_2}{\omega_1 \omega_2 (\sigma^2 - 1)} \sum_{j=1,2} \left[ A_j^{\mu\nu} e^{-i b_j \cdot k} \frac{K_0(\Omega_j)}{\sqrt{\sigma^2 - 1}} + B_j^{\mu\nu} e^{-i b_j \cdot k} K_1(\Omega_j) \right]$$

$$\frac{\tilde{W}_{0,\text{irr}}^{\mu\nu}}{\sqrt{8\pi G}} = \frac{G m_1 m_2}{(\sigma^2 - 1)^{5/2}} \int_0^1 e^{-i b(x) \cdot k} dx \left[ C^{\mu\nu} K_0(\Omega(x)) + D^{\mu\nu} \frac{K_1(\Omega(x))}{\Omega(x)} \right]$$

with  $A_j, B_j, C, D$  **polynomials** in  $u_1, u_2, k, b$ .

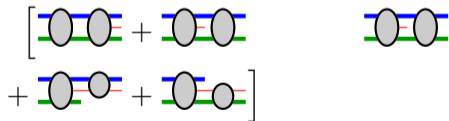
# Leading and Sub-Leading Waveform Kernels

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

$$\mathcal{W}_0^{\mu\nu} = \mathcal{B}_0^{\mu\nu} = \mathcal{A}_0^{\mu\nu}, \quad \mathcal{W}_1^{\mu\nu} = \mathcal{B}_1^{\mu\nu} + \frac{i}{2} \left[ \text{Diagram 1} + \text{Diagram 2} \right]$$

Indeed, 
$$-i \langle \text{out} | a_{\mu\nu}(k) | \text{out} \rangle = \langle \text{in} | a_{\mu\nu}(k) N | \text{in} \rangle + \frac{i}{2} \langle \text{in} | a_{\mu\nu}(k) N^2 | \text{in} \rangle - i \langle \text{in} | N a_{\mu\nu}(k) N | \text{in} \rangle + \dots$$



and (only) the “iteration” cuts cancel out.

# IR Divergences in the Waveform Kernel

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

- The IR divergence appearing in  $\frac{i}{2} \text{diagram} + \frac{i}{2} \text{diagram}$  obeys [Weinberg '65]

$$\mathcal{W}_1^{\mu\nu} = \mathcal{B}_1^{\mu\nu} - i\bar{\mu}^{2\epsilon} \frac{G}{\epsilon} (m_1\omega_1 + m_2\omega_2) \mathcal{B}_0^{\mu\nu} + i \text{Im} \mathcal{A}_1^{\mu\nu[-1,0]} + \mathcal{O}(\epsilon)$$

- Therefore, it exponentiates according to

$$\mathcal{W}^{\mu\nu} = e^{-i\frac{G}{\epsilon}(m_1\omega_1+m_2\omega_2)} [\mathcal{B}_0^{\mu\nu} + \mathcal{B}_1^{\mu\nu} + i\mathcal{M}_1^{\mu\nu}] + \mathcal{O}(\epsilon) + \mathcal{O}(G^{7/2})$$

where

$$\mathcal{M}_1^{\mu\nu} = G(m_1\omega_1 + m_2\omega_2) \mathcal{B}_0^{\mu\nu} \log \bar{\mu}^2 + \text{Im} \mathcal{A}_1^{\mu\nu[-1,0]}.$$



- The IR divergence and  $\log \bar{\mu}^2$  do not enter the spectral emission rate

$$d\tilde{\rho} = |\tilde{\mathcal{W}}|^2 \theta(k^0) 2\pi \delta(k^2) \frac{d^4 k}{(2\pi)^4}$$

# The Tail Effect

[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini '23; Herderschee, Teng, Roiban '23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]

[Georgoudis, CH, Vazquez-Holm '23]

- IR-divergent rescattering cuts  $\frac{i}{2}$   +  $\frac{i}{2}$   enter the waveform,

$$h^{\mu\nu} \sim \frac{1}{4\pi r} \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega \left( u - \frac{G}{\epsilon} (m_1 u_1 + m_2 u_2) \cdot n \right)} \left[ \tilde{\mathcal{B}}_0^{\mu\nu} + \tilde{\mathcal{B}}_1^{\mu\nu} + i\tilde{\mathcal{M}}_1^{\mu\nu} \right]_{k=\omega n} + (\text{c.c.}).$$

- The divergence can be formally reabsorbed via a constant shift of the detector's retarded time [Goldberger, Ross '12; Porto, Ross, Rothstein '12]

$$u \mapsto u - G \left( -\frac{1}{\epsilon} + \log \bar{\mu}^2 \right) (m_1 u_1 + m_2 u_2) \cdot n$$

- That is, one must fix experimentally the origin of the  $u$ -axis!  
(Tail effect [see e.g. Blanchet '13])

# Summary and Outlook

- An **operator eikonal exponentiation** emerges from  $2 \rightarrow 2$  and  $2 \rightarrow 3$  amplitudes
- It can be used to calculate **radiative** observables, like the **angular momentum** loss(es)
- The FT of the (suitably subtracted)  $2 \rightarrow 3$  amplitude is **the waveform**.
- Rescattering induces the **tail effect** in the subleading waveform kernel.

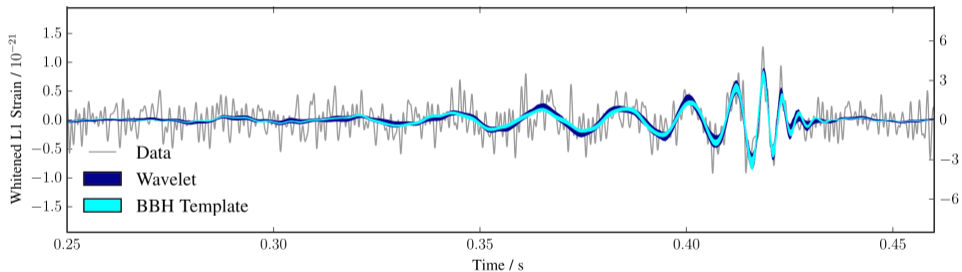
For the future:

- **Analytic results** for the subleading waveform?
- **Beyond coherent-state** approximation? E.g.  $a^\dagger a$ ,  $a^2$ ,  $a^{\dagger 2}$ ...
- **Differential** spectrum? **Integrated** ( $P_{\text{rad}}$  and)  $J_{\text{rad}}$  at  $\mathcal{O}(G^4)$ ? Nonlinear memory effect...
- Subleading  **$\log \omega$**  soft theorems?
- Comparison with **numerical** and **PN literature**?

# ADDITIONAL MATERIAL

# Waveform Templates

[LIGO Scientific Collaboration '16]



Inspiral  
Weak gravity

Merger  
Strong gravity

Ringdown  
Small oscillations

# Analytical Approximation Methods

- **Post-Newtonian (PN)**: expansion “for small  $G$  and small  $v$ ”

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1.$$

- **Post-Minkowskian (PM)**: expansion “for small  $G$ ”

$$\frac{Gm}{rc^2} \ll 1, \quad \text{generic } \frac{v^2}{c^2}.$$

- **Self-Force**: expansion in the near-probe limit  $m_2 \ll m_1$  or

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \ll 1.$$



# General Relativity from Scattering Amplitudes

## Idea

Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion  $\leftrightarrow$  PM expansion

Weak-coupling:  $\mathcal{A}_0 = \mathcal{O}(G)$     $\mathcal{A}_1 = \mathcal{O}(G^2)$     $\mathcal{A}_2 = \mathcal{O}(G^3)$     $\mathcal{A}_3 = \mathcal{O}(G^4)$

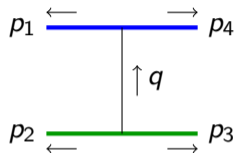
PM:      1PM                  2PM                  3PM                  4PM  
State of the art

...

- Lorentz invariance  $\leftrightarrow$  generic velocities
- Study scattering events, then export to bound trajectories  
( $V_{\text{eff}}$ , analytic continuation...)

# Example: the 1PM Eikonal

- Tree-level amplitude in  $D = 4 - 2\epsilon$  dimensions



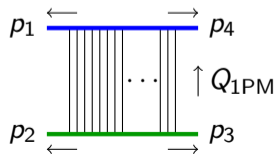
$$\mathcal{A}_0(s, q) = \frac{32\pi Gm_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \dots$$

$$\tilde{\mathcal{A}}_0(s, b) = \frac{4Gm_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}.$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al.'18]

$$e^{2i\delta_0} \xrightarrow{\text{"small } G"} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0.$$

- From  $Q = \partial_b 2\delta$ , we obtain the leading-order deflection



$$Q_{1\text{PM}} = \frac{4Gm_1 m_2 (\sigma^2 - \frac{1}{2})}{b\sqrt{\sigma^2 - 1}}$$

$$\Theta_{1\text{PM}} = \frac{4GE (\sigma^2 - \frac{1}{2})}{b(\sigma^2 - 1)}.$$

# Smoothness and Universality of $\text{Re } 2\delta_2$ at High Energy

[Di Vecchia, CH, Russo, Veneziano '20, '21]

At high energy, as  $\sigma \rightarrow \infty$  and  $s \sim 2m_1m_2\sigma$ , i.e. in the massless limit:

- the *complete* eikonal phase is smooth, **although** the conservative and radiation-reaction parts separately diverge like  $\log \sigma$ ,
- its expression is the same in  $\mathcal{N} = 8$  supergravity and in GR,

$$\text{Re } 2\delta_2 \sim Gs \frac{\Theta_s^2}{4}, \quad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

# The Initial State

[Kosower, Maybee, O'Connell '18]

- We model the **initial state** by  $|\text{in}\rangle = |1\rangle \otimes |2\rangle$ , with

$$|1\rangle = \int_{-p_1} \varphi_1(-p_1) e^{ib_1 \cdot p_1} | - p_1 \rangle$$

$$|2\rangle = \int_{-p_2} \varphi_2(-p_2) e^{ib_2 \cdot p_2} | - p_2 \rangle$$

and  $\int_{-p_i} = \int 2\pi \delta(p_i^2 + m_i^2) \theta(-p_i^0) \frac{d^D p_i}{(2\pi)^D}$  the LIPS measure.

- **Wavepackets**  $\varphi_i(-p_i)$  peaked around the classical incoming momenta.
- **Impact parameter**  $b^\mu = b_1^\mu - b_2^\mu$  lies in the transverse plane  $b \cdot p_1 = 0 = p_2 \cdot b$ .

# Elastic and Inelastic Fourier Transforms

- Elastic Fourier transform:

$$\begin{aligned}\text{FT } \mathcal{A}^{(4)} &= \int \frac{d^D q}{(2\pi)^D} 2\pi\delta(2\bar{m}_1 u_1 \cdot q) 2\pi\delta(2\bar{m}_2 u_2 \cdot q) e^{ib \cdot q} \mathcal{A}^{(4)}(q) \\ &\simeq \frac{1}{4Ep} \int \frac{d^{D-2} q}{(2\pi)^{D-2}} e^{ib \cdot q} \mathcal{A}(s, q) = \tilde{\mathcal{A}}^{(4)}.\end{aligned}$$

- Inelastic Fourier transform:

$$\begin{aligned}\text{FT } \mathcal{A}^{(5)} &= \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} (2\pi)^D \delta^{(D)}(q_1 + q_2 + k) \\ &\quad \times 2\pi\delta(2\bar{m}_1 u_1 \cdot q_1) 2\pi\delta(2\bar{m}_2 u_2 \cdot q_2) e^{ib_1 \cdot q_1 + ib_2 \cdot q_2} \mathcal{A}^{(5)}(q_1, q_2, k) \\ &\simeq \tilde{\mathcal{A}}^{(5)}(k).\end{aligned}$$

# L.O. Gravitational Waveform

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mouggiakakos, Riva, Vernizzi '21, '22; Cristofoli et al. '21]

- Even parity under time reversal implies

$$h_{\text{L.O.}}^{\mu\nu} \sim \frac{1}{4\pi r} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} \tilde{\mathcal{W}}_0^{\mu\nu}(k = \omega n)$$

- We isolate terms in which either  $1/q_1^2$  or  $1/q_2^2$  cancels, whose Fourier transform is simpler,

$$\tilde{\mathcal{W}}_0^{\mu\nu}(k) = \tilde{\mathcal{W}}_{0,12}^{\mu\nu}(k) + \tilde{\mathcal{W}}_{0,\text{irr}}^{\mu\nu}(k)$$

# Eikonal Final State

[Di Vecchia, CH, Russo, Veneziano '23 (review)] [Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White '21]

## Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$|\text{out}\rangle \simeq \int_{p_3} \int_{p_4} e^{-ib_1 \cdot p_4 - ib_2 \cdot p_3} \int \frac{d^D Q_1}{(2\pi)^D} \int \frac{d^D Q_2}{(2\pi)^D} \Phi_1(p_4 - Q_1) \Phi_2(p_3 - Q_2) \\ \times \int d^D x_1 \int d^D x_2 e^{i(b_1 - x_1) \cdot Q_1 + i(b_2 - x_2) \cdot Q_2} e^{2i\hat{\delta}(x_1, x_2)} |p_4, p_3, 0\rangle$$

with

$$e^{2i\hat{\delta}(x_1, x_2)} = e^{i \text{Re } 2\delta(b)} e^{i \int_k \left[ \tilde{\mathcal{A}}_j(x_1, x_2, k) a_j^\dagger(k) + \tilde{\mathcal{A}}_j^*(x_1, x_2, k) a_j(k) \right]}$$

Here

- $b$  is the projection of  $x_1 - x_2$  orthogonal to  $p_4 - Q_1/2$ ,  $p_3 - Q_2/2$ ,
- $\tilde{\mathcal{A}}_j = \varepsilon_{j\mu\nu} \tilde{\mathcal{A}}^{\mu\nu}$  is the impact-parameter space  $2 \rightarrow 3$  amplitude.

# Physical Polarizations

- Transverse polarization vectors,

$$\tilde{e}_\theta \cdot \tilde{e}_\theta = 1 = e_\phi \cdot e_\phi, \quad \tilde{e}_\theta \cdot e_\phi = 0, \quad \tilde{e}_\theta \cdot k = e_\phi \cdot k = 0$$

and

$$\tilde{e}_\theta \cdot b = 0, \quad e_\phi \cdot u_i = 0, \quad -\frac{u_1 \cdot \tilde{e}_\theta}{\sigma\omega_1 - \omega_2} = \frac{u_2 \cdot \tilde{e}_\theta}{\sigma\omega_2 - \omega_1} = \frac{1}{\sqrt{\mathcal{P}}}.$$

- Transverse-traceless polarization tensors

$$\varepsilon_{\times}^{\mu\nu} = \frac{1}{2}(\tilde{e}_\theta^\mu e_\phi^\nu + \tilde{e}_\theta^\nu e_\phi^\mu), \quad \varepsilon_{+}^{\mu\nu} = \frac{1}{2}(\tilde{e}_\theta^\mu \tilde{e}_\theta^\nu - e_\phi^\mu e_\phi^\nu)$$

so  $\tilde{\mathcal{W}}_{\times,+} = \varepsilon_{\times,+}{}_{\mu\nu} \tilde{\mathcal{W}}^{\mu\nu}$ .



## × Polarization

[Di Vecchia, CH, Russo, Veneziano '23 (review)]

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mouggiakakos, Riva, Vernizzi '21, '22]

For the × polarization,

$$\frac{\tilde{W}_{12,\times}}{\sqrt{8\pi G}} = -\frac{2iGm_1m_2c_0}{b(\sigma^2 - 1)} b \cdot e_\phi \left( e^{-ib_1 \cdot k} K_1(\Omega_1) u_1 \cdot \tilde{e}_\theta - e^{-ib_2 \cdot k} K_1(\Omega_2) u_2 \cdot \tilde{e}_\theta \right)$$

and

$$\frac{\tilde{W}_{\text{irr},\times}}{\sqrt{8\pi G}} = \frac{2iGm_1m_2}{\sqrt{\sigma^2 - 1}} \left( \frac{c_0\omega_1\omega_2}{\sqrt{\mathcal{P}}} - 2\sigma\sqrt{\mathcal{P}} \right) b \cdot e_\phi \int_0^1 e^{-ib(x) \cdot k} K_0(\Omega(x)) dx$$

with  $\mathcal{P} = -\omega_1^2 + 2\sigma\omega_1\omega_2 - \omega_2^2$  and  $c_0 = 2\sigma^2 - 1$ .

# + Polarization

[Di Vecchia, CH, Russo, Veneziano '23 (review)]

[D'Eath '78; Kovacs, Thorne '78; Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22]

For the + polarization,

$$\begin{aligned} \frac{\tilde{\mathcal{W}}_{12,+}}{\sqrt{8\pi G}} &= \frac{Gm_1 m_2}{\omega_1 \omega_2 (\sigma^2 - 1)} \left[ i \frac{b \cdot k}{b} c_0 \left( e^{-ib_2 \cdot k} K_1(\Omega_2) (u_2 \cdot \tilde{e}_\theta)^2 \omega_1 - e^{-ib_1 \cdot k} K_1(\Omega_1) (u_1 \cdot \tilde{e}_\theta)^2 \omega_2 \right) \right. \\ &+ \frac{e^{-ib_1 \cdot k} K_0(\Omega_1) u_1 \cdot \tilde{e}_\theta \omega_2 - e^{-ib_2 \cdot k} K_0(\Omega_2) u_2 \cdot \tilde{e}_\theta \omega_1}{\sqrt{\sigma^2 - 1} \sqrt{\mathcal{P}}} \\ &\left. \times \left( (\sigma^2 - 1) (4\mathcal{P}\sigma - c_0 \omega_1 \omega_2) - c_0 \mathcal{P}\sigma \right) \right] \end{aligned}$$

and

$$\begin{aligned} \frac{\tilde{\mathcal{W}}_{\text{irr},+}}{\sqrt{8\pi G}} &= \frac{Gm_1 m_2}{\sqrt{\sigma^2 - 1}} \int_0^1 dx e^{-ib(x) \cdot k} \left[ \frac{(b \cdot e_\phi)^2}{b^2} c_0 K_1(\Omega(x)) \Omega(x) - c_0 K_0(\Omega(x)) \right. \\ &+ \left. \frac{b^2 K_1(\Omega(x))}{\Omega(x) \mathcal{P}} (c_0 \omega_1^2 \omega_2^2 + 2\mathcal{P}^2 - 4\sigma \omega_1 \omega_2 \mathcal{P}) \right]. \end{aligned}$$