Demystifying the bound to boundary correspondence with Kerr geodesics

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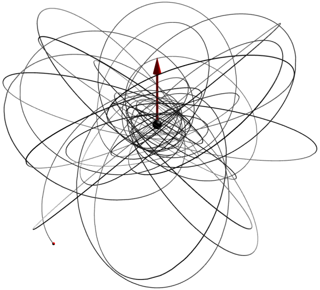

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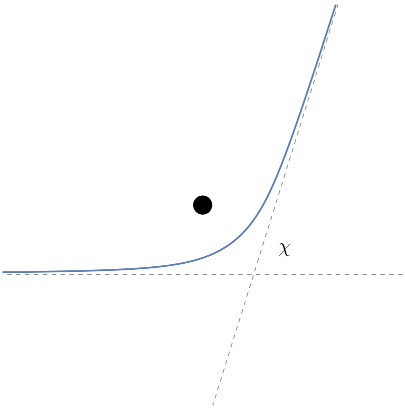
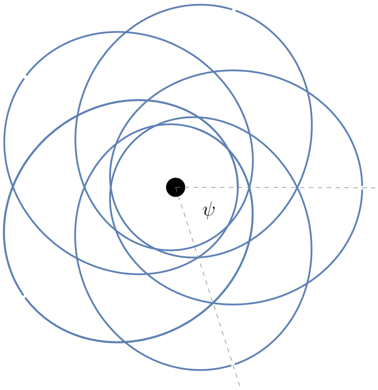
From Amplitudes to Gravitational Waves





Amplitude techniques

- Natural setting: scattering

Gravitational wave observations

- Need: Bound inspirals

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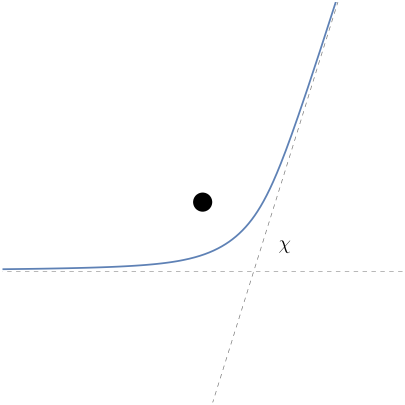
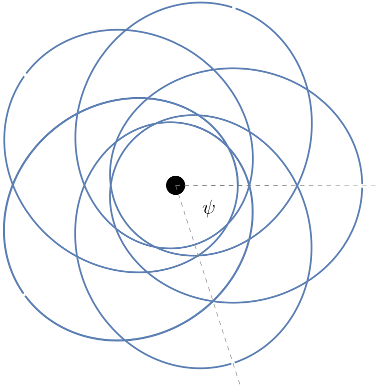
Bound-to-Boundary correspondence [Kälin&Porto, 2019+]





Scattering angle and Periapsis precession are related...

$$\psi(E, L) = \chi(E, L) + \chi(E, -L)$$

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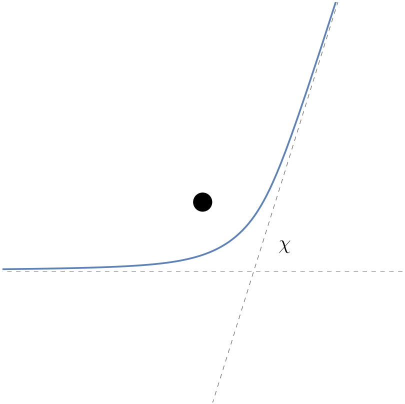

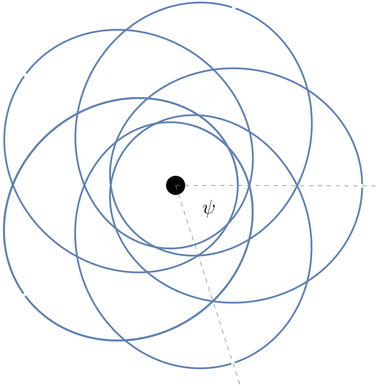

Scattering angle and Periapsis precession are related...

$$\psi(E, L, a) = \chi(E, L, a) + \chi(E, -L, -a)$$

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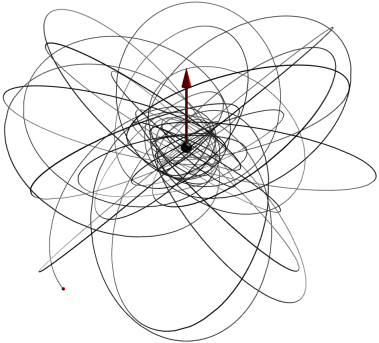

Geodesic Equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Why look at geodesics?

- All orders in G, $\frac{1}{c}$, M, and a
- "0th order" in secondary mass m and secondary spin s.
- Integrable system with explicit solutions available.

Goals

- Improve intuitive understand of B2B map
- Generalizations/alternative formulations

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Constants of Motion

Norm 4-velocity

$$-1 = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} g_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

Symmetries

$$\mathcal{E} := -\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} g_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\nu \qquad \text{energy}$$

$$\mathcal{L} := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} g_{\mu\nu} \left(\frac{\partial}{\partial \phi}\right)^\nu \qquad \text{angular momentum}$$

Hidden symmetry and Carter constant

$$Q := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \mathcal{K}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

First order form of geodesic equation:

$$\begin{aligned}
\Sigma^2 \left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 &= \Big(\mathcal{E} - a(\mathcal{L} - a\mathcal{E})u^2\Big)^2 \\
&\qquad - (1 - 2GMu + a^2u^2)\Big(1 + (Q + (\mathcal{L} - a\mathcal{E})^2)u^2\Big) \\
&= -a^2 Q(u-u_1)(u-u_2)(u-u_3)(u-u_4) =: U(u) \\
\Sigma^2 \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 &= Q - z^2\Big(a^2(1-\mathcal{E}^2)(1-z^2) + \mathcal{L}^2 + Q\Big) \\
&= a^2(1-\mathcal{E}^2)(z^2 - z_1^2)(z^2 - z_2^2) =: Z(z) \\
\Sigma \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= a\frac{\mathcal{E} - a(\mathcal{L} - a\mathcal{E})u^2}{1 - 2GMu + a^2u^2} + \frac{\mathcal{L}}{1 - z^2} - a\mathcal{E}, \\
\Sigma \frac{\mathrm{d}t}{\mathrm{d}\tau} &= \frac{(1 + a^2u^2)\Big(\mathcal{E} - a(\mathcal{L} - a\mathcal{E})u^2\Big)}{(1 - 2GMu + a^2u^2)u^2} - a^2\mathcal{E}(1 - z^2) + a\mathcal{L},
\end{aligned}$$

with

$$u := 1/r \qquad\qquad z := \cos\theta$$

$$\Sigma := r^2 + a^2 \cos^2\theta$$

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Mino Carter-Time

New (non-affine) parameter

$$d\lambda = \frac{1}{\Sigma} d\tau$$

Crucial bonus feature

Geodesics reach infinity in finite Mino time:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}u} = \pm\frac{1}{\sqrt{U(u)}} = \frac{1}{\sqrt{\mathcal{E}^2 - 1}} + \mathcal{O}(u)$$

Decoupled equations:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\lambda}\right)^2 = -a^2 Q(u-u_1)(u-u_2)(u-u_3)(u-u_4) = U(u)$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2 = a^2(1-\mathcal{E}^2)(z^2-z_1^2)(z^2-z_2^2) = Z(z)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = a\frac{\mathcal{E} - a(\mathcal{L} - a\mathcal{E})u^2}{1 - 2GMu + a^2u^2} + \frac{\mathcal{L}}{1-z^2} - a\mathcal{E},$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{(1+a^2u^2)\big(\mathcal{E} - a(\mathcal{L} - a\mathcal{E})u^2\big)}{(1 - 2GMu + a^2u^2)u^2} - a^2\mathcal{E}(1-z^2) + a\mathcal{L},$$

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Closed form solutions

Radial Solution

$$u = \frac{(u_2 - u_1) u_3 \, \mathsf{sn}^2(A q_r | k_r) - u_2 (u_3 - u_1)}{(u_2 - u_1) \, \mathsf{sn}^2(A q_r | k_r) - (u_3 - u_1)}$$

with

$$q_r = \Upsilon_r \lambda + q_{r,0}$$

Polar Solution

$$z = z_1 \, \mathsf{sn}(B q_z | k_z)$$

with

$$q_z = \Upsilon_z \lambda + q_{z,0}$$

Definitions:

- A, B, k_r, k_z, Υ_r, Υ_z, Υ_ϕ, Υ_t Functions of a, GM, $\mathcal{E}$, $\mathcal{L}$, and Q
- $\mathsf{K}(\cdot)$, $\mathsf{E}(\cdot)$, $\Pi(\cdot|\cdot)$: Complete elliptic functions
- $\mathsf{F}(\cdot|\cdot)$, $\mathsf{E}(\cdot|\cdot)$, $\Pi(\cdot;\cdot|\cdot)$: Incomplete elliptic functions
- $\mathsf{sn}(\cdot|\cdot)$,$\mathsf{am}(\cdot|\cdot)$ Jacobi elliptic sine and amplitude

Azimuthal solution

$\phi(q_\phi, q_r, q_z) = q_\phi + \phi_r(q_r) + \phi_z(q_z)$ with $q_\phi = \Upsilon_\phi \lambda + q_{\phi,0}$,and

$$\phi_r(q_r) := \tilde{\phi}_r\left(\mathsf{am}\left(\mathsf{K}(k_r)\frac{q_r}{\pi}\,\Big|\,k_r\right)\right) - \frac{\tilde{\phi}_r(\pi)}{2\pi} q_r,$$

$$\tilde{\phi}_r(\xi_r) := \frac{\mathcal{L} u_+ (u_3 - u_2)(u_+ - \frac{2GM\mathcal{E}}{a\mathcal{L}})\,\Pi(h_+; \xi_r | k_r)}{A(u_+ - u_2)(u_+ - u_3)(u_- - u_+)} + (+ \leftrightarrow -),$$

$$\phi_z(q_z) := \tilde{\phi}_z\left(\mathsf{am}\left(\mathsf{K}(k_z)\frac{2q_z}{\pi}\,\Big|\,k_z\right)\right) - \frac{\tilde{\phi}_z(\pi)}{\pi} q_z, \quad \tilde{\phi}_z(\xi_z) := -\frac{\mathcal{L}}{z_2}\Pi(z_1^2; \xi_z | k_z).$$

Time solution

$t(q_t, q_r, q_z) = q_t + \phi_r(q_r) + \phi_z(q_z)$ with $q_t = \Upsilon_t \lambda + q_{t,0}$,and

$$t_r(q_r) := \tilde{t}_r\left(\mathsf{am}\left(\mathsf{K}(k_r)\frac{q_r}{\pi}\,\Big|\,k_r\right)\right) - \frac{\tilde{t}_r(\pi)}{2\pi} q_r,$$

$$\tilde{t}_r(\xi_r) := \mathcal{E}\left(\frac{u_3 - u_2}{A}\left(\frac{2\mathcal{E}^2 - 3}{u_2 u_3(\mathcal{E}^2 - 1)}\Pi(h_r; \xi_r | k_r) - \frac{2}{a^2}\left\{\frac{u_+(4(GM)^2 - a(\mathcal{L}/\mathcal{E} + 2aGMu_+)}{(u_- - u_+)(u_+ - u_2)(u_+ - u_3)}\Pi(h_+; \xi_r | k_r) + (+ \leftrightarrow -)\right\}\right)\right.$$
$$\left. - \frac{2A}{GM(\mathcal{E}^2 - 1)}\left(\mathsf{E}(\xi_r | k_r) - h_r \frac{\sin\xi_r \cos\xi_r \sqrt{1 - k_r \sin^2\xi_r}}{1 - h_r \sin^2 \xi_r}\right)\right)$$

$$t_z(q_z) := \tilde{t}_z\left(\mathsf{am}\left(\mathsf{K}(k_z)\frac{2q_z}{\pi}\,\Big|\,k_z\right)\right) - \frac{\tilde{t}_z(\pi)}{\pi} q_z, \quad \tilde{t}_z(\xi_z) := -\frac{\mathcal{E}}{1 - \mathcal{E}^2} z_2\,\mathsf{E}(\xi_z | k_z),$$

Parametrizing geodesics

The Kerr background

Mass M and spin a

Initial conditions

$x^\mu(0)$ and $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(0)$

Constants of motion

$\mathcal{E}$, $\mathcal{L}$, and Q plus initial phases $q_{r,0}$, $q_{z,0}$, $q_{t,0}$, and $q_{\phi,0}$

Turning points

u_1, u_2, and z_1 plus initial phases $q_{r,0}$, $q_{z,0}$, $q_{t,0}$, and $q_{\phi,0}$

(p, e, x)

$p := \frac{2}{u_1+u_2}$, $e := \frac{u_2-u_1}{u_1+u_2}$, and $x := \mathrm{sign}(\mathcal{L})\sqrt{1-z_1^2}$

Scattering variables

Impact parameter b^μ, and velocity v^μ_∞

Notes

- All parameters linked by analytic relationships
- Geodesic solutions are analytic in all parameters
- $q_{t,0}$ and $q_{\phi,0}$ can be freely fixed using global symmetries
- $q_{r,0}$ can be fixed by letting $\lambda = 0$ at periapsis
- Relationship between (b^μ, v^μ_∞) and $(\mathcal{E}, \mathcal{L}, Q)$ features $q_{z,0}$.

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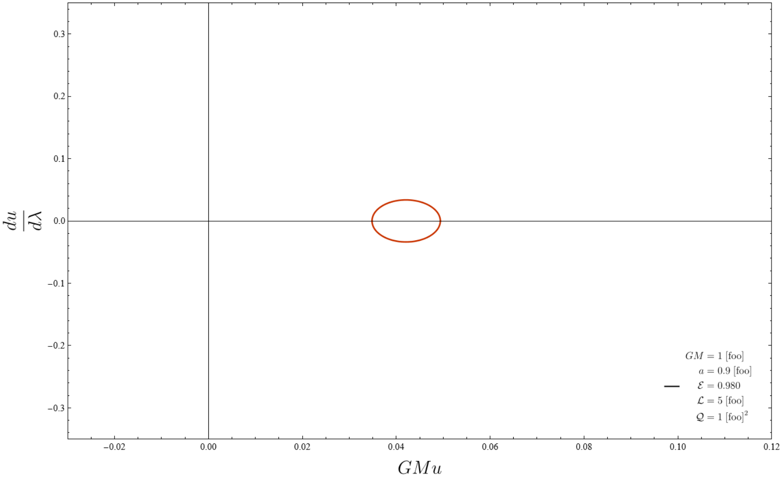
From bound to scatter

- Bound and scatter solutions belong to the same class of geodesic solutions with root structure:

$$u_1 < u_2 < u_3 < u_+ < u_- < u_4$$

- Can analytically deform bound ($u_1 > 0$, $\mathcal{E} < 1$, $e < 1$) to scatter ($u_1 < 0$, $\mathcal{E} > 1$, $e > 1$) solutions.



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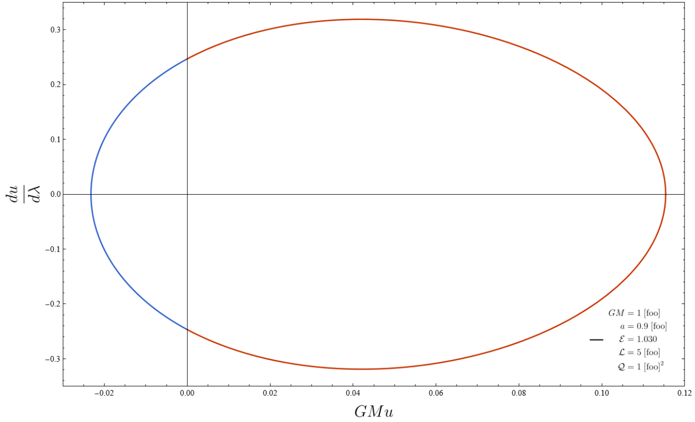
From scatter to bound



- Analytical continuation of bound orbit consists of two scattering events
- One event in $u > 0$ universe
- One event in $u < 0$ universe
- Need both to reconstruct bound solution!

Question:

Given knowledge of scattering in $u > 0$, can we reconstruct scattering in $u < 0$?

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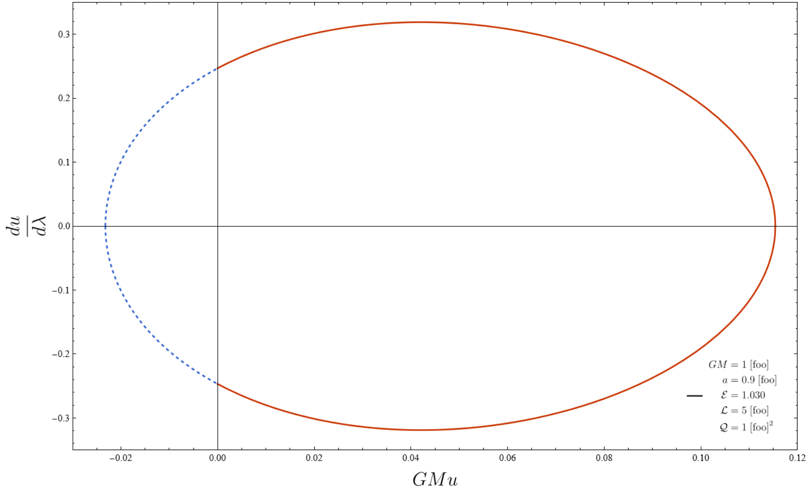
Option 1: Analytic continuation in λ



- Solutions are analytic in λ
- Given a partial solution on some interval, full solution can be recover through analytic continuation
- Beware branch cut for t (and τ) solution

Fix

M, a, $\mathcal{E}$, $\mathcal{L}$, Q

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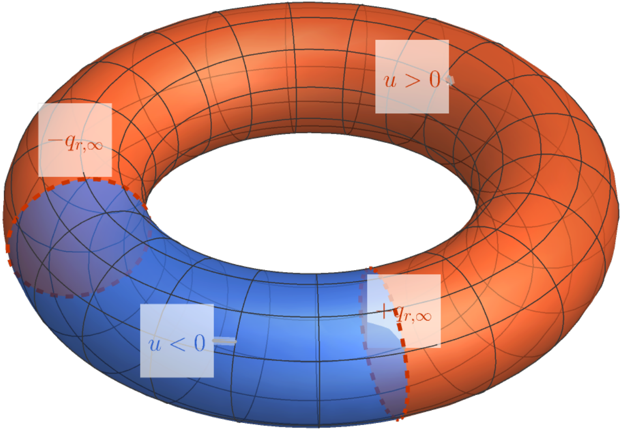
Option 1b: Analytic continuation on the (q_r, q_z)-torus

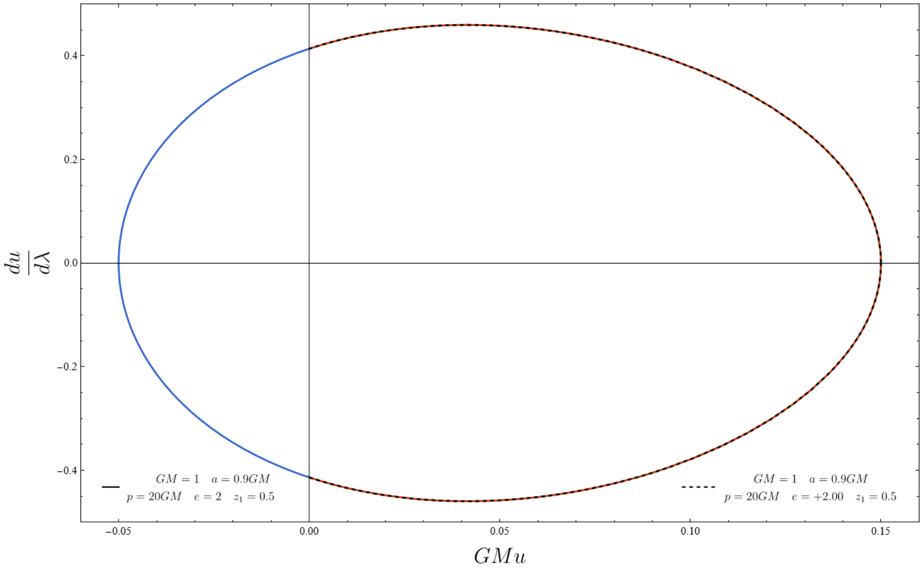
- Solution not periodic in λ
- Geodesics depends on λ through (q_t, q_r, q_z, q_ϕ)
- (q_t, q_ϕ) dependence from background symmetry
- Scattering in $u > 0$ gives solution for $-q_{r,\infty} < q_r < q_{r,\infty}$ and all q_z
- Full solution by anal. cont. on (q_r, q_z)-torus

Fix

M, a, $\mathcal{E}$, $\mathcal{L}$, Q



Option 2: Exchange of the radial roots



Exchange

$$u_1 \leftrightarrow u_2$$

Equivalent

$$e \leftrightarrow -e$$

Fix

$M,\ a,\ z_1$

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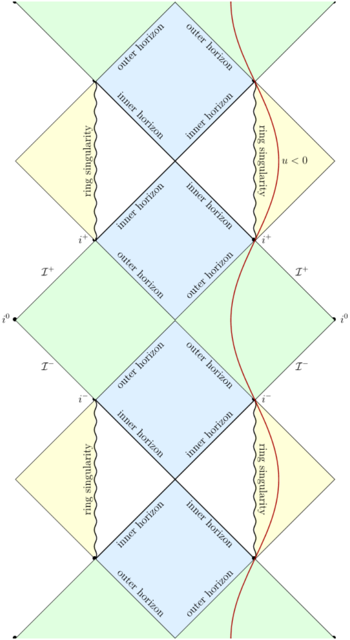
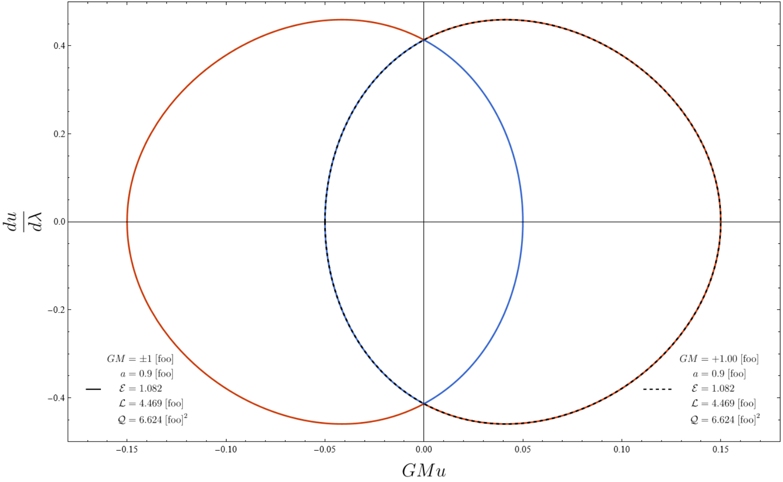
Option 3: Relating the universe to the anti-universe $GM \to -GM$





Fix

a, $\mathcal{E}$, $\mathcal{L}$, Q

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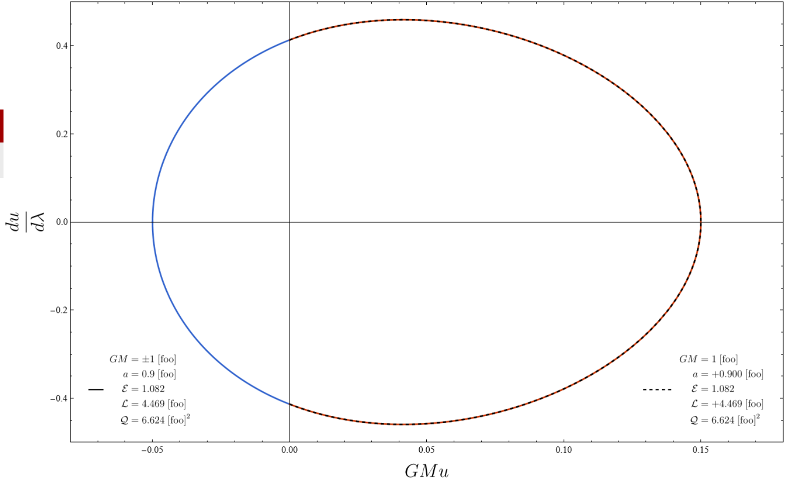
Option 4: Invert angular momenta ($1/\mathcal{L} \to -1/\mathcal{L}$ and $a \to -a$)

- As in [Kälin&Porto,2019+] prescription
- Needs $z_1 \to -z_1$ for precessing orbits

Fix
M, $\mathcal{E}$



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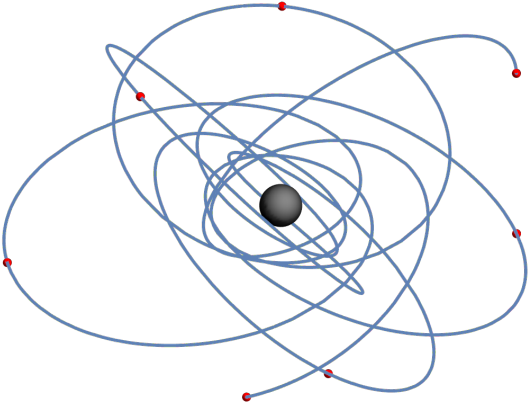
Periapsis precession for generic bound orbits



The accumulated azimuthal phase per radial period depends on q_z at radial turning points. Not coordinate independent!

Gauge invariant definition

$$
\begin{aligned}
\psi &:= \Lambda_r \langle \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \rangle \\
&= \Lambda_r \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\, \mathrm{d}\lambda \\
&= \frac{\Lambda_r}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\, \mathrm{d}q_r\, \mathrm{d}q_z
\end{aligned}
$$

[Drasco&Hughes, 2003]

B2B relation for misalgined spins

Scattering angle

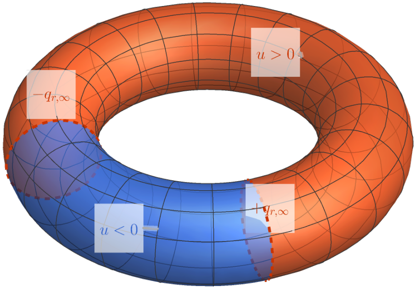
$$\chi = \int_{-\Lambda_\infty}^{\Lambda_\infty} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \frac{\Lambda_r}{2\pi}\int_{-q_{r,\infty}}^{q_{r,\infty}} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,\mathrm{d}q_r$$

Define:

$$\bar{\chi} = \frac{\Lambda_r}{(2\pi)^2}\int_{-q_{r,\infty}}^{q_{r,\infty}}\int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,\mathrm{d}q_r\,\mathrm{d}q_z$$

B2B relationship

$$\psi = \frac{\Lambda_r}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,\mathrm{d}q_r\,\mathrm{d}q_z = \frac{\Lambda_r}{(2\pi)^2}\int_{-q_{r,\infty}}^{q_{r,\infty}}\int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,\mathrm{d}q_r\,\mathrm{d}q_z + \frac{\Lambda_r}{(2\pi)^2}\int_{q_{r,\infty}}^{2\pi-q_{r,\infty}}\int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,\mathrm{d}q_r\,\mathrm{d}q_z = \bar{\chi}(u>0) + \bar{\chi}(u<0)$$



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Beyond geodesics: The GSF formalism

Gravitational Self Force (GSF) Formalism

Expansion of relativistic two body dynamics around geodesic solutions

$$\begin{aligned}\frac{\mathrm{d}\vec{q}}{\mathrm{d}\lambda} &= \vec{\Upsilon}(\vec{P}) + \epsilon \vec{f}_1(\vec{P},\vec{q}) + \epsilon^2 \vec{f}_2(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^3) \\ \frac{\mathrm{d}\vec{P}}{\mathrm{d}\lambda} &= \quad 0 \quad + \epsilon \vec{F}_1(\vec{P},\vec{q}) + \epsilon^2 \vec{F}_2(\vec{P},\vec{q}) + \mathcal{O}(\epsilon^3)\end{aligned}$$

- Geodesic results lift to general dynamics
- Possible obstruction: $\vec{f}_i$ and $\vec{F}_i$ may not be analytic everywhere (tails?!)

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Example: Waveform

Waveform observed at null infinity at $t(\lambda) - r_*(\lambda)$:

$$h(\lambda, \Theta_{\rm obs}, \Phi_{\rm obs}) = \frac{1}{D} \sum_{\ell m n k} \mathcal{A}_{\ell m n k}(\vec{P}) e^{-i(n q_r + k q_z + m q_\phi)} {}_{-2}Y_{\ell m}(\Theta_{\rm obs}, \Phi_{\rm obs})$$

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Summary

2 distinct relations:

- 1 relating bound and unbound
- 1 relating scattering and anti-scattering

4-ways to relate (anti)-scattering

- Analytic continuation of Mino time
- Exchange of radial roots ($e \leftrightarrow -e$)
- Inversion of gravitational constant ($G \to -G$)
- Reversal of angular momenta ($1/\mathcal{L} \to -1/\mathcal{L}$, $a \to -a$ and $z_1 \to -z_1$)