What's so special about black holes

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From Amplitudes to Gravitational Waves

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- Binary black holes at $\mathcal{O}(G^2)$ and higher orders in spin

  —— Consensus through S^4!

  —— Proposals to single out black hole dynamics at S^5 and beyond

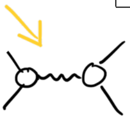
  —— Confrontation with black hole perturbation theory

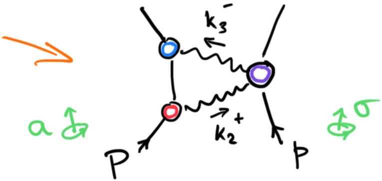
- Conservation laws for a quadrupolar test black hole in a Kerr background from Kerr's "hidden symmetry" (Killing-Yano tensor)

  —— with an intriguing relation to $G^2 S^4$

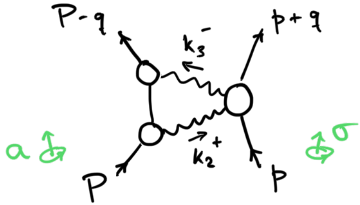
Conservative dynamics of binary BHs in the post-Minkowskian (post-Newtonian) expansion

spin^0	1PM (0PN)	2PM (1PN)	3PM (2PN)	4PM (3PN)	5PM (4PN)	6PM (5PN)	7PM (6PN)
spin^1		LO S^1 (1.5PN)	NLO S^1 (2.5PN)	N^{2}LO S^1 (3.5PN)	N^{3}LO S^1 (4.5PN)	N^{4}LO S^1 (5.5PN)	N^{5}LO S^1 (6.5PN)
spin^2			LO S^2 (2PN)	NLO S^2 (3PN)	N^{2}LO S^2 (4PN)	N^{3}LO S^2 (5PN)	N^{4}LO S^2 (6PN)
spin^3				LO S^3 (3.5PN)	NLO S^3 (4.5PN)	N^{2}LO S^3 (5.5PN)	N^{3}LO S^3 (6.5PN)
...					LO S^4 (4PN)	NLO S^4 (5PN)	N^{2}LO S^4 (6PN)
						LO S^5 (5.5PN)	NLO S^5 (6.5PN)
							LO S^6 (6PN)





$$\mathcal{M}^{\triangle +-} = \oint \mathrm{d}\zeta\,(\dots)\, e^{-k_2\cdot a}\, e^{-k_3\cdot a}\, \langle 3|p|2]^4 \exp\left[\left(2\frac{p\cdot k_2}{\langle 3|p|2]}|3\rangle[2| - k_2 - k_3\right)\cdot\sigma\right]$$



$$\mathcal{M}^{\triangle+-} = \oint \mathrm{d}\zeta\,(\ldots)\,\exp\left[-(k_2+k_3)\cdot a + \left(2\frac{p\cdot k_2}{\langle 3|p|2]}|3\rangle[2| - k_2 - k_3\right)\cdot \sigma\right] + \mathcal{O}(\sigma^5)$$

- Arkani-Hamed, Huang, Huang; Guevara, Ochirov, JV; Chung, Huang, Kim, Lee ... Aoude+; Bern+; Bjerrum-Bohr+; Porto+ ... Chen, Chung, Huang, Kim ... Bautista ...
- Aoude, Haddad, Helset; Bern, Kosmopoulos, Luna, Roiban, Teng

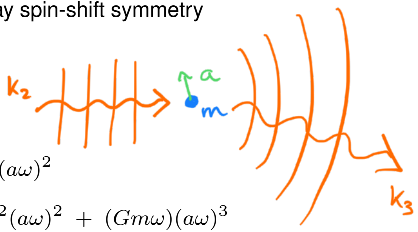
  — **spin-shift symmetry for BHs**: $a^\mu \to a^\mu + q^\mu/q^2$ (and $a \to \sigma$) $(q = k_2 - k_3)$

  (and better-behaved high-energy limit for BHs)

  [— *extra Wilson coeff.s ... extra d.o.f.s (electric dipole in EM toy model)* $\to$ *Roiban's talk*]
- Cangemi, Chiodaroli, Johansson, Ochirov, Pichini, Skvortsov

  — **massive higher-spin gauge symmetry for BHs** ($\to$ *Cangemi's talk*)
- Levi, Teng; Kim, Levi, Yin — **PN EFT** — **zero Love numbers for BHs** [... Chia ...] (motivation to push Compton-from-worldline-action [Saketh+ to a^3] to a^4, a^5 ...)

- Siemonsen, JV — circular equatorial **self-force** in Kerr (analytical Teukolsky: MST): redshift and precession invariants [Kavanagh, Ottewill, Wardell; ... van de Meent+ ...] $\to$ "First Law" (Le Tiec+) $\to$ constraints on aligned-spin scattering angle at NLO a^3, a^4
—— need more data from eccentric equatorial (not to mention precessing) orbits
[see Munna's redshift paper out yesterday!]
- Bautista, Guevara, Kavanagh, JV — tree-level (G^1) classical **Compton from Teukolsky**
— unambiguous confirmation of exponential Compton through a^4 (**!!!**)
— ambiguities starting at a^5 ... preliminary results (from analytically continuing to $a \gg Gm$)
do not display spin-shift symmetry



$$
\begin{aligned}
\mathcal{A} \sim\ & Gm\omega \\
& + (Gm\omega)^2 + (Gm\omega)(a\omega) \\
& + (Gm\omega)^3 + (Gm\omega)^2(a\omega) + (Gm\omega)(a\omega)^2 \\
& + (Gm\omega)^4 + (Gm\omega)^3(a\omega) + (Gm\omega)^2(a\omega)^2 + (Gm\omega)(a\omega)^3 \\
& + (Gm\omega)^5 + (Gm\omega)^4(a\omega) + (Gm\omega)^3(a\omega)^2 + (Gm\omega)^2(a\omega)^3 + (Gm\omega)(a\omega)^4 \\
& + (Gm\omega)^6\left[\ldots + (\ldots)\,\psi\!\left(\frac{2ia}{\sqrt{(Gm)^2-a^2}}\right) + (\ldots)\,\log\left(1-\frac{a^2}{(Gm)^2}\right) + \left(\frac{a}{Gm}\right)^5\right] \\
& + \ldots
\end{aligned}
$$

—— **absorption (!)** —— Goldberger, Li, Rothstein ... Saketh, Zhou, Ivanov ...

2PM contribution to eikonal action (one of the two triangles)

$$\chi^{\triangle+-} \propto \int \mathrm{d}^2 q\, e^{iq\cdot b} \mathcal{M}^{\triangle+-} \propto \oint \mathrm{d}\zeta\,(\ldots)\left[\int \frac{\mathrm{d}^2 q}{|q|} \exp\Big(iq\cdot\beta + \alpha|q|\Big) \propto \frac{1}{\sqrt{|\beta|^2+\alpha^2}}\right]$$

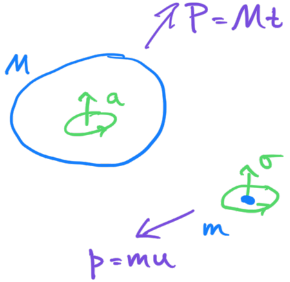
$$\frac{\chi^{\mathrm{2PM}(\triangle_{+-})}}{\pi(GM)^2 m} = \oint \frac{\mathrm{d}\zeta}{4\pi i} \frac{\zeta^4\left(1-2\gamma\zeta+\zeta^2\right)^{-3/2}}{\sqrt{\mathcal{Q}+2aj_z\zeta-\dfrac{2\sigma_\ell}{\zeta}+(1-2\gamma\zeta+\zeta^2)\left(a^2+\dfrac{\sigma^2}{\zeta^2}\right)}} + \mathcal{O}(\sigma^5)$$

remarkably, depends only on $\gamma = -u\cdot t$,

$$aj_z = \ell\cdot a - \gamma a\cdot\sigma - (u\cdot a)(t\cdot\sigma),$$

$$\sigma_\ell = \ell\cdot\sigma - \gamma a\cdot\sigma - (u\cdot a)(t\cdot\sigma),$$

$$\begin{aligned}\mathcal{Q} = {}& \ell^2 - 2\gamma(\ell\cdot a) + (\gamma^2-1)a^2 - (u\cdot a)^2 \\ & + 2\gamma(\ell\cdot\sigma) - 2(\gamma^2+1)(a\cdot\sigma) - 2\gamma(u\cdot a)(t\cdot\sigma) \\ & + (\gamma^2-1)\sigma^2 - (t\cdot\sigma)^2,\end{aligned}$$



and σ^2 (and a^2), which we will shortly identify
as the constants of motion for
a quadrupolar test black hole (velocity u, ring-radius σ)
in a Kerr background (velocity t, ring-radius a),
evaluated for an incoming scattering state at infinity.

$$\ell^a = t^d \epsilon_d{}^a{}_{bc} b^b u^c, \quad \ell^2 = (\gamma^2-1)b^2.$$

Conserved quantities for Kerr geodesics

Killing vector fields $\xi^a = \{t^a, \phi^a\}$ generate spacetime isometries:

time translation $t^a := (\frac{\partial}{\partial t})^a$, and axial rotation $\phi^a := (\frac{\partial}{\partial \phi})^a$ (in BL coord.).

Killing vectors satisfy $\nabla_a \xi_b = \nabla_{[a}\xi_{b]}$. Thus, the geodesic equation, $\frac{\mathrm{D}}{\mathrm{d}\lambda} p^a = p^b \nabla_b p^a = 0$,

has a constant of motion $(p_a \xi^a)$ for every ξ^a of the background, $\frac{\mathrm{D}}{\mathrm{d}\lambda}(p_a \xi^a) = p^a p^b \nabla_a \xi_b = 0$.

A Killing-Yano tensor is a 2-form $Y_{ab} = Y_{[ab]}$ satisfying $\nabla_c Y_{ab} = \nabla_{[c} Y_{ab]}$

(encoding "hidden symmetry").

Thus, for geodesics, the 1-form $l_a = Y_{ab} p^b$ is parallel-transported,

$$\frac{\mathrm{D}}{\mathrm{d}\lambda} l_a = p^c \nabla_c (Y_{ab} p^b) = p^c p^b \nabla_c Y_{ab} = 0,$$

and so its magnitude $Q = l_a l^a$ is a constant of motion, $\frac{\mathrm{D}}{\mathrm{d}\lambda} Q = 0$.

For the (nontrivial) Killing-Yano tensor in Kerr (with $\nabla_c Y_{ab} = \epsilon_{abcd} t^d$),

this Q is the Carter constant, completing the list of four constants of motion,

$$-m^2 = p_a p^a, \quad E = -p_a t^a, \quad J_z = p_a \phi^a, \quad Q = Y_{ac} Y_b{}^c p^a p^b,$$

to make the motion fully integrable.

Now consider the **Mathisson-Papapetrou-Dixon eqs.** for multipolar test body:

$$\frac{\mathrm{D}}{\mathrm{d}\tau}p_a + \frac{1}{2}R_{abcd}\dot{x}^b S^{cd} = -\frac{1}{6}\nabla_a R_{bcde}J^{bcde} + \ldots,$$
$$\frac{\mathrm{D}}{\mathrm{d}\tau}S^{ab} - 2p^{[a}\dot{x}^{b]} = \frac{4}{3}R^{[a}{}_{cde}J^{b]cde} + \ldots.$$

In a background with a Killing vector ξ^a, the quantity $\mathcal{P}_\xi = p_a\xi^a + \frac{1}{2}S^{ab}\nabla_a\xi_b$

is exactly conserved, to all orders in the multipole expansion, for arbitrary multipoles.

In Kerr, there is a generalization of the Carter constant for pole-dipole MPD,

$$Q = Y_{ac}Y_b{}^c p^a p^b + 4t^e\epsilon_{ecd[a}Y_{b]}{}^d p^a S^{bc} + \mathcal{O}(S^2),$$

and a further constant $S_\ell = \frac{1}{2}{}^*Y_{ab}S^{ab}$, which are conserved $+\,\mathcal{O}(S^2)$,

first found by Rüdiger ['81,'83] and later by Gibbons+ in “SUSY in the sky” [hep-th/9303112].

(((((The radial action from Witzany's seperated Hamilton-Jacobi eq. for a pole-dipole in Kerr [1903.03651], $E = -\mathcal{P}_t, \quad J_z = \mathcal{P}_\phi, \quad \Delta = r^2 + a^2 - 2GMr$

$$I_r = \oint \frac{\mathrm{d}r}{\Delta}\sqrt{\Big[(r^2+a^2)E - aJ_z\Big]^2 - \Delta\Big[r^2m^2 + Q - 2\frac{(r^2+a^2)E - aJ_z}{r^2m^2+Q}m^2S_\ell\Big]} + \mathcal{O}(S^2)\,,$$

coincides with the eikonal action from amplitudes through $G^2a^\infty S^1$.)))))

Covariant building blocks for Kerr: (all that's needed to prove the conservation laws)

complex scalar $\mathcal{R} = r + ia\cos\theta$, timelike Killing vector t^a,

and anti-self-dual 2-form $(\frac{1}{2}\epsilon_{ab}{}^{cd}N_{cd} = -iN_{ab})$

$$N_{ab} = -2i(l_{[a}n_{b]} + m_{[a}\bar{m}_{b]}) = -iG_{abcd}l^c n^d,$$

where $\{l, n, m, \bar{m}\}$ is Kinnersly's null tetrad (l, n are the principal null directions),

and $G_{ab}{}^{cd} = 2\delta_a{}^{[c}\delta_b{}^{d]} + i\epsilon_{ab}{}^{cd}$ is (4 times) the anti-self-dual projector.

Closed covariant differential relations ($\nabla G = 0$):

$$i\nabla_a\mathcal{R} = N_{ab}t^b, \quad i\nabla_c(\mathcal{R}N_{ab}) = G_{abcd}t^d, \quad i\nabla_a t_b = -\frac{GM}{2}\left(\frac{N_{ab}}{\mathcal{R}^2} - \frac{\bar{N}_{ab}}{\bar{\mathcal{R}}^2}\right).$$

Killing-Yano and Riemann:

$$Y_{ab} = \frac{1}{2}(\mathcal{R}N_{ab} + \bar{\mathcal{R}}\bar{N}_{ab}), \qquad R_{abcd} = \frac{GM}{2\mathcal{R}^3}(3N_{ab}N_{cd} - G_{abcd}) + c.c.$$

Algebraic identities: $N_a{}^c N_{bc} = g_{ab}$, $N_a{}^c\bar{N}_{bc} = h_{ab} = \bar{h}_{ab} = h_{(ab)}$, $N_{ab}\bar{N}^{ab} = 0$, $\ldots$

Now consider the quadrupolar MPD equations, with the covariant SSC, $p_a S^{ab} = 0$,

$$\frac{D}{d\tau} p_a + \frac{1}{2} R_{abcd} \dot{x}^b S^{cd} = -\frac{1}{6} \nabla_a R_{bcde} J^{bcde} + \mathcal{O}(S^3),$$
$$\frac{D}{d\tau} S^{ab} - 2p^{[a} \dot{x}^{b]} = \frac{4}{3} R^{[a}{}_{cde} J^{b]cde} + \mathcal{O}(S^3),$$

with a spin-induced quadrupole ($C_{ES^2} = 1$ for a black hole),

$$J^{abcd} = C_{ES^2} \frac{3p \cdot \dot{x}}{(p^2)^2} p^{[a} S^{b]}{}_e p^{[c} S^{d]e}.$$

Only for $C_{ES^2} = 1$ do there exist a generalized Carter constant,

$$Q = Y_{ac} Y_b{}^c p^a p^b + 4t^e \epsilon_{ecd[a} Y_{b]}{}^d p^a S^{bc} + \mathcal{M}_{abcd} S^{ab} S^{cd},$$

$$\mathcal{M}_{abcd} = g_{ac}\Big(t_b t_d - \frac{1}{2} g_{bd} t^2\Big) - \frac{1}{2} Y_a{}^e \Big(Y_c{}^f R_{ebfd} + \frac{1}{2} Y_e{}^f R_{fbcd}\Big),$$

and $S_\ell = \frac{1}{2} Y_*^{ab} S_{ab}$ (unmodified) which are conserved $+\mathcal{O}(S^3)$,

— Compère, Druart, JV [2302.14549]

as are $E = -p_a t^a - \frac{1}{2} S^{ab} \nabla_a t_b$, $\quad J_z = p_a \phi^a + \frac{1}{2} S^{ab} \nabla_a \phi_b$, and $S^2 := \frac{1}{2} S_{ab} S^{ab}$.

In Kerr at infinity, in an effective Minkowski vector space,

— the timelike Killing vector t^a becomes a constant timelike unit vector,

— the axial Killing vector ϕ^a is given by $a\phi^a = t^b \epsilon_b{}^a{}_{cd} a^c x^d \quad (a\boldsymbol{\phi} = \boldsymbol{a} \times \boldsymbol{x})$,

— the Killing-Yano tensor becomes $Y_{ab}(x) = -\epsilon_{abcd} t^c x^d - 2t_{[a} a_{b]}$,

where x^a is the displacement from the origin at the center of the black hole,
and a^a is in the spin (z) direction with magnitude $a = S_{\mathrm{Kerr}}/M$.

From this, we can evaluate for an incoming scattering state at infinity
the constants of motion for a quadrupolar test black hole
with velocity $u^a = \dfrac{p^a}{m}$, ring-radius $\sigma^a = -\dfrac{1}{2m}\epsilon^a{}_{bcd} u^b S^{cd}$, and $\dfrac{L^a}{m} = \ell^a = t^d \epsilon_d{}^a{}_{bc} b^b u^c$,

$$\frac{aJ_z}{m} = aj_z = \ell \cdot a - \gamma a \cdot \sigma - (u \cdot a)(t \cdot \sigma), \qquad \frac{E}{m} = \gamma,$$

$$\frac{S_\ell}{m} = \sigma_\ell = \ell \cdot \sigma - \gamma a \cdot \sigma - (u \cdot a)(t \cdot \sigma), \qquad \frac{S^2}{m^2} = \sigma^2,$$

$$\begin{aligned}\frac{Q}{m^2} - a^2 - \sigma^2 = \mathcal{Q} = \ell^2 - 2\gamma(\ell \cdot a) + (\gamma^2 - 1)a^2 - (u \cdot a)^2 \\ + 2\gamma(\ell \cdot \sigma) - 2(\gamma^2 + 1)(a \cdot \sigma) - 2\gamma(u \cdot a)(t \cdot \sigma) \\ + (\gamma^2 - 1)\sigma^2 - (t \cdot \sigma)^2,\end{aligned}$$

which we recognize in

$$\frac{\chi^{2\mathrm{PM}(\triangle_{+-})}}{\pi(GM)^2 m} = \oint \frac{\mathrm{d}\zeta}{4\pi i} \frac{\zeta^4 \left(1 - 2\gamma\zeta + \zeta^2\right)^{-3/2}}{\sqrt{\mathcal{Q} + 2aj_z\zeta - \dfrac{2\sigma_\ell}{\zeta} + \left(1 - 2\gamma\zeta + \zeta^2\right)\left(a^2 + \dfrac{\sigma^2}{\zeta^2}\right)}} + \mathcal{O}(\sigma^5).$$

- Amid several proposals to specify the effective (conservative) dynamics of spinning black holes at higher orders in spin, we need more guidance from black hole perturbation theory (and should probably not expect such a clean split from absorptive effects)

- Black holes display uniqueness in having hidden-symmetry conservation laws in Kerr backgrounds, at least in the probe limit, at least to quadrupolar/spin-squared order ...

  —– What about octupolar/spin-cubed order? ...

  —– What about beyond the probe limit?
  (Generic binary BHs are integrable at 2PN — Tanay, Stein, Ghersi [2012.06586])

  —– Is the dependence of the 2PM-S^4 eikonal action only on the QTBH-in-Kerr constants of motion a hint of structure at higher orders?