An on-shell approach to self-force

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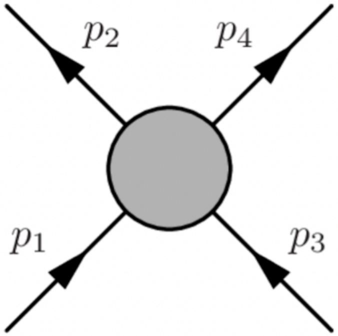
July 25th, 2023

The two-body problem in general relativity

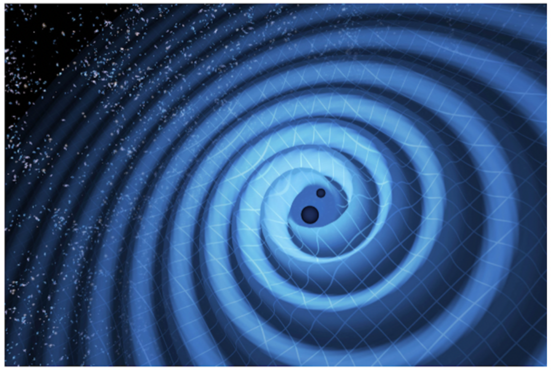
- No exact solution is known for the two body problem

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad , \quad \ddot{x}^{\mu}_{i=1,2} = -\Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}_i \dot{x}^{\beta}_i + \dots$$

- The post-Minkowskian approximation (PM) has gained a renewed attention after a remarkable state of the art calculation (Bern, Cheung, Roiban, Shen, Solon, Zeng)



$\Rightarrow$



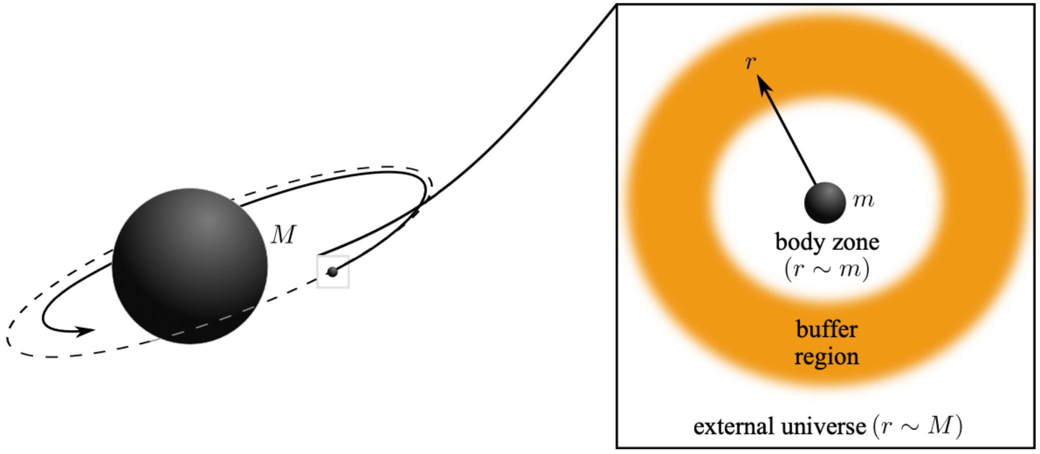
Credit: Tim Pyle

- Another perturbative scheme relevant to the two-body problem is the self-force expansion (see talks by Ruf and Pound)

Definition

The SF approximation is a perturbation theory where observables are computed in powers of m/Λ. Λ is a scale associated to a background

- (E.g. $\Lambda = M$, Schwarzschild). At LO the motion is geodesics



Credit: Leor Barack and Adam Pound

- At NLO in m/M, the motion is corrected by back-reaction

$$\frac{D^2 x^\mu}{d\tau^2}(\tau) = 0 \quad \Rightarrow \quad \underbrace{\frac{\tilde{D} x^\mu}{\tilde{d}\tau^2}(\tau) = \mathcal{O}\left(\frac{m^2}{M^2}\right)}_{\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{m}{M} h^R_{\mu\nu}(x)}$$

Comparison

The SF approximation for scattering orbits has gained interested only recently (Hopper, Cardoso, Long, Barack). Can we understand this perturbative scheme using on-shell scattering amplitudes?

The PM approximation

- Small parameter: G
- Resum PN results
- Impulse:

$$\Delta p^\mu = \underbrace{\Delta p_0^\mu}_{\sim G} + \underbrace{\Delta p_1^\mu}_{\sim G^2} + \ldots$$

- Scattering waveform:

$$W^{\mu\nu\rho\sigma} = \underbrace{W_0^{\mu\nu\rho\sigma}}_{G^{3/2}} + \underbrace{W_1^{\mu\nu\rho\sigma}}_{G^{5/2}}$$

The SF approximation

- Small parameter m/M
- Resum PM results
- Impulse:

$$\Delta p^\mu = \underbrace{\Delta p_0^\mu}_{geodesic} + \underbrace{\Delta p_1^\mu}_{\sim m/M} + \ldots$$

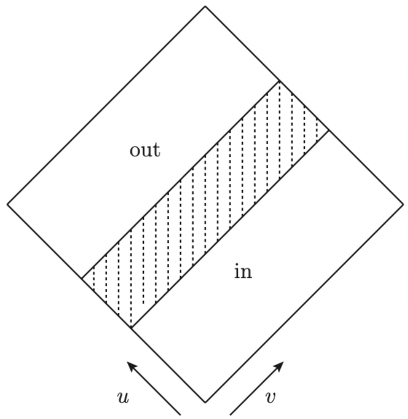
- Scattering waveform

$$W^{\mu\nu\rho\sigma} = \underbrace{W_0^{\mu\nu\rho\sigma}}_{geodesic} + \underbrace{W_1^{\mu\nu\rho\sigma}}_{\sim m/M}$$

The SF approximation on plane waves

- Consider a gravitational plane wave background

$$ds^2 = 2\,\mathrm{d}u\mathrm{d}v - \underbrace{H_{ab}(u)\,x^a x^b}_{\textit{compact support}}\;\mathrm{d}u^2 - \mathrm{d}x^{\perp}\mathrm{d}x^{\perp}$$



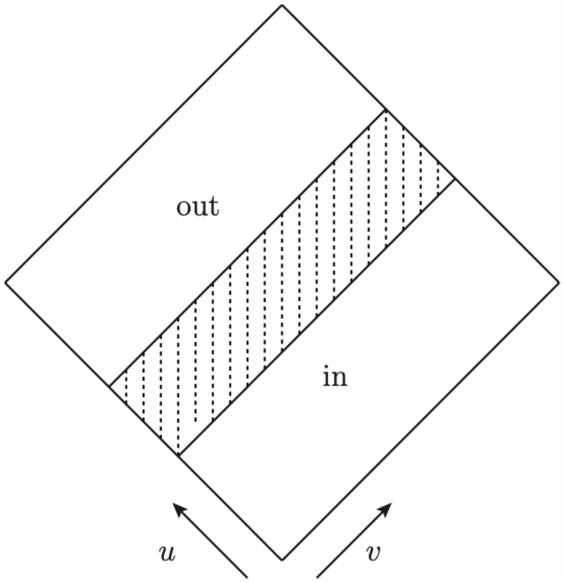
- Why studying the self-force expansion on plane waves?
  1. Penrose limit: any spacetimes along a null geodesic can be viewed as a gravitational plane wave (Penrose)
  2. Analytic results for observables (Harte, Flanagan, Fransen)
  3. Existence of compact formulae for scattering amplitudes on plane wave backgrounds (Adamo, Mason, Sharma, Casali)

Leading SF on a plane wave

- The exact geodesic motion is captured by the impulse

$$\Delta p^{\perp} = p_+ \sqrt{G} \dot{E}_j^{\perp}(u=\infty) b^j$$

where the zweibein satisfies $\ddot{E}_{ai} = H_{ab} E_i^b$ (Shore).
Memory effects: If H_{ab} has compact support $[x_i, x_f]$, then
$E_i^a(u > x_f) = \delta_i^a + u\sqrt{G} c_i^a$ even if the background is flat



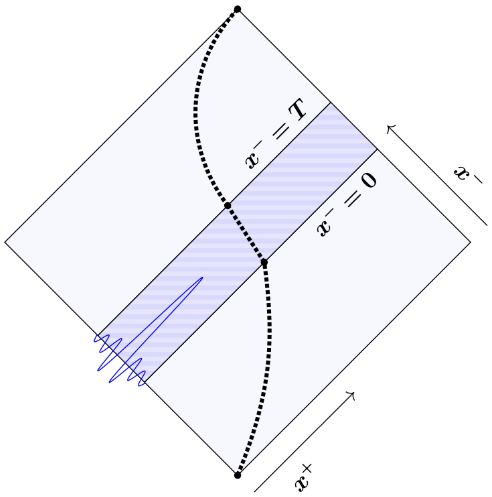
- Can we describe the exact geodesic motion from amplitudes?

- The in region is flat. Neglecting emission (0SF)

$$\underbrace{\mathcal{S}_\alpha}_{S \text{ on } g^0} |\psi\rangle = \int d\Phi(p,p')\,\phi(p)\underbrace{\langle p'|\,\mathcal{S}_\alpha\,|p\rangle}_{2-point}|p'\rangle + \ldots$$

- 2-point on a plane wave background are non vanishing. We can compute them using LSZ on a curved background

$$\langle p'|\,\mathcal{S}_\alpha\,|p\rangle = \hat{\delta}^+(p'-p)\frac{4\pi e^{-\frac{i}{2\sqrt{G}p_+}(p'-p)_\perp c^{-1}(p'-p)_\perp}}{\sqrt{\det(G\,c)}\,\hbar}$$



- Taking a stationary phase approximation in the final state, we can then compute the exact impulse via a mean value (✓)

$$\mathcal{S}_\alpha|\psi\rangle = \int d\Phi(p)\, e^{i\delta(q^*)/\hbar}\phi(p_+, p_\perp + \underbrace{\sqrt{G}p_+ c^i_\perp b_i}_{\textit{shift}})\ |p\rangle$$

$$\langle\psi|\, S^\dagger_\alpha \mathbb{P}^\mu S_\alpha\, |\psi\rangle = p^\mu + \sqrt{G}p_+\, \delta^\mu_a\, c^a_i\, b^i$$
$$-n^\mu\left(\frac{2\sqrt{G}p_+\, p_a c^a_i b^i - Gp_+^2\, c^a_i b^i c^b_j b^j \delta_{ab}}{2p_+}\right)$$

- 2-point amplitude on a plane wave equivalent to an eikonal resummation of perturbative 3-point amplitudes in vacuum, non vanishing due to memory (C., Ilderton, Elkhidir, O'Connell)

$$\langle p'|\, \mathcal{S}_\alpha\, |p\rangle = 2p_+\delta_+(p'-p)\delta_\perp(p'-p-ea(\infty))e^{ie\mathcal{A}_3+ie^2\mathcal{A}_4+e^2\mathcal{A}_3^2/2}$$

- Higher points define classical radiation in the SF approximation. The "simplest" one is

$$\underbrace{\langle p', k^{\eta} | \mathcal{S}_{\alpha} | p \rangle}_{3-point} \sim \frac{2i\kappa}{\hbar^{3/2}} \int_{\mathbb{R}} dx^{-} \frac{e^{i\mathcal{V}(x^{-})}}{\sqrt{|E(x^{-})|}} \mathcal{E}^{\eta}_{\mu\nu}(k; x^{-}) P^{\mu}(x^{-}) P'^{\nu}(x^{-})$$

$$\underbrace{\mathcal{V}(x^{-})}_{Volkov\ exponent} := \int_{-\infty}^{x^{-}} dy^{-} \frac{P^{\mu}(y^{-}) g_{\mu\nu,0}(y^{-}) K^{\nu}(y^{-})}{p_{+} - k_{+}}$$

$$\underbrace{\mathcal{E}^{\eta}_{\mu\nu}(k; x^{-})}_{Dressed\ polarization} := \left[\mathbb{P}_{\mu\rho}(x^{-}) \mathbb{P}_{\nu\sigma}(x^{-}) - \frac{i n_{\mu} n_{\nu} \sigma_{\rho\sigma}(x^{-})}{k_{+}} \right] \underbrace{\varepsilon^{\sigma\nu}_{\eta}(k)}_{free\ part}$$

$$\underbrace{P^{\mu}(x^{-})}_{Geodesic\ motion\ (known\ exactly!)}$$

- The weak field limit is a Compton amplitude in vacuum

- 3-points define the LO waveform in the SF approximation. Radiation on $\mathcal{I}^+$ as (see talks by Heissenberg and Travaglini)

$$\mathbb{W}_{\vec{\mu}}(u,\hat{x}) = -\frac{i\hbar^2}{4\pi r}\int_0^\infty \hat{\mathrm{d}}\omega\, \mathrm{e}^{-i\omega u}\, \underbrace{C^{\eta}_{\vec{\mu}}(k)}_{helicity}\, a_\eta(k)\Big|_{k=\hbar\omega\hat{x}} + \text{c.c.}$$

- The waveform follows from averaging on the final state

$$W_{\mu\nu\sigma\rho}(u,\hat{x}) = \frac{i\kappa}{2\pi\hbar^{\frac{1}{2}}}\int_0^\infty \hat{\mathrm{d}}\omega \mathrm{e}^{-i\omega u} k_{[\mu}\varepsilon^{-\eta}_{\nu]} k_{[\sigma}\varepsilon^{-\eta}_{\rho]}$$

$$\times \int \mathrm{d}\Phi(p)\, \underbrace{\langle\psi|\,\mathcal{S}^\dagger\,|p\rangle\,\langle p,k^\eta|\,\mathcal{S}\,|\psi\rangle}_{2-point\,\times\,3-point}|_{k=\hbar\omega\hat{x}} + \text{c.c.}$$

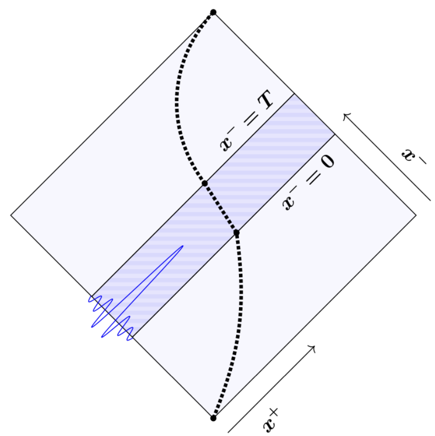
All order waveforms from amplitudes (Adamo, C. , Ilderton, Klisch)

$$W_{\mu\nu\sigma\rho}(u,\hat{x})_{-AF} = \hat{x}_{[\mu}\hat{x}_{[\sigma}\int_y \delta(u-\overline{\mathcal{V}}(y))\mathcal{D}^2 T^0_{\rho]\nu]}(\hat{x},y)$$

- Choosing $H_{ab}(u) = \delta(u)d(\lambda,-\lambda)$ and defining $\nu := \kappa\lambda|u|$

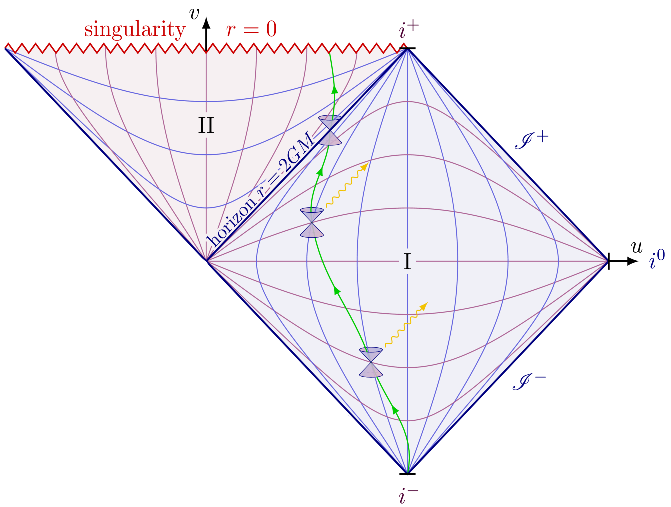
$$W_{\mu\nu\sigma\rho} = -\frac{\kappa^2 p_+}{\pi^2\sqrt{8}}\delta^+_{[\mu}\delta^+_{[\sigma}(-1)^{(a)}\delta^a_{\rho]}\delta^a_{\nu]}\frac{\partial^2}{\partial u^2}\left(\frac{\nu\log\left(\nu+\sqrt{\nu^2-1}\right)}{\sqrt{\nu^2-1}}\right)$$

- The LO waveform in the SF approximation on a plane wave is an explicit resummation of contributions in the coupling κ



The SF expansion on Schwarzschild

- Consider QFT on a static background (e.g. Schwarzschild)



$$ds^2 = \left(\eta_{\mu\nu} + k_\mu k_\nu \frac{2GM}{r}\right) dx^\mu dx^\nu$$

- The geodesic motion is captured by the classical limit of a 2-point (equivalent to an eikonal amplitude in vacuum)

$$\underbrace{\mathcal{S}}_{S\ on\ g} |\psi\rangle = \int d\Phi\left(p, p'\right)\phi\left(p\right) \underbrace{\langle p'|\,\mathcal{S}\,|p\rangle}_{2-point} |p'\rangle + \ldots$$

$$\underbrace{\langle p'|\,\mathcal{S}\,|p\rangle}_{Boundary\ i^0} = \delta(E'-E)\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell\left(\hat{p}\cdot\hat{p}'\right) e^{2\mathrm{i}\,\overbrace{I_\ell(r=\infty)}^{radial\ action} - \mathrm{i}\pi\ell}$$

- 2-points on Schwarzschild are function of the radial action $I(r)$

$$\underbrace{\langle p'|\,\mathcal{S}\,|p\rangle}_{\textit{geodesic motion}} = \delta(E'-E)N\int d^2x^\perp\, e^{ix^\perp\cdot(\hat{p}-\hat{p}\,')}\,\left(2I(|x^\perp|)-\pi\,|\vec{p}|\,|x^\perp|\right)$$

$$I(r) = \int_{r_{\text{turn}}}^{r} ds\,\sqrt{\frac{p^2\,s^2+2GMm\,s-\frac{s-2GM}{s}\ell^2}{(s-2GM)^2}}$$

- As example, the exact geodesic scattering angle for a scalar particle on Schwarzschild is (O'Connell, Ofri, Telem)

$$\Delta\varphi = \frac{2L}{3GMr_m\sqrt{pm}}\,F_1\big(1;1/2,1/2,3/2,\;\underbrace{r_1/r_m,r_2/r_m}_{\textit{characteristic zeroes}}\;\big)$$

- Some work is required to evaluate a 3-point amplitude on a Schwarzschild background $g_{\mu\nu}$. Using the pertubiner approach

$$\underbrace{S_3(\phi_{in}, \phi_{out}, h^{out})}_{\textit{Trilinear action}} := \kappa \int d^4x \sqrt{-|g|}\, h^{\rm out}_{k\,\mu\nu} \left[2\,\partial^\mu \phi^{\rm out}_{p'}\, \partial^\nu \bar{\phi}^{\rm in}_{p}\right.$$

$$\left. \text{-g}^{\mu\nu} \left(\partial_\alpha \phi^{\rm out}_{p'}\, \partial^\alpha \bar{\phi}^{\rm in}_{p} - \frac{m^2}{2}\, \phi^{\rm out}_{p'}\, \bar{\phi}^{\rm in}_{p}\right)\right]$$

- The incoming scalar, and outgoing scalar and graviton on the background $g_{\mu\nu}$ are determined via a matching condition on i^0

$$\phi^{\rm in}_{p} = \int d\Phi(l)\, \Lambda^{p}(l)\, e^{-i\, S_l}\Big|_{l^0 = p^0}$$

$$\phi^{\rm out}_{p'} = \int d\Phi(l')\, \Lambda^{p'}(l')\, e^{i\, S_{l'}}\Big|_{(l')^0 = (p')^0} \quad \Rightarrow \quad \underbrace{g_{\mu\nu}\partial^\mu S \partial^\nu S = m^2}_{\textit{Hamilton Jacobi}}$$

$$h^{\rm out}_{k\,\mu\nu} = \int d\Phi(k')\, \Lambda^{k\ \ \rho\sigma}_{\mu\nu}(k')\, \mathcal{E}_{\rho\sigma}\, e^{i\, S_{k'}}\Big|_{(k')^0 = k^0}$$

- The classical limit of a 3-point is determined by Hamilton's principal function $S(t, r, \phi, \theta)$, not the radial action alone

$$\underbrace{\langle p', k^{\eta} | \mathcal{S} | p \rangle}_{radiation} = -\kappa \int_{\mathbb{R}^{1,3} \backslash \overline{B(r_S)}} d^4x \sqrt{-|g|} \int_{l,l',k'} \Lambda^{*,p}(l) \, \Lambda^{p'}(l')$$

$$\times \Lambda^{k}_{\mu\nu}{}^{\rho\sigma}(k') \, \mathcal{E}^{\eta}_{\rho\sigma} \left[2\partial^{\mu} S_{l'} \, \partial^{\nu} S_{l} - g^{\mu\nu} \left(\partial S_{l'} \cdot \partial S_{l} + \frac{m^2}{2} \right) \right] e^{i(S_{k'} + S_{l'} - S_{l})}$$

- The weak field limit of a 3-point on Schwarzschild reproduces the probe limit of a 5-point amplitude on a flat background

$$\underbrace{\langle p', k | \mathcal{S} | p \rangle}_{S-matrix\ on\ g} = \frac{\hat{\delta}(U \cdot (q + k))}{2M} \underbrace{\mathcal{A}_5(p, P \to p + q, P - q - k, k)}_{S-matrix\ on\ \eta} + \ldots$$

Weak field limit $\Rightarrow$ linearized black hole

$$\Lambda^{p}(l) = \delta(p_0 - l_0) f^{eik.}(p, l) \quad , \quad \Lambda^{k'}_{\mu\nu}{}^{\rho\sigma}(k) = -2\, \delta^{\rho}_{(\mu} \, \delta^{\sigma}_{\nu)} \Lambda^{k'}(k)$$

Conclusions

- We can define the SF expansion to classical observables in terms of the classical limit of amplitudes on a background.
- Relevant on-shell building blocks for the SF expansion on a plane wave and Schwarzschild (2-points and 3-points).
- Plane wave case: impulse and waveform from amplitudes.
- Schwarzschild case: the weak field limit of a 3-point is a 5-point in vacuum: using Hamilton's principal function is key.

Main message

We can address the self-force approximation using scattering amplitudes