Thermodynamic nature of irreversibility in active matter

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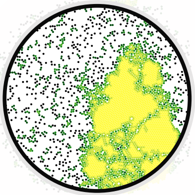

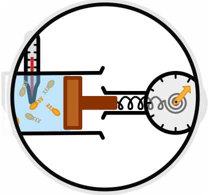

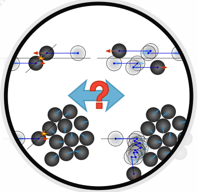

21 October 2024
Nordita program “Measuring and manipulating nonequilibrium systems”

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- particles/entities that can **consume energy** from their environment and use it to **move persistently and autonomously**

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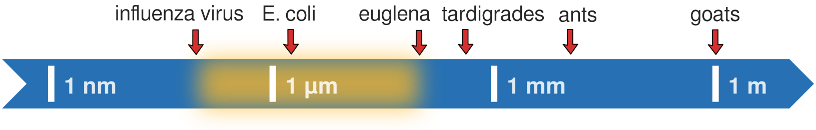

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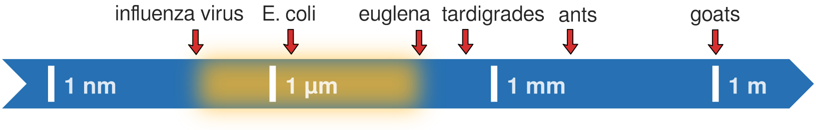

- emerging collective behavior (clustering, swarming, communication, ...)

Modeling active matter at mesoscopic scales

- N interacting, spherical Brownian particles in $d = 2$ or 3 which convert energy from surroundings into directed motion

$$\dot{\boldsymbol{x}}_i(t) = \frac{1}{\gamma}\boldsymbol{F}_i(\boldsymbol{x}(t)) + \sqrt{2D}\,\boldsymbol{\xi}_i(t) + \boldsymbol{v}_i(t)$$

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- hard-core contact interactions via steep pair potential $U(r)$,

$$\boldsymbol{F}_i(\boldsymbol{x}) = -\sum_{j(\neq i)} \boldsymbol{F}(\boldsymbol{x}_i - \boldsymbol{x}_j)\,, \quad \boldsymbol{F}(\boldsymbol{r}) = -U'(|\boldsymbol{r}|)\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\,,$$

such that $U(r) \equiv 0$ if $r > 2R$ (R: particle radius)

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- collisions with surrounding molecules lead to
  - dissipation (friction coefficient γ)
  - random forces ("thermal fluctuations"), modeled by Gaussian white-noise processes:

$$\langle \boldsymbol{\xi}_i(t) \rangle = 0$$

$$\langle \xi_i^\mu(t)\, \xi_j^\nu(t') \rangle = \delta_{ij}\delta_{\mu\nu}\delta(t - t')$$

  $\rightarrow$ fluctuation-dissipation relation: $D = k_{\mathrm{B}} T/\gamma$
  - overdamped approximation: inertia negligible compared to friction/dissipation

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- self-propulsion also erratic, but persistent

→ $\boldsymbol{v}_i(t)$ independent random processes ("active fluctuations") with correlation function

$$\langle v_i^\mu(t)\, v_j^\nu(t')\rangle = \delta_{ij}\,\delta_{\mu\nu}\,\frac{D_\mathrm{a}}{\tau_\mathrm{a}}\mathrm{e}^{-|t-t'|/\tau_\mathrm{a}}$$

  - τ_a: correlation time
  - D_a: "active diffusion coefficient" (speed/intensity of self-propulsion)

→ effective, phenomenological model, disregarding details of self-propulsion mechanism

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$\rightarrow$ entropy introduces a (thermodynamic) "arrow of time"

Thermodynamics of irreversibility: passive systems

- **irreversibility**: log-ratio of path probabilities for forward- and backward-in-time trajectories/histories,

$$\Sigma := \ln \frac{P[\{\boldsymbol{x}(t)\}_{t=0}^{\tau}]}{P[\{\tilde{\boldsymbol{x}}(t)\}_{t=0}^{\tau}]}, \quad \tilde{\boldsymbol{x}}(t) := \boldsymbol{x}(\tau - t)$$

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- passive systems:

$$\dot{\boldsymbol{x}}_i(t) = \frac{1}{\gamma}\boldsymbol{F}_i(\boldsymbol{x}(t)) + \sqrt{2D}\,\xi_i(t) + \cancel{\boldsymbol{v}_i(t)}$$

$\rightarrow$ Σ related to **thermodynamic entropy** S via stochastic energetics and fluctuation theorems [Seifert, PRL **95** (2005)]:

$$k_{\mathrm{B}}\,\langle\Sigma\rangle = \frac{Q}{T} = S \qquad (Q\text{: dissipated heat})$$

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- how does Σ relate to thermodynamics in active matter?

Irreversibility in active matter

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- explicitly known for active Ornstein-Uhlenbeck particles (when $\boldsymbol{v}(t)$ is an Ornstein-Uhlenbeck process):

$$\Sigma = \frac{1}{k_{\mathrm{B}}T}\int_0^{\tau}\mathrm{d}t\int_0^{\tau}\mathrm{d}t' \sum_{i=1}^{N}\dot{\boldsymbol{x}}_i(t)\cdot\boldsymbol{F}_i(t')\left[\delta(t-t')-\Gamma(t,t')\right]+\Delta S_{\mathrm{sys}}$$

[LD, Bo, Eichhorn, PRX **9** (2019)]

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- steady-state "irreversibility production rate":

$$\sigma := \lim_{\tau\to\infty} \frac{\langle\Sigma\rangle}{\tau}$$

Irreversibility for AOUPs

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with $\Gamma(t, t') \sim \frac{\tau_* D_\mathrm{a}/D}{2\tau_\mathrm{a}^2} \mathrm{e}^{-|t-t'|/\tau_*}$ asymptotically ($\tau \to \infty$), $\tau_* = \frac{\tau_\mathrm{a}}{\sqrt{1+D_\mathrm{a}/D}}$

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- partial integration and $\tau \to \infty$:

$$\sigma = \frac{C}{\gamma} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s/\tau_*} \sum_{i=1}^N \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0+s) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0-s) \rangle\right],$$

where t_0 is an arbitrary reference time and $C := \frac{D_\mathrm{a}}{2D^2\tau_\mathrm{a}^2}$

Pressure in active matter

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- three contributions: $p = p_{\rm o} + p_{\rm int} + p_{\rm a}$
  - osmotic/ideal-gas pressure: $p_{\rm o} = Nk_{\rm B}T/V$
  - interaction pressure: $p_{\rm int} = \frac{1}{dV}\sum_i \langle \boldsymbol{x}_i \cdot \boldsymbol{F}_i(\boldsymbol{x})\rangle$
  - active or **swim pressure**: $p_{\rm a} = \frac{\gamma}{dV}\sum_i \langle \boldsymbol{x}_i \cdot \boldsymbol{v}_i\rangle$

Irreversibility as a thermodynamic state function

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$$\boxed{\sigma = \left(\frac{2^{d-2}V_\mathrm{p}}{k_\mathrm{B}\tau_\mathrm{a}}\right)\frac{N\,p_\mathrm{a}}{T}\,\left[1+O(\phi)\right]}$$

(V_p: particle volume; $\phi = \frac{NV_\mathrm{p}}{V}$: packing fraction)

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- low density *and* high activity:

$$\sigma = \left(\frac{2^{d-2} V_{\mathrm{p}}}{k_{\mathrm{B}} \tau_{\mathrm{a}}}\right) \frac{N\, p}{T} \left[1 + O(\phi, \mathrm{Pe}^{-1})\right]$$

Derivation

$$\sigma = \frac{C}{\gamma}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s/\tau_*} \sum_{i=1}^{N} \left[\langle \boldsymbol{x}_i(t_0)\cdot \boldsymbol{F}_i(t_0+s)\rangle - \langle \boldsymbol{x}_i(t_0)\cdot \boldsymbol{F}_i(t_0-s)\rangle\right]$$

key ingredients and assumptions:

- exactly known expression for σ for AOUPs
- particles are hard spheres (steep interaction potential)
- "most" interactions are pair collisions ($\phi \ll 1$)

main steps:

- replace interaction forces by self-propulsion in correlator:
  $\sum_i \langle \boldsymbol{x}_i(t_0)\cdot \boldsymbol{F}_i(t_0 \pm s)\rangle = -\frac{2^{d-2}\gamma}{d}\phi \sum_i \langle \boldsymbol{x}_i(t_0)\cdot \boldsymbol{v}_i(t_0 \pm s)\rangle \,[1+O(\phi)]$
- evaluate s integral using Novikov's theorem:
  $\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s/\tau_*}\langle \boldsymbol{x}_i(t_0)\cdot \boldsymbol{v}_i(t_0 \pm s)\rangle = \frac{\tau_\mathrm{a}}{\sqrt{1+\frac{D_\mathrm{a}}{D}} \pm 1}\langle \boldsymbol{x}_i(t_0)\cdot \boldsymbol{v}_i(t_0)\rangle$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$

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  - positions: $\boldsymbol{r}_{ij} := \boldsymbol{x}_i - \boldsymbol{x}_j$
  - fluctuations: $\boldsymbol{\alpha}_{ij} := \boldsymbol{\alpha}_i - \boldsymbol{\alpha}_j$ with $\boldsymbol{\alpha}_i := \boldsymbol{v}_i + \sqrt{2D}\boldsymbol{\xi}_i$

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- focus on **pair collisions** (e.g., low particle density):
  at any given time, a fixed particle i interacts at most with one other particle j

$\rightarrow$ $\dot{\boldsymbol{r}}_{ij} = \frac{2}{\gamma}\boldsymbol{F}(\boldsymbol{r}_{ij}) + \boldsymbol{\alpha}_{ij}$

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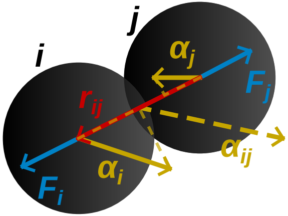

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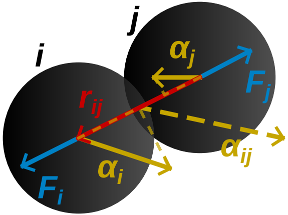

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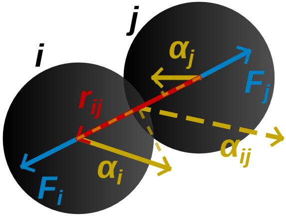

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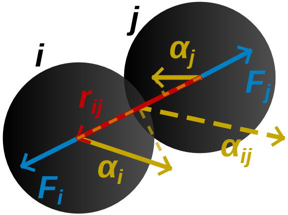

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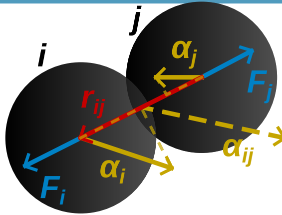
$\rightarrow$ $\frac{1}{\gamma} \boldsymbol{F}(\boldsymbol{r}_{ij}) \approx -\frac{1}{2} \boldsymbol{\alpha}_{ij}^{\parallel} \chi(r_{ij})$ with $\chi(r) = \Theta(2R - r)$

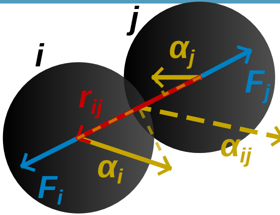
$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle =$
$-\frac{\gamma}{2} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s)) \chi(r_{ij}(t_0 \pm s)) \rangle$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (3)

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle =$
$-\frac{\gamma}{2} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle$



Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (3)



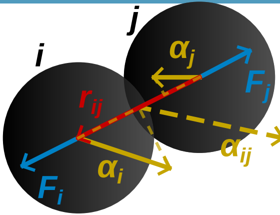
$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle =$
$-\frac{\gamma}{2} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$

- inspect

$$\langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$$

$$= \langle [\boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)][\boldsymbol{\alpha}_{ij}(t_0 \pm s) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)]\, \chi(r_{ij}(t_0 \pm s))\rangle$$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (3)



→ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle =$
$-\frac{\gamma}{2} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s) \rangle \chi(r_{ij}(t_0 \pm s)) \rangle$

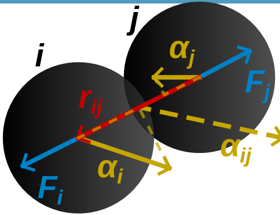
- inspect

$$\langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s)) \chi(r_{ij}(t_0 \pm s)) \rangle$$

$$= \langle [\boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)][\boldsymbol{\alpha}_{ij}(t_0 \pm s) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)] \chi(r_{ij}(t_0 \pm s)) \rangle$$

- *before* collision, particles move around freely, hence orientations of $\boldsymbol{r}_{ij}$ and $\boldsymbol{\alpha}_{ij}$ approximately uncorrelated

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (3)



→ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle =$
$-\frac{\gamma}{2} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}^{\parallel}_{ij}(t_0 \pm s) \rangle \chi(r_{ij}(t_0 \pm s)) \rangle$

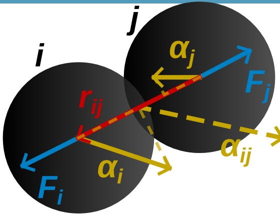
- inspect

$$\langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}^{\parallel}_{ij}(t_0 \pm s)) \chi(r_{ij}(t_0 \pm s)) \rangle$$

$$= \langle [\boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)][\boldsymbol{\alpha}_{ij}(t_0 \pm s) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)] \chi(r_{ij}(t_0 \pm s)) \rangle$$

- *before* collision, particles move around freely, hence orientations of $\boldsymbol{r}_{ij}$ and $\boldsymbol{\alpha}_{ij}$ approximately uncorrelated
- to *initiate* and *maintain* collision, need $\boldsymbol{n}_{ij} \cdot \boldsymbol{\alpha}_{ij} < 0$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (3)



$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle =$
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- inspect

$$\langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)\rangle$$

$$= \langle [\boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)][\boldsymbol{\alpha}_{ij}(t_0 \pm s) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)] \, \chi(r_{ij}(t_0 \pm s)\rangle$$

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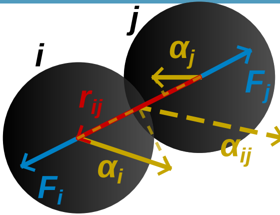
$\rightarrow$ $\langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)\rangle$
$\approx \frac{1}{2d} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)\rangle$ up to errors $O(\frac{V_\mathrm{p}}{V})$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (3)



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$$\langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}^{\parallel}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$$
$$= \langle [\boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)][\boldsymbol{\alpha}_{ij}(t_0 \pm s) \cdot \boldsymbol{n}_{ij}(t_0 \pm s)]\, \chi(r_{ij}(t_0 \pm s))\rangle$$

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$\approx \frac{1}{2d} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$ up to errors $O(\frac{V_\mathrm{p}}{V})$

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
$= -\frac{\gamma}{4d} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (4)

$$\rightarrow \sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$$
$$= -\frac{\gamma}{4d} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (4)

$$\to \sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle = -\frac{\gamma}{4d} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$$

- switch back to absolute positions and fluctuations:

$$\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle = -\frac{\gamma}{2d} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle]$$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (4)

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$
$= -\frac{\gamma}{4d} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle$

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$$\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle = -\frac{\gamma}{2d} \sum_{i<j} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle \right]$$

- $\langle \cdots \rangle$ is steady-state average over all pair configurations, but $\chi(r_{ij}(t_0 \pm s))$ picks out those where particles overlap

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (4)

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$
$= -\frac{\gamma}{4d} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle$

- switch back to absolute positions and fluctuations:
  $$\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle = -\frac{\gamma}{2d} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle]$$
- $\langle \cdots \rangle$ is steady-state average over all pair configurations, but $\chi(r_{ij}(t_0 \pm s))$ picks out those where particles overlap
- to leading order (up to corrections $O(\frac{V_\mathrm{p}}{V})$), pair distribution is uniform

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (4)

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$

$$= -\frac{\gamma}{4d} \sum_{i<j} \langle \boldsymbol{r}_{ij}(t_0) \cdot \boldsymbol{\alpha}_{ij}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle$$

- switch back to absolute positions and fluctuations:

$$\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle = -\frac{\gamma}{2d} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle]$$

- $\langle \cdots \rangle$ is steady-state average over all pair configurations, but $\chi(r_{ij}(t_0 \pm s))$ picks out those where particles overlap
- to leading order (up to corrections $O(\frac{V_\mathrm{p}}{V})$), pair distribution is uniform

$\rightarrow$ $\chi(r_{ij})$ picks out "interaction volume" (sphere of radius $2R$):

$$\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s)) \rangle = \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \rangle [1 + O(\tfrac{V_\mathrm{p}}{V})]$$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (5)

- $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$
  $= -\frac{\gamma}{2d} \sum_{i<j} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle \right]$
- $\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle$
  $= \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \rangle [1 + O(\frac{V_\mathrm{p}}{V})]$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$ (5)

- $$\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle = -\frac{\gamma}{2d} \sum_{i<j} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle \right]$$
- $$\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle = \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \rangle [1 + O(\tfrac{V_\mathrm{p}}{V})]$$

$\rightarrow$ $$\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle = -\frac{2^{d-1} V_\mathrm{p} \gamma}{dV} \sum_{i<j} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s)) \rangle \right] [1 + O(\tfrac{V_\mathrm{p}}{V})]$$

Rewriting $\sum_i \langle x_i(t_0) \cdot F_i(t_0 \pm s) \rangle$ (5)

- $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$
  $= -\frac{\gamma}{2d} \sum_{i<j} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle \right.$
  $\left. - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle \right]$
- $\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \, \chi(r_{ij}(t_0 \pm s)) \rangle$
  $= \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s)) \rangle [1 + O(\frac{V_\mathrm{p}}{V})]$

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s) \rangle$
  $= -\frac{2^{d-1} V_\mathrm{p} \gamma}{dV} \sum_{i<j} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)) \rangle \right.$
  $\left. - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s)) \rangle \right] [1 + O(\frac{V_\mathrm{p}}{V})]$

- packing fraction: $\phi = \frac{N V_\mathrm{p}}{V}$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (5)

- $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
  $$= -\frac{\gamma}{2d} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle]$$
- $\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$
  $$= \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\rangle [1 + O(\tfrac{V_\mathrm{p}}{V})]$$

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
$$= -\frac{2^{d-1} V_\mathrm{p} \gamma}{dV} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\rangle]\, [1 + O(\tfrac{V_\mathrm{p}}{V})]$$

- packing fraction: $\phi = \frac{N V_\mathrm{p}}{V}$
- first term: sum over j gives factor $\frac{N-1}{2}$

Rewriting $\sum_i \langle x_i(t_0) \cdot F_i(t_0 \pm s)\rangle$ (5)

- $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
  $= -\frac{\gamma}{2d} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$
  $- \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle]$
- $\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$
  $= \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\rangle [1 + O(\frac{V_\mathrm{p}}{V})]$

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
$= -\frac{2^{d-1} V_\mathrm{p} \gamma}{dV} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\rangle$
$- \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\rangle] [1 + O(\frac{V_\mathrm{p}}{V})]$

- packing fraction: $\phi = \frac{NV_\mathrm{p}}{V}$
- first term: sum over j gives factor $\frac{N-1}{2}$
- second term: $\langle \boldsymbol{x}_i \cdot \boldsymbol{\alpha}_j\rangle = O(\frac{V_\mathrm{p}}{V})$ $\rightarrow$ overall $O(\phi^2)$

Rewriting $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$ (5)

- $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
$$= -\frac{\gamma}{2d} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle \\ -\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle]$$
- $\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\, \chi(r_{ij}(t_0 \pm s))\rangle$
$$= \frac{2^d V_\mathrm{p}}{V} \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_{i/j}(t_0 \pm s))\rangle [1 + O(\tfrac{V_\mathrm{p}}{V})]$$

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle$
$$= -\frac{2^{d-1} V_\mathrm{p} \gamma}{dV} \sum_{i<j} [\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s))\rangle \\ -\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_j(t_0 \pm s))\rangle]\,[1 + O(\tfrac{V_\mathrm{p}}{V})]$$

- packing fraction: $\phi = \frac{N V_\mathrm{p}}{V}$
- first term: sum over j gives factor $\frac{N-1}{2}$
- second term: $\langle \boldsymbol{x}_i \cdot \boldsymbol{\alpha}_j \rangle = O(\frac{V_\mathrm{p}}{V}) \rightarrow$ overall $O(\phi^2)$

$\rightarrow$ $\sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{F}_i(t_0 \pm s)\rangle = -\frac{2^{d-2}\gamma}{d} \phi \sum_i \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0 \pm s)\rangle [1 + O(\phi)]$

Integrate over time delay s

- recall: $\sigma = \frac{C}{\gamma} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s/\tau_*} \sum_i \langle \boldsymbol{x}_i(t_0) \cdot [\boldsymbol{F}_i(t_0+s) - \boldsymbol{F}_i(t_0-s)] \rangle$

Integrate over time delay s

- recall: $\sigma = \frac{C}{\gamma} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s/\tau_*} \sum_i \langle \boldsymbol{x}_i(t_0) \cdot [\boldsymbol{F}_i(t_0+s) - \boldsymbol{F}_i(t_0-s)] \rangle$

$\rightarrow$ to leading order in ϕ,

$$\sigma = -\frac{2^{d-2} C}{d} \phi \sum_i \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s/\tau_*} \left[\langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0+s) \rangle - \langle \boldsymbol{x}_i(t_0) \cdot \boldsymbol{\alpha}_i(t_0-s) \rangle \right]$$

Integrate over time delay s

- recall: $\sigma = \frac{C}{\gamma}\int_0^\infty ds\, e^{-s/\tau_*} \sum_i \langle \boldsymbol{x}_i(t_0)\cdot[\boldsymbol{F}_i(t_0+s) - \boldsymbol{F}_i(t_0-s)]\rangle$

$\rightarrow$ to leading order in ϕ,

$$\sigma = -\frac{2^{d-2}C}{d}\phi\sum_i\int_0^\infty ds\, e^{-s/\tau_*}\left[\langle \boldsymbol{x}_i(t_0)\cdot\boldsymbol{\alpha}_i(t_0+s)\rangle - \langle \boldsymbol{x}_i(t_0)\cdot\boldsymbol{\alpha}_i(t_0-s)\rangle\right]$$

- Novikov's theorem [Novikov, Sov. Phys. JETP **20** (1965)]:
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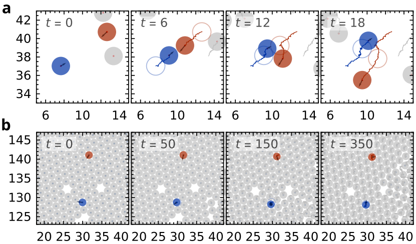

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low density ($\phi \ll 1$):

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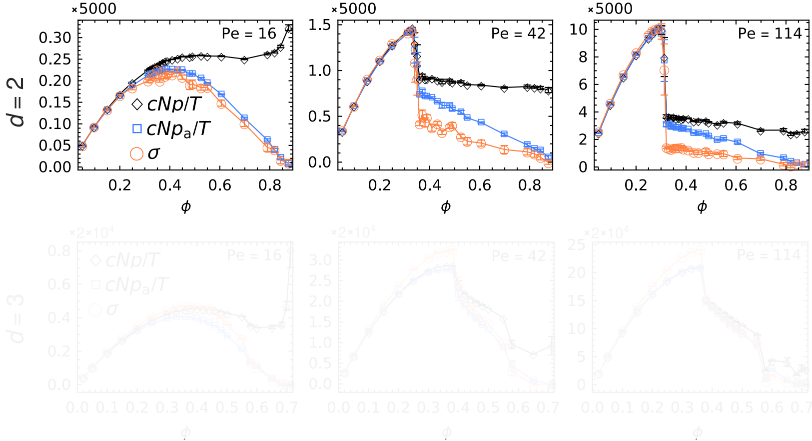

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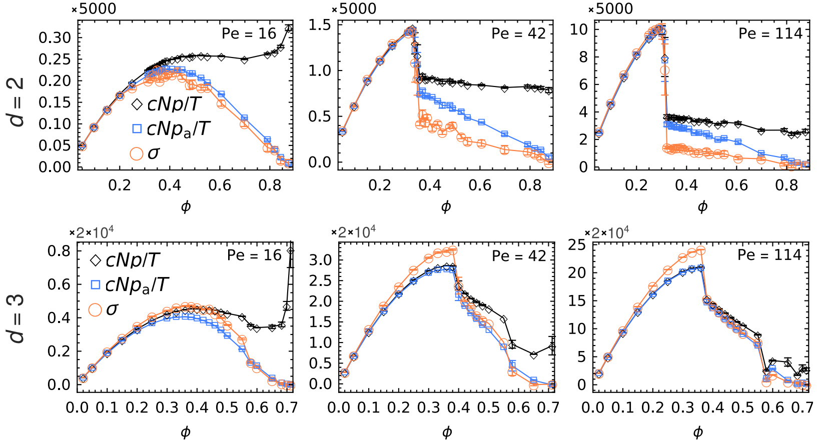

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arXiv:2308.03625 **Thank you!**

Phase diagram ($d = 2$)