# Magneto-Seismology: written in stellar atmospheric waves

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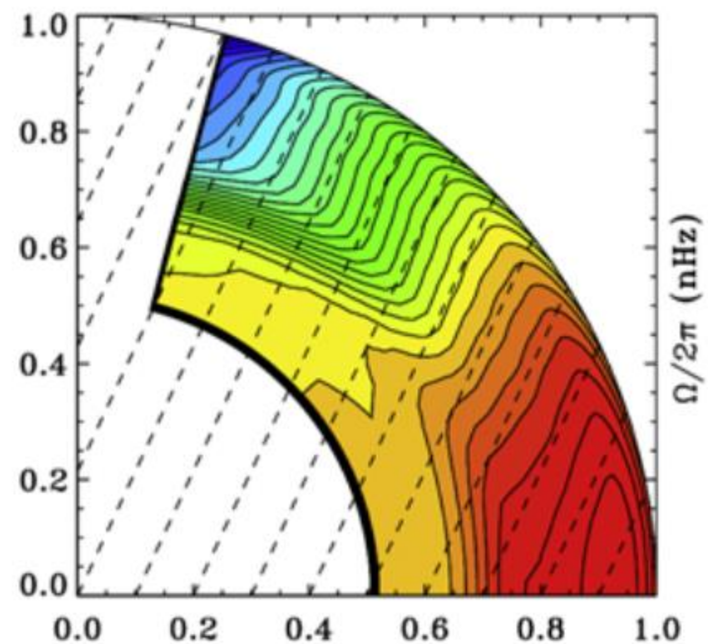
# Helio/astro seismology

REVIEWS OF MODERN PHYSICS, VOLUME 93, JANUARY–MARCH 2021

**Probing the interior physics of stars through asteroseismology**

C. Aerts

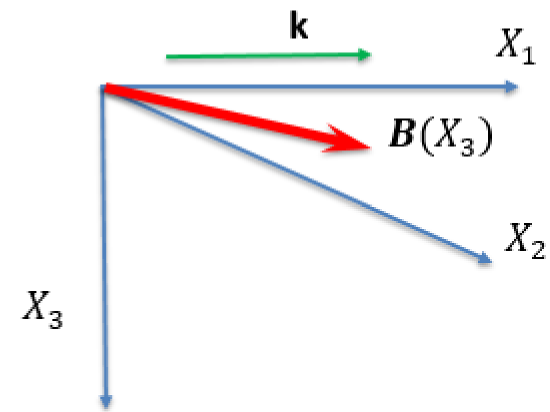
- Finding out the flows in the interior of the Sun from its surface fluctuations.
- What about magnetic fields ?



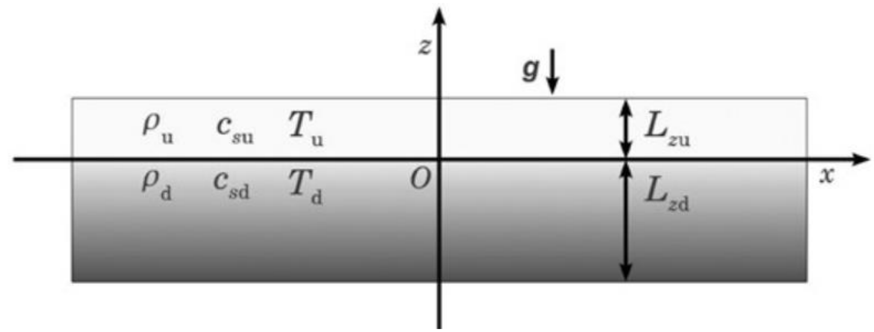

Loren I. Matilsky *et al* 2020 *ApJ* **898** 111

# Magnetic effect on stratified plasma : theory

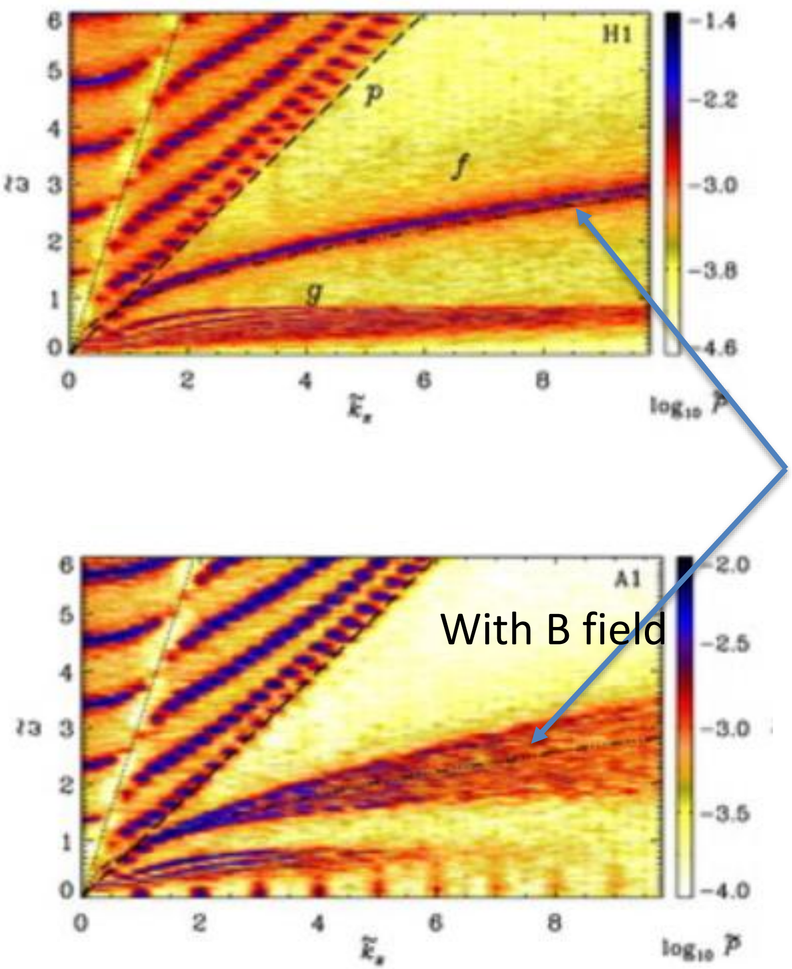
- Constant Alfven speed ( Yu 1965), exact solution.
- Constant magnetic field ( Nye & Thomas 1976; Adam 1977; Thomas 1983; Campos 1983), exact solution, waves along magnetic field.
- Layered model ( Lee & Roberts 1986; Miles & Roberts 1992; Jain & Roberts 1991, Fullon & Roberts 2005), matched asymptotics.
- Waves along the magnetic field (Cally 2007)
- Waves perpendicular to both gravity & magnetic field (Campos & Marta 2015).
- Magnetic effects upshift the frequencies of acoustic modes of stellar oscillations (Gough & Thompson 1990; Goldreich et al. 1991; Dziembowski & Goode 2004). Perturbative calculation in spherical coordinate.



# Magnetic effects : simulations



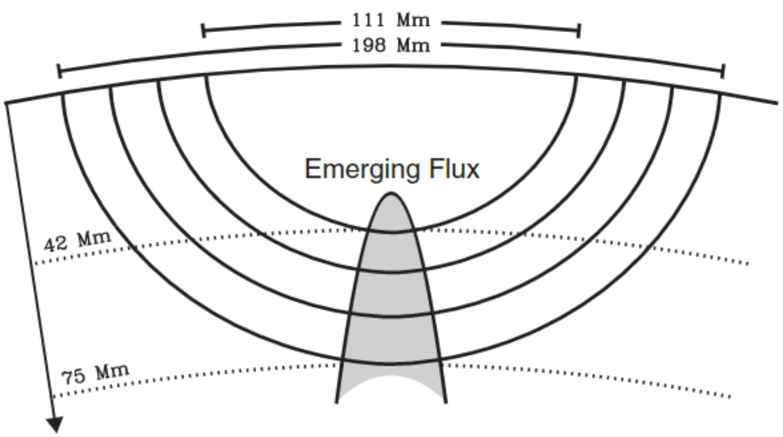

No B field




- The distinguishing effect of the imposed field is strengthening, fanning of f-mode

Singh et al. ApjL (2014), MNRAS (2015), ApJ (2016), GAFd (2020)

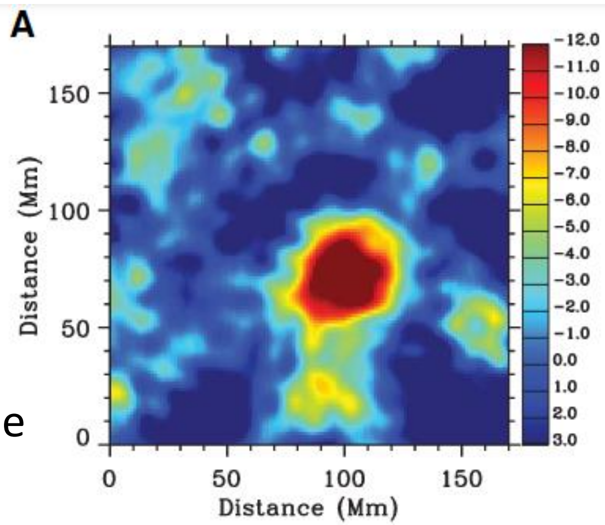
# Magnetic effects: observation



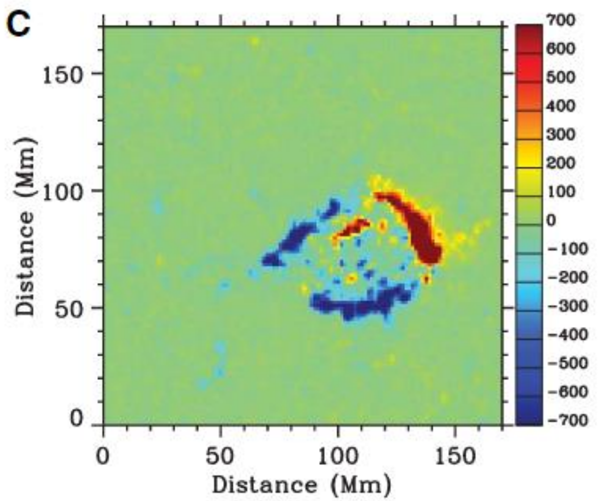

Ilonidis et al 2011, Science

To the best of my knowledge
this study has not been reproduced.

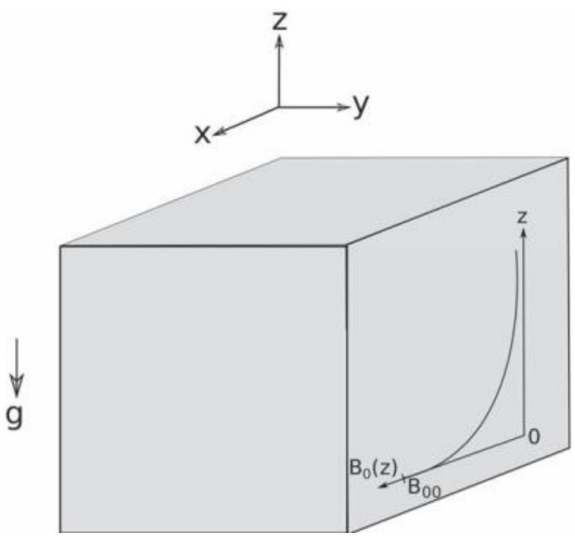
Travel-time map




B field 24 hr later



# Magnetic effects : theory



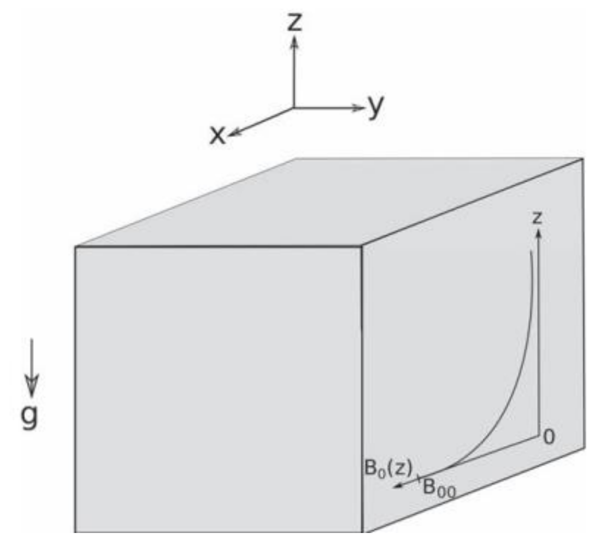

- The plasma is gravitationally stratified, i.e., density is a function of the vertical
- Isothermal gas
- There is a uniform magnetic field pointing along the horizontal direction.
- Even this “simple” problem is very difficult. It has not been solved in full generality.
- If we only study waves along the magnetic field, the solutions are Hypergeometric functions.

# Sketch of exact soln : isothermal atmosphere

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \frac{1}{\rho_0}\nabla(\rho_0 c_s^2 \nabla \cdot \boldsymbol{u}) + \nabla(\boldsymbol{u} \cdot \boldsymbol{g}) - \boldsymbol{g}(\nabla \cdot \boldsymbol{u})$$
$$- \frac{1}{\mu_0\rho_0}\nabla\left[\boldsymbol{u} \cdot \left\{\nabla\left(\frac{B_0^2}{2}\right) - \boldsymbol{B}_0 \cdot \nabla \boldsymbol{B}_0\right\}\right]$$
$$+ \frac{1}{\mu_0\rho_0}[\nabla \times \{\nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0)\}] \times \boldsymbol{B}_0$$
$$+ \frac{1}{\mu_0\rho_0}[(\nabla \times \boldsymbol{B}_0) \times \{\nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0)\}].$$

1. Waves propagating along the $x$-axis (along the background magnetic field);
2. Waves propagating along the $z$-axis (along the direction of the gravity); and
3. Waves propagating along the $y$-axis (orthogonal to the direction of both the gravity and the background magnetic field)

$$M_{ij}\hat{u}_j + N_{ij}D\hat{u}_j + \delta_{i,3}\left(1 + \frac{B_0{}^2}{\rho_0 \mathrm{M}_\mathrm{A}^2}\right)D^2\hat{u}_3 = 0,$$



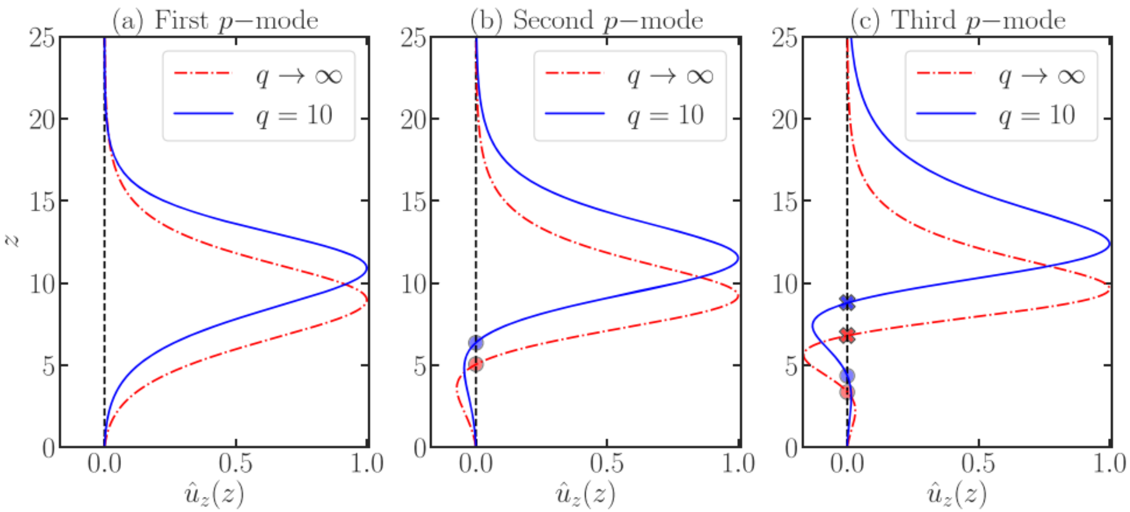

$$[A_2 e^s + B_2]\frac{d^2\hat{u}_z}{ds^2} + [A_1 e^s + B_1]\frac{d\hat{u}_z}{ds} + [A_0 e^s + B_0]\hat{u}_z = 0,$$

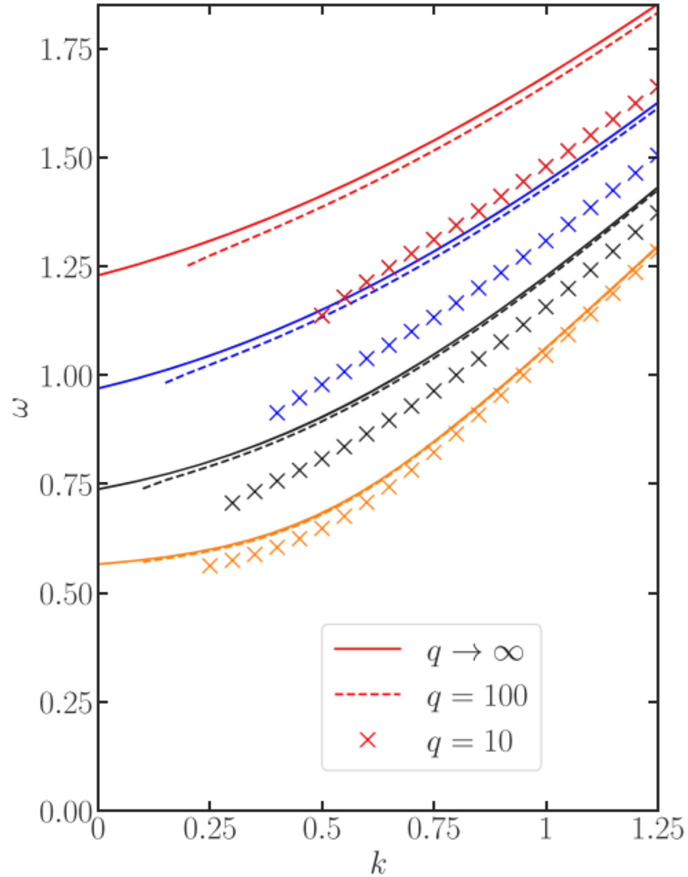
2nd order ODE with exponential coefficients -> can be mapped to hypergeometric

# Isothermal atmosphere : p modes

$$\xi(1-\xi)\frac{d^2W}{d\xi^2} + \left[C - (A+B+1)\,\xi\right]\frac{dW}{d\xi} - ABW = 0.$$

- Waves parallel and perpendicular to the magnetic field.
- Exact solution of the wave equation.
- Also approximate (WKB) solution.
- Leakage of eigenfunctions



# Polytropic problem

- For unmagnetized polytrope this the is old Lamb problem:

$$\frac{\Omega^2}{2K} - \frac{(n+1)(n+1-\gamma n)K}{2\gamma^2\Omega^2} = m + \frac{n}{2} \qquad \Omega^2 \sim K\left(2m+n\right).$$

- Gough & Thomson, Spruit & Bogdan, Cally & Bogdan 93, Bogdan & Cally, Cally & Bogdan 97

# Cally & Bogdan 97

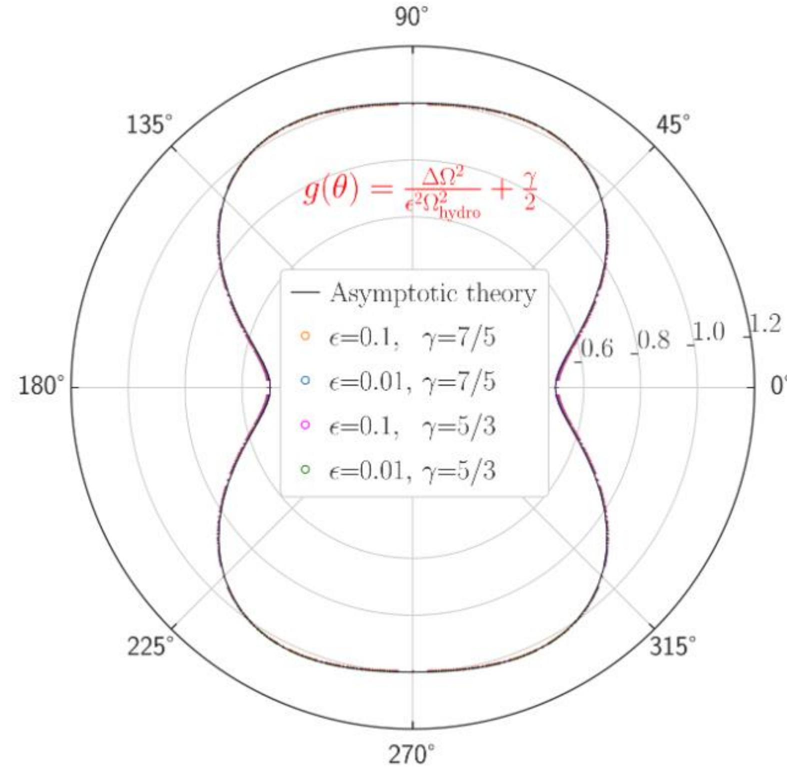
"*Ideally, we would wish to proceed by writing down an equation analogous to Lamb's formula for the magnetized polytrope. Unfortunately, this approach is not feasible and for the most part one must instead be content with a numerically derived visual comparison of how the allowed oscillation frequencies depend upon the choice of the horizontal wavenumber* $k$."

# Perturbative & Numerical Solution

$$P_0 \sim \rho_0^{1+1/n},$$

$$P_0 = c^2 \rho_0 / \gamma$$

- The small parameter is inverse Alfvenic Mach number, proportional to magnetic field strength divided by speed of sound




$$\boxed{\frac{\Omega^2}{K}\left[\frac{1}{2} - \frac{\epsilon^2}{4}\left\{2 - \gamma + \frac{\cos^2\theta}{2} - \frac{3\cos^4\theta}{2}\right\}\right] \sim \left(m + \frac{n}{2}\right);\quad m = 0, 1, 2, \dots .}$$

$$\begin{aligned} g(\theta) &\equiv \gamma/2 + \Delta\Omega^2/(\epsilon^2\Omega^2_{\rm hydro}) \\ &= 1 + \frac{1}{4}\left(\cos^2\theta - 3\cos^4\theta\right), \end{aligned}$$

# Linearized MHD in Polytropic atmosphere

$$\partial_z P_0 = \rho_0 g - \frac{1}{2\mu_0}\partial_z B_0^2. \qquad P_0 \sim \rho_0^{1+1/n}. \qquad B_0^2(z) \sim z^{n+1}. \qquad c^2 = \frac{\gamma g z}{(n+1)(1+\gamma \mathrm{M}_\mathrm{A}^{-2}/2)}.$$

Force-balance demands that sound speed a linear function of depth for a magnetic field that is a power-law in depth.

$$\partial_t^2 \widetilde{u}_x = c^2 \partial_x \widetilde{\chi} + g\partial_x \widetilde{u}_z, \tag{17a}$$

$$\partial_t^2 \widetilde{u}_y = c^2 \partial_y \widetilde{\chi} + g\partial_y \widetilde{u}_z + \frac{c^2}{\mathrm{M}_\mathrm{A}^2}\left(\partial_{xx}\widetilde{u}_y + \partial_y \widetilde{\chi} - \partial_{xy}\widetilde{u}_x\right), \tag{17b}$$

$$\partial_t^2 \widetilde{u}_z = c^2 \partial_z \widetilde{\chi} - g\partial_x \widetilde{u}_x \left[1 + \frac{\gamma}{\mathrm{M}_\mathrm{A}^2(1+\gamma \mathrm{M}_\mathrm{A}^{-2}/2)}\right] - g\partial_y \widetilde{u}_y + \frac{\gamma(1+\mathrm{M}_\mathrm{A}^{-2}) g \widetilde{\chi}}{(1+\gamma \mathrm{M}_\mathrm{A}^{-2}/2)} + \frac{c^2}{\mathrm{M}_\mathrm{A}^2}\left(\partial_{xx}\widetilde{u}_z + \partial_z \widetilde{\chi} - \partial_{xz}\widetilde{u}_x\right). \tag{17c}$$

div of velocity

Reduces to Lamb's equations for zero magnetic field.

# Fourier space + non-dimensionalization

$$\widetilde{u}_x(Z) = \int dK_X dK_Y d\Omega \, \hat{u}_x \exp\left[i(K_X X + K_Y Y + \Omega T)\right],$$

$$\left[\frac{-\Omega^2(1+\gamma\epsilon^2/2)}{iK_X Z}\right]\hat{u}_x + \left[\frac{-(n+1)(1+\gamma\epsilon^2/2)}{\gamma Z}\right]\hat{u}_z = \hat{\chi}, \tag{21a}$$

$$\begin{aligned}\left[-i\epsilon^2 K_X\right]\hat{u}_x + \left[\frac{-\Omega^2(1+\gamma\epsilon^2/2)}{iK_Y Z} + \frac{\epsilon^2 K_X^2}{iK_Y}\right]\hat{u}_y \\ + \left[\frac{-(n+1)(1+\gamma\epsilon^2/2)}{\gamma Z}\right]\hat{u}_z = \hat{\chi}(1+\epsilon^2),\end{aligned} \tag{21b}$$

$$\begin{aligned}&\left[iK_X\left\{\frac{1+\gamma\epsilon^2/2}{\gamma} + \epsilon^2\left(1+\frac{Z\partial_Z}{n+1}\right)\right\}\right]\hat{u}_x \\ &+ \left[\frac{iK_Y(1+\gamma\epsilon^2/2)}{\gamma}\right]\hat{u}_y + \left[\frac{-\Omega^2(1+\gamma\epsilon^2/2)+\epsilon^2 Z K_X^2}{n+1}\right]\hat{u}_z \\ &\qquad = \left(\hat{\chi} + \frac{Z\partial_Z\hat{\chi}}{n+1}\right)(1+\epsilon^2).\end{aligned} \tag{21c}$$

# Structure of problem

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \end{bmatrix} = \begin{bmatrix} h_x(\hat{\chi}) \\ h_y(\hat{\chi}) \\ h_z(\hat{\chi}, \partial_Z \hat{\chi}) \end{bmatrix},$$

$$\hat{u}_\nu = f_\nu(Z)\hat{\chi} + g_\nu(Z)\partial_Z \hat{\chi},$$

$$\partial_Z^2 \hat{\chi} + P(Z,\epsilon)\partial_Z \hat{\chi} + R(Z,\epsilon)\hat{\chi} = 0,$$

# WKB form

$$\hat{\chi}(Z) = \hat{\psi}(Z) \exp\left[-\frac{1}{2}\int_Z dZ\, P(Z)\right],$$

$$\partial_Z^2 \hat{\psi} + \Gamma^2(Z, \epsilon, \delta)\hat{\psi} = 0.$$

$$\delta = K/\Omega^2, \text{ with } K = \sqrt{K_X^2 + K_Y^2}$$

# WKB leading order

$$\zeta^2 \partial_Z^2 \hat{\psi} + \Gamma^2(Z, \epsilon, \delta)\hat{\psi} = 0.$$

$$\frac{1}{\pi} \int_{Z_1(\epsilon,\delta)}^{Z_2(\epsilon,\delta)} \Gamma(Z, \epsilon, \delta)\, dZ \sim \left(m + \frac{1}{2}\right); \;\; m = 0, 1, 2, ...,$$

$$\frac{1}{\pi} \int_{\alpha}^{\beta} \Gamma_0(Z)\, dZ \equiv$$

$$I_{\text{Lamb}} = \frac{(n+1)}{2} \left[ \frac{\Omega^2}{K(n+1)} + \frac{(\gamma n - n - 1)K}{\gamma^2 \Omega^2} - 1 \right] + 1.$$

# Keep perturbing

$$\Gamma(Z,\epsilon,\delta) = \Gamma_0(Z,\delta) + \epsilon\Gamma_1(Z,\delta) + \epsilon^2\Gamma_2(Z,\delta) + \epsilon^3\Gamma_3(Z,\delta) + \mathcal{O}(\epsilon^4).$$

$$\Gamma_0(Z,\delta) = \frac{K\sqrt{(Z-\alpha)(\beta-Z)}}{Z},$$

$$\Gamma_2(Z,\delta) = \frac{(b_2Z^2 + b_1Z + b_0)}{\sqrt{(Z-\alpha)(\beta-Z)}},$$

$$Z_j(\epsilon) = Z_j^{(0)} + \epsilon Z_j^{(1)} + \epsilon^2 Z_j^{(2)} + \mathcal{O}(\epsilon^3),$$

$$\frac{1}{\pi}\int_{Z_1^{(0)}+\epsilon^2 Z_1^{(2)}+\mathcal{O}(\epsilon^3)}^{Z_2^{(0)}+\epsilon^2 Z_2^{(2)}+\mathcal{O}(\epsilon^3)} \left[\Gamma_0(Z) + \epsilon^2\Gamma_2(Z) + \mathcal{O}(\epsilon^3)\right]\, dZ \sim \left(m+\frac{1}{2}\right);\;\; m = 0,1,2,....$$

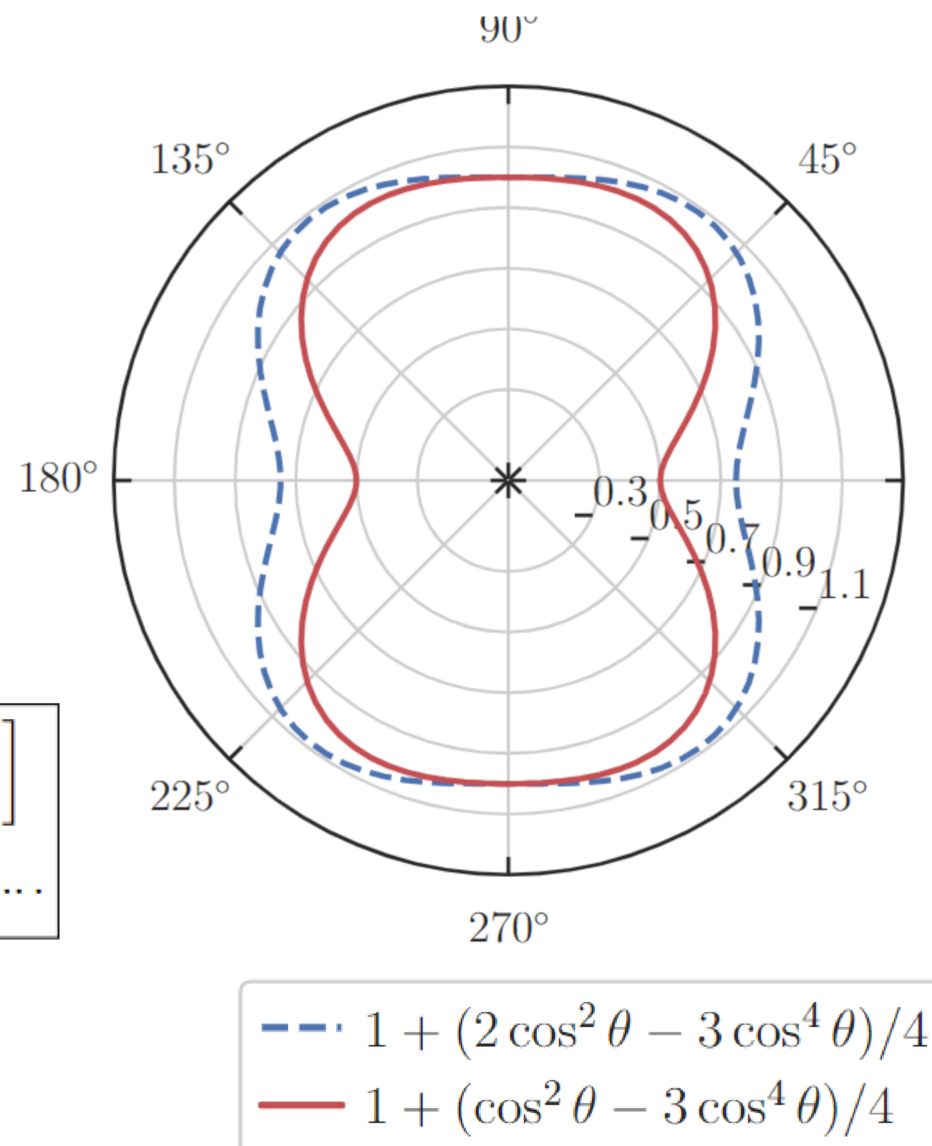
$$I_{\mathrm{Lamb}} + \frac{\epsilon^2}{\pi}\int_\alpha^\beta \Gamma_2(Z)dZ + \mathcal{O}(\epsilon^3) \sim \left(m+\frac{1}{2}\right);\;\; m = 0,1,2,...,$$

# Are we there ?

$$\frac{\Omega^2}{K}\left[\frac{1}{2}-\frac{\epsilon^2}{4}\left\{2-\gamma+\cos^2\theta-\frac{3\cos^4\theta}{2}\right\}\right]$$
$$\sim\left(m+\frac{n}{2}\right);\quad m=0,1,2,\dots.$$

$$g(\theta)\equiv\gamma/2+\Delta\Omega^2/(\epsilon^2\Omega^2_{\mathrm{hydro}})$$

$$\boxed{\begin{aligned}&\frac{\Omega^2}{K}\left[\frac{1}{2}-\frac{\epsilon^2}{4}\left\{2-\gamma+\frac{\cos^2\theta}{2}-\frac{3\cos^4\theta}{2}\right\}\right]\\&\qquad\sim\left(m+\frac{n}{2}\right);\quad m=0,1,2,\dots.\end{aligned}}$$



# Conclusion

- We get excellent agreement between numerics and our formula.
- How to improve our asymptotic calculation ?
- Both the isothermal and polytropic calculation assumes a magnetic field that is known as function of depth. This is unrealistic.
- The mathematically tractable models are not so useful in practice. But they are useful to obtain insight.
- Magneto-seismology is a treasure trove of interesting problems.

# Sketch of WKB

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \end{bmatrix} = \begin{bmatrix} h_x(\hat{\chi}) \\ h_y(\hat{\chi}) \\ h_z(\hat{\chi}, \partial_z \hat{\chi}) \end{bmatrix}$$

$$\hat{u}_x = f_x(Z)\hat{\chi} + g_x(Z)\partial_z\hat{\chi},$$
$$\hat{u}_y = f_y(Z)\hat{\chi} + g_y(Z)\partial_z\hat{\chi},$$
$$\hat{u}_z = f_z(Z)\hat{\chi} + g_z(Z)\partial_z\hat{\chi},$$

$$iK_X\hat{u}_x + iK_Y\hat{u}_y + \partial_Z\hat{u}_z = \hat{\chi}$$

$$\partial_Z^2\hat{\chi} + P(Z,\epsilon)\partial_Z\hat{\chi} + R(Z,\epsilon)\hat{\chi} = 0,$$

$$\hat{\chi}(Z) = \hat{\psi}(Z)\exp\left[-2^{-1}\int_Z dZ\, P(Z)\right],$$

$$\boxed{\partial_Z^2\hat{\psi} + \Gamma^2(Z,\epsilon)\hat{\psi} = 0.}$$

$$\Gamma(Z,\epsilon) = \Gamma_0(Z) + \epsilon\Gamma_1(Z) + \epsilon^2\Gamma_2(Z) + \epsilon^3\Gamma_3(Z) + \mathcal{O}(\epsilon^4).$$

Turning points

$$\frac{1}{\pi}\int_{Z_1(\epsilon)}^{Z_2(\epsilon)} \Gamma(Z,\epsilon)\, dZ \sim \left(m + \frac{1}{2}\right); \quad m = 0, 1, 2, \ldots,$$

$$Z_j(\epsilon) = Z_j^{(0)} + \epsilon Z_j^{(1)} + \epsilon^2 Z_j^{(2)} + \mathcal{O}(\epsilon^3),$$

$$I_{\text{Lamb}} - \epsilon^2\left[b_0 + \left(\frac{\alpha+\beta}{2}\right)b_1 + \left(\frac{3\alpha^2 + 2\alpha\beta + 3\beta^2}{8}\right)b_2\right] \sim \left(m + \frac{1}{2}\right)$$

$$\frac{\Omega^2}{K}\left[\frac{1}{2} - \frac{\epsilon^2}{4}\left\{2 - \gamma + \cos^2\theta - \frac{3\cos^4\theta}{2}\right\}\right] \sim \left(m + \frac{n}{2}\right);$$

# Acknowledgement

- Swedish research council.
- IISER-Kolkata and SPARC project.
- Dibyendu Nandi
- Srijan Bharati Das

Recipe for inferring sub-surface solar magnetism via local mode-coupling using Slepian basis functions, Srijan Bharati Das, ArXiv 2209.07979

# Conclusion and further reading

- We can calculate how the presence of magnetic field changes the p-modes. For a magnetic field that decays exponential with depth.
- We have an exact solution for the isothermal case.
- We have perturbative solution for the polytropic atmosphere.


MHD equations :

Choudhuri A. R., 1998, The physics of fluids and plasmas: an introduction for astrophysicists. Cambridge University Press

Magnetiseismology:

Singh, N. K., Brandenburg, A., Rheinhardt, M., 2014, The Astrophysical Journal Letters, 795, L8

Singh, N. K., Brandenburg, A., Chitre, S.M., Rheinhardt, M., 2015, Monthly Notices of the Royal Astronomical Society, 447(4), 3708-3722

Singh N. K., Raichur H., Brandenburg A., 2016, The Astrophysical Journal, 832, 120

# Further reading

Waves : Tolstoy I., 1963, Reviews of Modern Physics, 35, 207

Inverse problems:

**Inverse Problems**

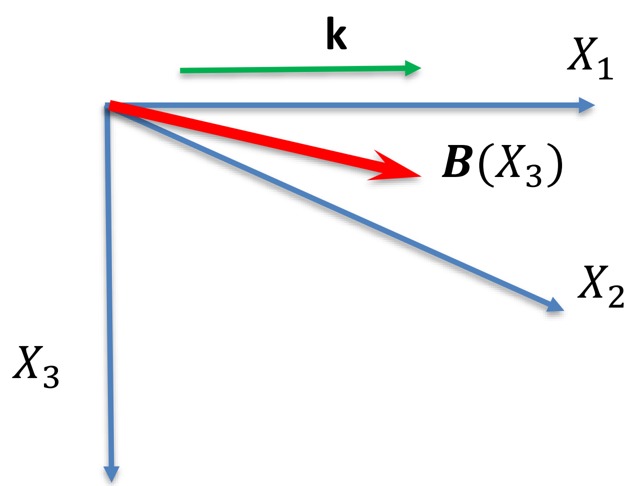
Joseph B. Keller

**The American Mathematical Monthly**

Volume 83, 1976 - Issue 2

WKB and other Perturbative methods

https://youtu.be/LYNOGk3ZjFM



k
$X_1$
$\boldsymbol{B}(X_3)$
$X_2$
$X_3$

# Polytropic atmosphere

$$\partial_z P_0 = \rho_0 g - \frac{1}{2\mu_0}\partial_z B_0^2. \qquad P_0 \sim \rho_0^{1+1/n}. \qquad B_0^2(z) \sim z^{n+1}. \qquad c^2 = \frac{\gamma g z}{(n+1)(1+\gamma \mathrm{M}_\mathrm{A}^{-2}/2)}.$$

Force-balance demands that sound speed a linear function of depth for a magnetic field that is a power-law in depth.

$$\left[\frac{-\Omega^2(1+\gamma\epsilon^2/2)}{iK_X Z}\right]\hat{u}_x + \left[\frac{-(n+1)(1+\gamma\epsilon^2/2)}{\gamma Z}\right]\hat{u}_z = \hat{\chi}, \tag{18a}$$

$$\left[-i\epsilon^2 K_X\right]\hat{u}_x + \left[\frac{-\Omega^2(1+\gamma\epsilon^2/2)}{iK_Y Z} + \frac{\epsilon^2 K_X^2}{iK_Y}\right]\hat{u}_y + \left[\frac{-(n+1)(1+\gamma\epsilon^2/2)}{\gamma Z}\right]\hat{u}_z = \hat{\chi}(1+\epsilon^2), \tag{18b}$$

$$\left[iK_X\left\{\frac{1+\gamma\epsilon^2/2}{\gamma} + \epsilon^2\left(1+\frac{Z\partial_Z}{n+1}\right)\right\}\right]\hat{u}_x + \left[\frac{iK_Y(1+\gamma\epsilon^2/2)}{\gamma}\right]\hat{u}_y + \left[\frac{-\Omega^2(1+\gamma\epsilon^2/2)+\epsilon^2 Z K_X^2}{n+1}\right]\hat{u}_z = \left(\hat{\chi} + \frac{Z\partial_Z\hat{\chi}}{n+1}\right)(1+\epsilon^2).$$

One over Alfvenic Mach no.

div of velocity

Non-dimensionalised. Reduces to Lamb's equations for zero magnetic field.