New Aspects in Mean-Field Dynamo Theory

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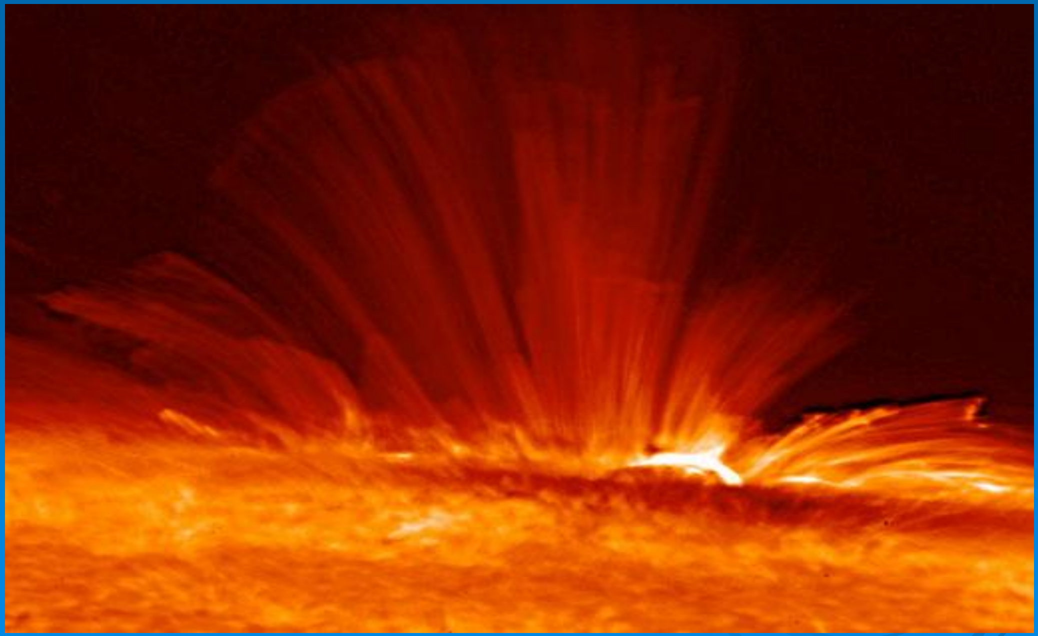
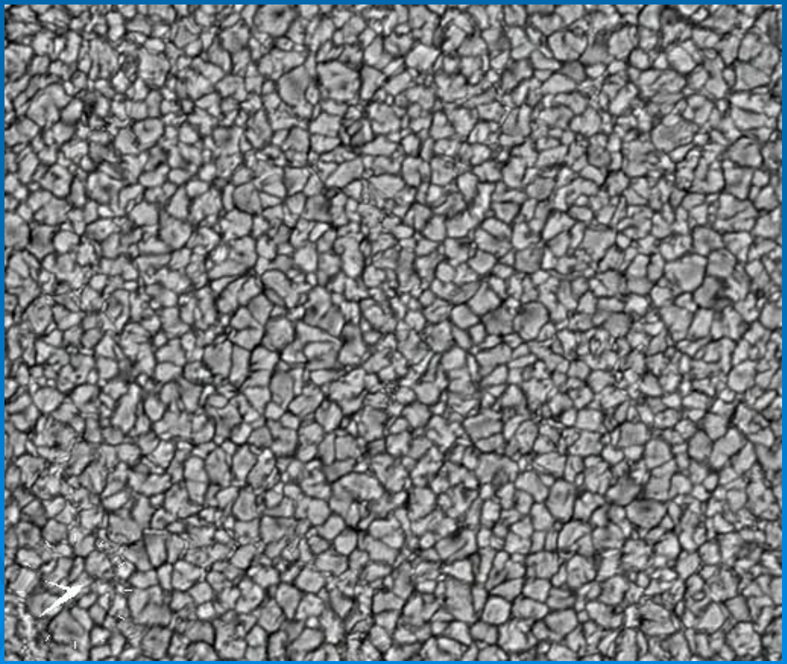




Mean-Field Dynamo

Mean-Field Approach: $B = \overline{B} + b, \quad U = \overline{U} + u,$

- **Induction equation for mean magnetic field:**

$$\frac{\partial \overline{B}}{\partial t} = \nabla \times \left(\overline{U} \times \overline{B} + \langle u \times b \rangle \right) + \eta \Delta \overline{B},$$

$$\overline{B} = \langle B \rangle$$

$$\overline{U} = \langle U \rangle$$

- **Turbulent electromotive force:** $\mathcal{E} = \langle u \times b \rangle,$

$$\mathcal{E} = \alpha \overline{B} + V^{\rm eff} \times \overline{B} - \eta_{_T} (\nabla \times \overline{B}),$$

$$\alpha = -\frac{\tau_0}{3} \langle u \cdot (\nabla \times u) \rangle .$$

$$V^{\rm eff} = -\frac{1}{2} \nabla \eta_{_T} .$$

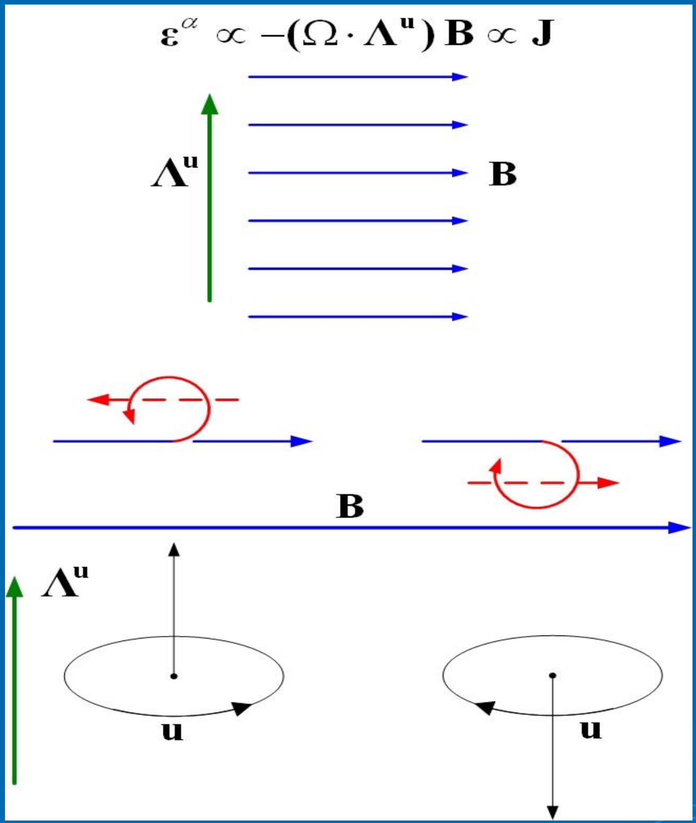
$$\eta_{_T} = \frac{\tau_0}{3} \langle u^2 \rangle .$$

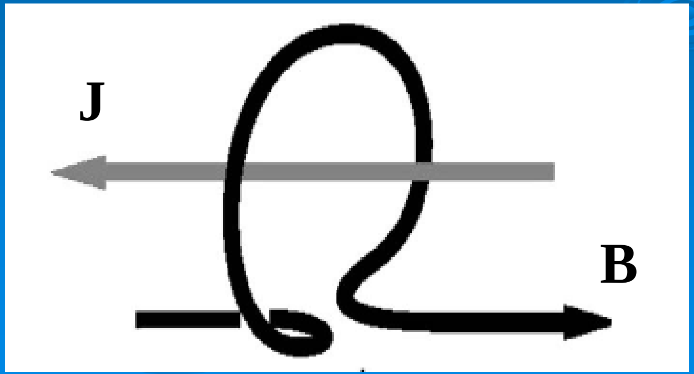
Steenbeck, Krause, Rädler (1966)

Physics of the Kinetic Alpha-Effect

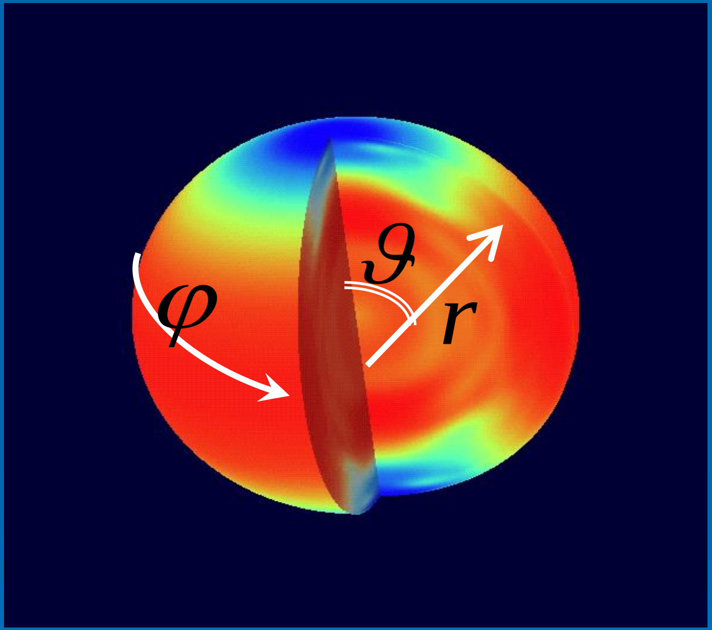
Parker (1955); Steenbeck, Krause, Rädler (1966)

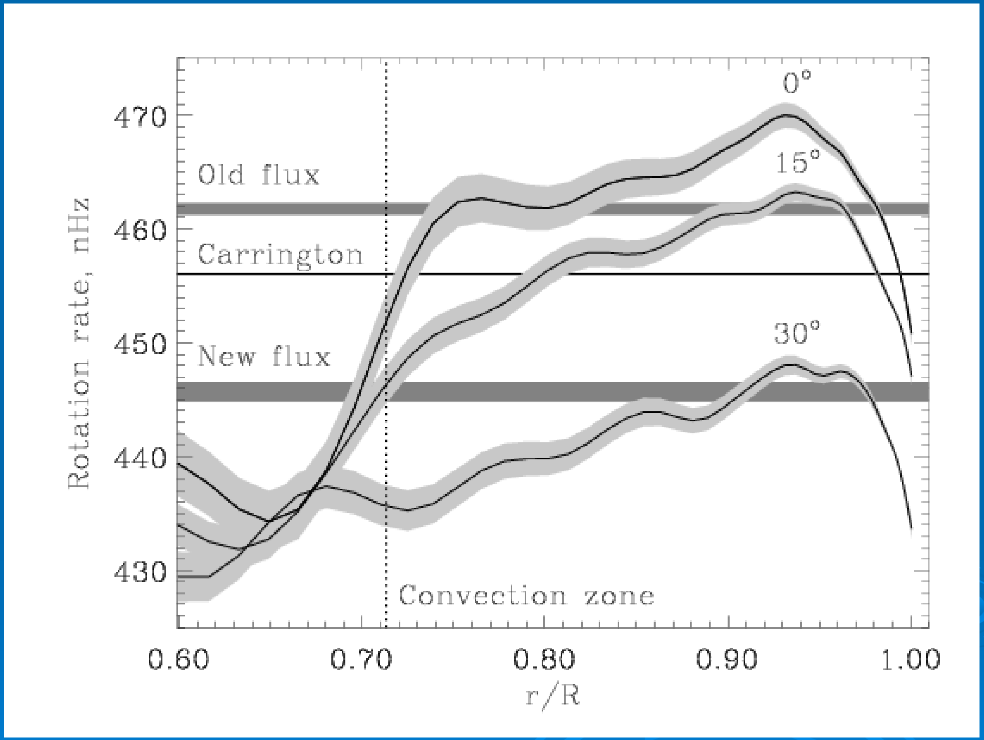
- The α - effect is related to the kinetic helicity in a rotating density stratified convective turbulence or a rotating inhomogeneous turbulence.
- The deformations of the magnetic field lines are caused by upward and downward rotating turbulent eddies.
- The stratification of turbulence or turbulence inhomogeneity breaks a symmetry between the upward and downward eddies.
- Therefore, the total effect of the upward and downward eddies on the mean magnetic field does not vanish and it creates the mean electric current parallel to the original mean magnetic field.





Differential Rotation



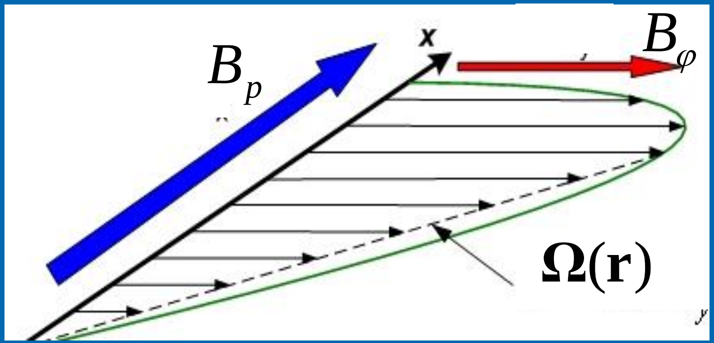


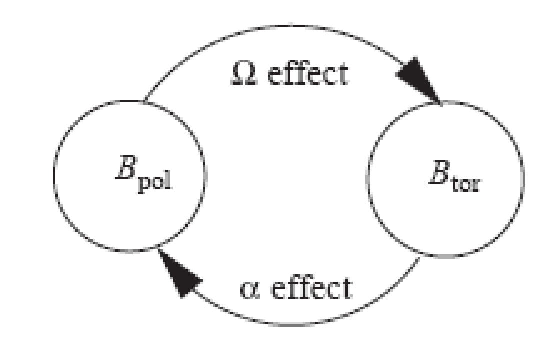
Generation of the mean magnetic field due to the $\alpha\Omega$ dynamo

Parker (1955)

Mean magnetic field:

$$\overline{B}(t,x,z) = \overline{B}_y(t,x,z)e_y + \nabla\times\left[\overline{A}(t,x,z)e_y\right],$$





$$\frac{\partial \overline{A}(t,x,z)}{\partial t} = \alpha\overline{B}_y + \eta_T\Delta\overline{A},$$

$$\frac{\partial \overline{B}_y(t,x,z)}{\partial t} = -\alpha\Delta\overline{A} - S\nabla_z A + \eta_T\Delta\overline{B}_y.$$

Algebraic and Dynamic Nonlinearities in Mean-Field Dynamo

- **Induction equation for mean magnetic field:**

$$\frac{\partial \mathrm{B}}{\partial t} = \nabla \times (\mathrm{U} \times \mathrm{B} + \mathcal{E} - \eta \nabla \times \mathrm{B}),$$

- **Nonlinear electromotive force:** $\mathcal{E}(\mathrm{B}) = \langle \mathbf{u} \times \mathbf{b} \rangle$

$$\mathcal{E}(\mathrm{B}) = \alpha(\mathrm{B})\,\mathrm{B} - [\mathbf{V}^{A}(B) \cdot \nabla]\,\mathbf{A} - \eta_{T}^{A,B}(B)\,(\nabla \times \mathbf{B})$$

- **Total (kinetic + magnetic) nonlinear alpha effect :**

$$\alpha(\mathrm{B}) = \alpha^{v} + \alpha^{m} = \chi^{v}\,\Phi_{v}(B) + \chi^{c}(\mathbf{B})\,\Phi_{m}(B)$$

$$\chi^{c}(\mathrm{B}) = \frac{\tau}{3\rho}\langle \mathrm{b} \cdot (\nabla \times \mathrm{b}) \rangle = \frac{2}{9\eta_{T}\rho}\langle \mathbf{a} \cdot \mathbf{b} \rangle + O\Big(\frac{l_0^2}{L_u^2}\Big)$$

Methods for Derivation of EMF

- **Quasi-Linear Approach** or Second-Order Correlation Approximation (SOCA) or First-Order Smoothing Approximation (FOSA)

  Rm << 1 , Re << 1

  Steenbeck, Krause, Rädler (1966); Roberts, Soward (1975); Moffatt (1978)

- **Path-Integral Approach** (delta-correlated in time random velocity field or short yet finite correlation time)

  Zeldovich, Molchanov, Ruzmaikin, Sokoloff (1988)
  Rogachevskii, Kleeorin (1997)

  $$\mathrm{St} = \frac{\tau}{\ell/u} \ll 1$$

- **Tau-approaches** (spectral tau-approximation, minimal tau-approximation) – third-order or high-order closure

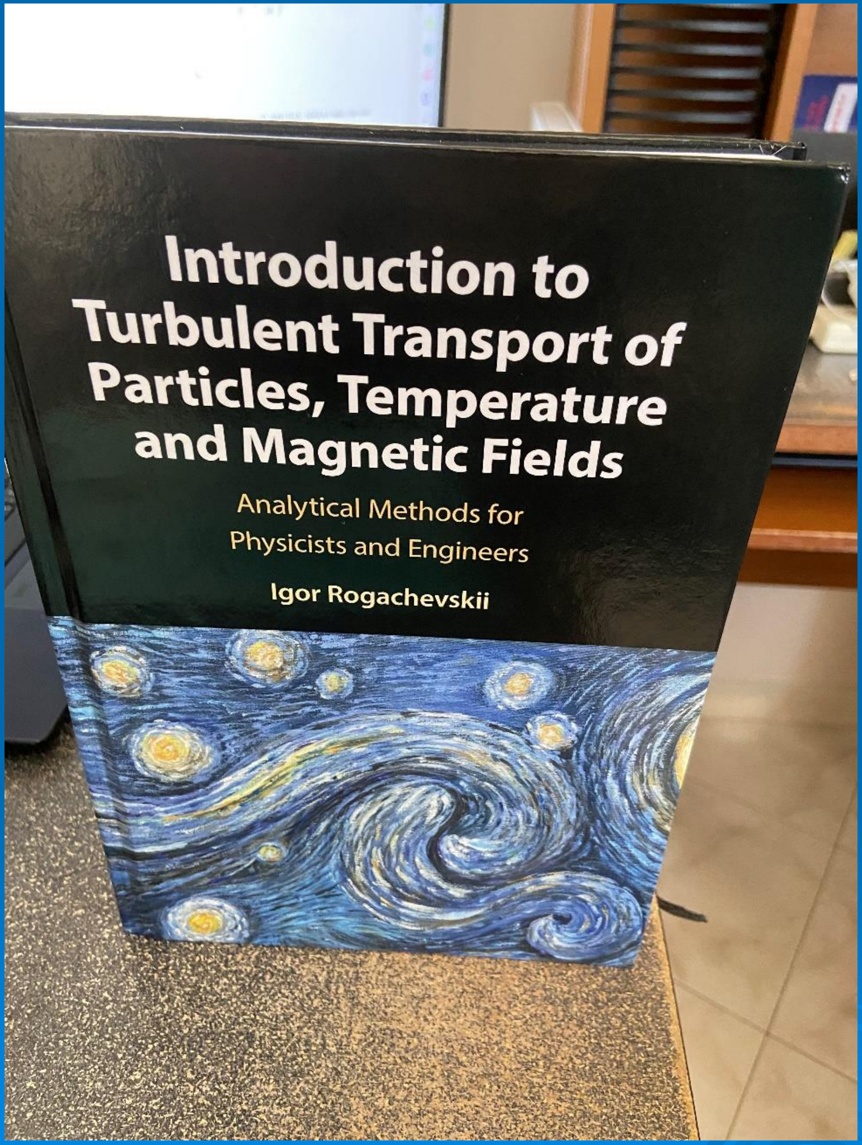
  Re >> 1 and Rm >> 1

  Pouquet, Frisch, Leorat (1976);
  Rogachevskii, Kleeorin (2000; 2001; 2003); Blackman, Field (2002);
  Rädler, Kleeorin, Rogachevskii (2003)

- **Renormalization Procedure** (renormalization of viscosity, diffusion, electromotive force and other turbulent transport coefficients) - there is no separation of scales

  Moffatt (1981; 1983); Kleeorin, Rogachevskii (1994)

I. Rogachevskii, "Introduction to Turbulent Transport of Particles, Temperature and Magnetic Fields" (Cambridge University Press, Cambridge, 2021).



- Various analytical methods are applied in this book:
- Mean-field approach;
- Multi-scale approach;
- Dimensional analysis;
- Quasi-linear approach;
- Tau approach;
- Path-integral approach;
- Analyses based on the budget equations.
- One-way and two-way couplings between turbulence and particles, or temperature, or magnetic fields are described.

- Table of Contents:

Algebraic Nonlinearities in Mean-Field Dynamo

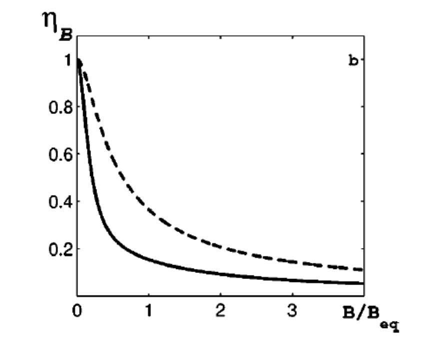
- **Induction equation for mean magnetic field:**

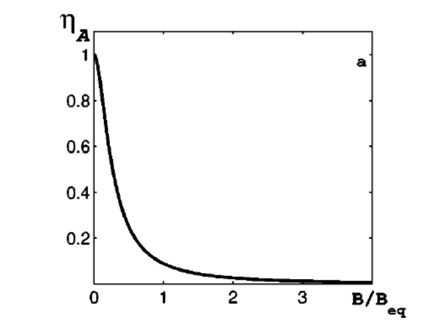
$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla}\times\left(\overline{\boldsymbol{U}}\times\overline{\boldsymbol{B}} + \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\right) + \eta\,\Delta\overline{\boldsymbol{B}},$$

- **Nonlinear turbulent electromotive force (algebraic nonlinearity):**

$$\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right) = \boldsymbol{\alpha}\left(\overline{\boldsymbol{B}}\right)\overline{\boldsymbol{B}} + \boldsymbol{V}^{\rm eff}\left(\overline{\boldsymbol{B}}\right)\times\overline{\boldsymbol{B}} - \eta_{_T}\left(\overline{\boldsymbol{B}}\right)\left(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}\right),$$

Turbulent diffusion





Iroshnikov (1970); Rüdiger (1974)

$$\alpha(\overline{\boldsymbol{B}}) = \frac{\alpha_{\rm K}}{1+\overline{\boldsymbol{B}}^2/\overline{B}_{\rm eq}^2},$$

$$\overline{B}_{\rm eq}^{\,2} = \mu_0\,\overline{\rho}\langle \boldsymbol{u}^2\rangle,$$

Nonlinear turbulent electromotive force:

Rüdiger and Kichatinov (1993; 1994) $\mathrm{Rm} \ll 1$

Field, Blackman, Chou (1999)

Rogachevskii, Kleeorin (2000; 2001; 2004; 2006) $\mathrm{Rm} \gg 1$

Nonlinear Effect: Magnetic Part of Alpha effect

- Induction equation for **mean magnetic field**:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{U} \times \mathbf{B} + \boldsymbol{\varepsilon} - \eta \nabla \times \mathbf{B} \right)$$

- **Electromotive force**:

$$\boldsymbol{\varepsilon} \equiv \langle \mathbf{u} \times \mathbf{b} \rangle = \boldsymbol{\alpha} \mathbf{B} - \boldsymbol{\eta}_{\mathbf{T}} \nabla \times \mathbf{B} + \ldots$$

$$\boldsymbol{\alpha} = -\frac{\tau}{3} \langle \mathbf{u} \cdot \mathbf{rot}\, \mathbf{u} \rangle + \frac{\tau}{12\pi\rho} \underbrace{\langle \mathbf{b} \cdot \mathbf{rot}\, \mathbf{b} \rangle}_{\sim \mathbf{a} \bullet \mathbf{b}}$$

A. Pouquet, U. Frisch, and J. Leorat, J. Fluid Mech. 77, 321 (1976)

Magnetic Helicity

Total magnetic helicity is conserved for very large magnetic Reynolds numbers

$$\chi^m_{\text{total}}(\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + \langle \mathbf{a}\cdot\mathbf{b}\rangle \to const$$

Magnetic part of alpha effect:

$$\alpha^m = \chi^c(\mathbf{B})\,\Phi_m(B)$$

The dynamic nonlinearity:

$$\chi^c(\mathbf{B}) \sim \frac{2}{9\eta_T\rho}\langle \mathbf{a}\cdot\mathbf{b}\rangle$$

The evolutionary equation: $\chi^{(m)}(\mathbf{B}) = \langle \mathbf{a}\cdot\mathbf{b}\rangle$

Dynamic Nonlinearity

$$\alpha^{(\rm M)} = \frac{\tau_c}{3\mu_0\,\overline{\rho}}\,\langle \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle,$$

$$\frac{\partial\alpha^{(\rm M)}}{\partial t} + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{F}}^{(\rm m)} = -\frac{2}{9\mu_0\,\eta_{_T}\,\overline{\rho}}\,\boldsymbol{\mathcal{E}}\cdot\overline{\boldsymbol{B}} - \frac{\alpha^{(\rm M)}}{\tau_c\,{\rm Rm}},$$

$$\langle \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle = \ell_0^{-2}\,\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle + {\rm O}(\ell_0^2/L_B^2),$$

$$\boldsymbol{\mathcal{E}}\cdot\overline{\boldsymbol{B}} = \alpha\,\overline{\boldsymbol{B}}^2 - \eta_{_T}\,\overline{\boldsymbol{B}}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}),$$

$$\alpha\left(\overline{\boldsymbol{B}}\right) = -\frac{\tau_c}{3}\,\langle\boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle + \frac{\tau_c}{3\mu_0\,\overline{\rho}}\,\langle\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle,$$

$$\eta_{_T} = \frac{\tau_c\,\langle\boldsymbol{u}^2\rangle}{3},$$

Kleeorin and Ruzmaikin (1982) – for isotropic turbulence

Kleeorin and Rogachevskii (1999) – for anisotropic turbulence

In the absence of the magnetic helicity fluxes

catastrophic quenching

$$\alpha(\overline{\boldsymbol{B}}) = \frac{\alpha_{\rm K}}{1 + {\rm Rm}\,\overline{\boldsymbol{B}}^2/\overline{B}_{\rm eq}^2},$$

$$\overline{B}_{\rm eq}^2 = \mu_0\,\overline{\rho}\langle\boldsymbol{u}^2\rangle,$$

Vainshtein and Cattaneo (1992);
Gruzinov and Diamond (1994);
Cattaneo, Hughes (1996)

Open Boundary Conditions

Blackman and Field (2000);

Effect of Magnetic Helicity Fluxes

Kleeorin, Moss, Rogachevskii and Sokoloff (2000);

$${\rm Rm} \gg 1$$

$$\alpha(\overline{\boldsymbol{B}}) = -\frac{\tau_c\,\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{F}}^{(\rm m)}}{\overline{\boldsymbol{B}}^2/\overline{B}_{\rm eq}^2},$$

Different Forms of Magnetic Helicity Fluxes:

Kleeorin and Rogachevskii (1999);
Kleeorin, Moss, Rogachevskii and Sokoloff (2000, 2002, 2003),
Vishniac and Cho (2001); Brandenburg and Subramanian (2005)

Kleeorin and Rogachevskii (2022), Gopalakrishnan and Subramanian (2023)

Algebraic Nonlinearities in Mean-Field Dynamo

$$\overline{\boldsymbol{B}} = \overline{B}_y(x,z)\,\boldsymbol{e}_y + \nabla \times [\overline{A}(x,z)\,\boldsymbol{e}_y],$$

$$\alpha(\overline{\boldsymbol{B}}) = \alpha_{\rm K}(\overline{\boldsymbol{B}}) + \alpha_{\rm M}(\overline{\boldsymbol{B}}),$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix} = \hat{N}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix},$$

$$\hat{N} = \begin{pmatrix}\eta_T^{(A)}\left(\overline{B}\right)\Delta & \alpha\left(\overline{\boldsymbol{B}}\right)\\ R_\alpha R_\omega\,\hat{\Omega} & \nabla_j\,\eta_T^{(B)}\left(\overline{B}\right)\nabla_j\end{pmatrix}$$

$$D_{\rm L} = R_\alpha R_\omega,$$

$$R_\alpha = \alpha_* L/\eta_T^{(0)},$$

$$R_\omega = (\delta\Omega)\,L^2/\eta_T^{(0)}$$

$$D_{\rm N}\left(\overline{B}\right) = \frac{\alpha\left(\overline{B}\right)\,\delta\Omega\,L^3}{\eta_T^{(B)}\left(\overline{B}\right)\,\eta_T^{(A)}\left(\overline{B}\right)} \propto \overline{B}/\overline{B}_{\rm eq}$$

$$\overline{B} \ll \overline{B}_{\rm eq}/4$$

$$\overline{B} \gg \overline{B}_{\rm eq}/4$$

$$\beta = \sqrt{8}\,\overline{B}/\overline{B}_{\rm eq},$$

$$\alpha^{(\rm K)}(\beta) = \alpha_{\rm K}^{(0)}(1-\epsilon)\left(1-\frac{12\beta^2}{5}\right),$$

$$\alpha^{(\rm M)}\left(\overline{\boldsymbol{B}}\right) = \frac{\tau_0}{3\mu_0\,\overline{\rho}}\,H_{\rm c}\left(\overline{\boldsymbol{B}}\right)\left(1-\frac{3\beta^2}{5}\right),$$

$$\eta_T^{(A)}(\beta) = \eta_T^{(0)}\left(1-\frac{12}{5}\beta^2\right),$$

$$\eta_T^{(B)}(\beta) = \eta_T^{(0)}\left(1-\frac{4}{5}(5-4\epsilon)\,\beta^2\right),$$

$$\alpha^{(\rm K)}(\beta) = \frac{\alpha_{\rm K}^{(0)}}{\beta^2}(1-\epsilon),$$

$$\alpha^{(\rm M)}\left(\overline{\boldsymbol{B}}\right) = \frac{\tau_0}{\mu_0\,\overline{\rho}}\,\frac{H_{\rm c}\left(\overline{\boldsymbol{B}}\right)}{\beta^2},$$

$$\eta_T^{(A)}(\beta) = \frac{\eta_T^{(0)}}{\beta^2}, \qquad \eta_T^{(B)}(\beta) = \frac{2\eta_T^{(0)}}{3\beta}(1+\epsilon).$$

Nonlinear Electromotive Force

$$\frac{\partial f_{ij}(\boldsymbol{k})}{\partial t} = i(\boldsymbol{k}\cdot\overline{\boldsymbol{B}})\Phi_{ij} + I^f_{ij} + F_{ij} + f^N_{ij},$$

$$\frac{\partial h_{ij}(\boldsymbol{k})}{\partial t} = -i(\boldsymbol{k}\cdot\overline{\boldsymbol{B}})\Phi_{ij} + I^h_{ij} + h^N_{ij},$$

$$\frac{\partial g_{ij}(\boldsymbol{k})}{\partial t} = i(\boldsymbol{k}\cdot\overline{\boldsymbol{B}})[f_{ij}(\boldsymbol{k}) - h_{ij}(\boldsymbol{k}) - h^{(H)}_{ij}] + I^g_{ij} + g^N_{ij}$$

$$\hat{L}(a;c) = \int \langle a(\mathbf{k}+\mathbf{K}/2)c(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K},$$

$$f_{ij}(\mathbf{k},\mathbf{R}) = \hat{L}(u_i;u_j), \quad h_{ij}(\mathbf{k},\mathbf{R}) = \hat{L}(b_i;b_j),$$

$$g_{ij}(\mathbf{k},\mathbf{R}) = \hat{L}(b_i;u_j).$$

For turbulent convection:

$$F_i(\boldsymbol{k}) = \hat{L}(u_i, s), \quad \Theta(\boldsymbol{k}) = \hat{L}(s,s), \quad W_i(\boldsymbol{k}) = \hat{L}(b_i, s),$$

Multi-scale-approach (Roberts and Soward, 1975):

$$\mathbf{R} = (\mathbf{x}+\mathbf{y})/2, \mathbf{r} = \mathbf{x}-\mathbf{y}, \quad \mathbf{K} = \mathbf{k}_1+\mathbf{k}_2, \mathbf{k} = (\mathbf{k}_1-\mathbf{k}_2)/2,$$

$$\langle u_i(\mathbf{x})u_j(\mathbf{y})\rangle = \int \langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle \exp\{i(\mathbf{k}_1\cdot\mathbf{x} + \mathbf{k}_2\cdot\mathbf{y})\}d\mathbf{k}_1 d\mathbf{k}_2$$
$$= \int f_{ij}(\mathbf{k},\mathbf{K})\exp(i\mathbf{k}\cdot\mathbf{r} + i\mathbf{K}\cdot\mathbf{R})d\mathbf{k}\, d\mathbf{K},$$
$$= \int f_{ij}(\mathbf{k},\mathbf{R})\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k},$$

$$I^f_{ij} = \frac{1}{2}(\overline{\boldsymbol{B}}\cdot\nabla)\Phi^{(P)}_{ij} + [g_{qj}(\boldsymbol{k})(2P_{in}(k) - \delta_{in}) + g_{qi}(-\boldsymbol{k})(2P_{jn}(k) - \delta_{jn})]\overline{B}_{n,q} - \overline{B}_{n,q}k_n\Phi^{(P)}_{ijq},$$

$$I^h_{ij} = \frac{1}{2}(\overline{\boldsymbol{B}}\cdot\nabla)\Phi^{(P)}_{ij} - [g_{iq}(\boldsymbol{k})\delta_{jn} + g_{jq}(-\boldsymbol{k})\delta_{in}]\overline{B}_{n,q} - \overline{B}_{n,q}k_n\Phi^{(P)}_{ijq}$$

$$I^g_{ij} = \frac{1}{2}(\overline{\boldsymbol{B}}\cdot\nabla)(f_{ij} + h_{ij}) + h_{iq}(2P_{jn}(k) - \delta_{jn})\overline{B}_{n,q} - f_{nj}\overline{B}_{i,n} - \overline{B}_{n,q}k_n(f_{ijq} + h_{ijq})$$

$$\Phi^{(P)}_{ij}(\boldsymbol{k}) = g_{ij}(\boldsymbol{k}) + g_{ji}(-\boldsymbol{k}),$$

$$f_{ijq} = (1/2)\partial f_{ij}/\partial k_q$$

Tau-approach (spectral tau-approximation, minimal tau-approximation) – third-order or high-order closure

$$\hat{\mathcal{M}}F^{(III)}(\boldsymbol{k}) - \hat{\mathcal{M}}F^{(III,0)}(\boldsymbol{k}) = -\frac{1}{\tau_r(k)}\left[F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})\right],$$

Background Turbulence

Velocity Fluctuations in Density Stratified Turbulence with Non-Uniform Kinetic Helicity

$$f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{R}) = \langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k})\rangle^{(0)}$$

$$f_{ij}^{(0)} = \frac{E_u(k)}{8\pi k^2}\left\{\left[(\delta_{ij} - k_{ij}) + \frac{\mathrm{i}}{k^2}\left(\tilde{\lambda}_i k_j - \tilde{\lambda}_j k_i\right)\right]\langle \boldsymbol{u}^2\rangle - \frac{1}{k^2}\left[\mathrm{i}\varepsilon_{ijp}\, k_p + (\varepsilon_{jpm}\, k_{ip} + \varepsilon_{ipm}\, k_{jp})\tilde{\lambda}_m\right] H_{\mathrm{u}}\right\}$$

$$\tilde{\lambda}_m = \lambda_m - \nabla_m/2$$

$$\boldsymbol{\lambda} = -\boldsymbol{\nabla} \ln \overline{\rho}$$

$$H_{\mathrm{u}} = \langle \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle$$

$$k_{ij} = k_i k_j / k^2$$

Magnetic Fluctuations with Non-Uniform Current and Magnetic Helicities

$$h_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{R}) = \langle b_i(\boldsymbol{k})\, b_j(-\boldsymbol{k})\rangle^{(0)}$$

$$h_{ij}^{(0)} = \frac{1}{8\pi k^2}\left\{E_b(k)\,(\delta_{ij} - k_{ij})\,\langle \boldsymbol{b}^2\rangle - \frac{1}{k^2}\left[\mathrm{i}\varepsilon_{ijp}\, k_p - \frac{1}{2}(\varepsilon_{jpm}\, k_{ip} + \varepsilon_{ipm}\, k_{jp})\nabla_m\right] H_{\mathrm{c}}\,\delta(k - k_0)\right\}$$

$$H_{\mathrm{c}} = \langle \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle$$

Possible Solution of the Problem

- The dissipation of the generated strong large-scale magnetic field increases:

- (i) the turbulent kinetic energy of the background turbulence
- (ii) the turbulent magnetic diffusion coefficient.

- This non-linear effect is taken into account by means of the budget equations for the turbulent kinetic energy and the turbulent total energy for the background turbulence.

- This additional non-linear effect decreases the non-linear dynamo number with increase of a large-scale magnetic field and causes a saturation of the dynamo growth of large-scale magnetic field.

Budget Equations

The density of turbulent kinetic energy:

$$E_K = \overline{\rho}\,\langle \boldsymbol{u}^2 \rangle / 2$$

$$\frac{\partial E_K}{\partial t} + \operatorname{div} \boldsymbol{\Phi}_K = \Pi_K - \varepsilon_K ,$$

The density of turbulent magnetic energy:

$$E_M = \langle \boldsymbol{b}^2 \rangle / 2\mu_0$$

$$\frac{\partial E_M}{\partial t} + \operatorname{div} \boldsymbol{\Phi}_M = \Pi_M - \varepsilon_M ,$$

Production of TKE:

$$\Pi_K = -\frac{1}{\mu_0}\Big[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}} + \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\Big] + \overline{\rho}\Big[g\,F_z - \langle u_i u_j\rangle\,\nabla_j \overline{U}_i + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Big]$$

Production of TME:

$$\Pi_M = \frac{1}{\mu_0}\Big[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}} + \langle b_i\, b_j\rangle\,\nabla_j \overline{U}_i - \langle \boldsymbol{b}^2\rangle\,(\boldsymbol{\nabla}\cdot\overline{\boldsymbol{U}})\Big]$$

The density of total turbulent energy:

$$E_T = E_K + E_M$$

$$\boldsymbol{\mathcal{E}}(\overline{\boldsymbol{B}}) = \langle \boldsymbol{u}\times\boldsymbol{b}\rangle$$

$$\frac{\partial E_T}{\partial t} + \operatorname{div} \boldsymbol{\Phi}_T = \Pi_T - \varepsilon_T ,$$

$$\mathcal{E}_i = \alpha\,\overline{B}_i - \eta_{ij}^{(T)}(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_j$$

$$\Pi_T = \Big[\Big(\langle b_i\, b_j\rangle - \mu_0\,\overline{\rho}\,\langle u_i u_j\rangle\Big)\nabla_j \overline{U}_i - \langle \boldsymbol{b}^2\rangle\,(\boldsymbol{\nabla}\cdot\overline{\boldsymbol{U}}) - \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\Big]\mu_0^{-1} + \overline{\rho}\Big(g\,F_z + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Big).$$

$$E_K + E_M \sim \tau\,\Pi_T$$

The Dissipation of the Generated Large-Scale Magnetic Field

Production of TKE:

$$\Pi_{\rm K} = -\frac{1}{\mu_0}\Big[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}} + \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\Big] + \overline{\rho}\Big[g\,F_z - \langle u_i u_j\rangle\,\nabla_j\overline{U}_i + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Big]$$

Turbulent electromotive force (EMF):

$$\boldsymbol{\mathcal{E}}(\overline{\boldsymbol{B}}) = \langle \boldsymbol{u}\times\boldsymbol{b}\rangle$$

$$E_{\rm K} = -\frac{\tau}{\mu_0}\,\boldsymbol{\mathcal{E}}(\overline{\boldsymbol{B}})\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}).$$

$$\mathcal{E}_i = \alpha\,\overline{B}_i - \eta_{ij}^{(\rm T)}(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_j$$

$$\eta_{ij}^{(\rm T)}(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_j(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_i = \eta_{_T}^{(A)}\,(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})^2_\varphi + \eta_{_T}^{(B)}\,(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})^2_{\rm p}$$

$$|(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_\varphi| \sim |\overline{B}_{\rm p}|/L_B$$

$$|(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_{\rm p}| \sim |\overline{B}_\varphi|/L_B$$

$$\eta_{_T}^{(A)}(\beta) \sim \eta_{_T}^{(0)}/\beta^2, \text{ while } \eta_{_T}^{(B)}(\beta) \sim \eta_{_T}^{(0)}/\beta,$$

$$\beta = \sqrt{8}\,\overline{B}/\overline{B}_{\rm eq},$$

$$-\boldsymbol{\mathcal{E}}(\overline{\boldsymbol{B}})\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) \sim \frac{\eta_{_T}^{(B)}}{L_B^2}\,\overline{B}_\varphi^2 \sim \frac{\eta_{_T}^{(0)}}{4L_B^2}\,\overline{B}_\varphi\,\overline{B}_{\rm eq},$$

$$E_{\rm K}(\overline{B}) \sim \frac{E_{\rm K}^{(0)}}{6}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right).$$

Turbulent magnetic Diffusion and Nonlinear Dynamo Number

$$E_{\rm K}\left(\overline{B}\right) \sim \frac{E_{\rm K}^{(0)}}{6}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right).$$

$$\eta_T^{(B)}\left(\overline{B}\right) = \eta_T^{(0)}\,\phi_\eta^{(B)}\,E_{\rm K}\left(\overline{B}\right)/E_{\rm K}^{(0)}$$

$$\frac{\eta_T^{(B)}\left(\overline{B}\right)}{\eta_T^{(0)}} \approx \frac{1}{24}\left(\frac{\ell_0}{L_B}\right)^2 = const,$$

$$\frac{\eta_T^{(A)}\left(\overline{B}\right)}{\eta_T^{(B)}\left(\overline{B}\right)} \approx \frac{1}{2}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}.$$

$$\frac{\alpha\left(\overline{B}\right)}{\alpha_{\rm K}^{(0)}} \propto \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-2}$$

$$D_{\rm N}\left(\overline{B}\right) = \frac{\alpha\left(\overline{B}\right)\,\delta\Omega\,L^3}{\eta_T^{(B)}\left(\overline{B}\right)\,\eta_T^{(A)}\left(\overline{B}\right)},$$

$$\frac{D_{\rm N}\left(\overline{B}\right)}{D_{\rm L}} \approx 2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}\left(\frac{\eta_T^{(B)}}{\eta_T^{(0)}}\right)^{-2} \propto \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}.$$

$$D_{\rm L} = \alpha_{\rm K}\,\delta\Omega\,L^3/\eta_T^2$$

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Budget equations and astrophysical non-linear mean-field dynamos

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ABSTRACT

Solar, stellar and galactic large-scale magnetic fields are originated due to a combined action of non-uniform (differential) rotation and helical motions of plasma via mean-field dynamos. Usually, non-linear mean-field dynamo theories take into account algebraic and dynamic quenching of alpha effect and algebraic quenching of turbulent magnetic diffusivity. However, the theories of the algebraic quenching do not take into account the effect of modification of the source of turbulence by the growing large-scale magnetic field. This phenomenon is due to the dissipation of the strong large-scale magnetic field resulting in an increase of the total turbulent energy. This effect has been studied using the budget equation for the total turbulent energy (which takes into account the feedback of the generated large-scale magnetic field on the background turbulence) for (i) a forced turbulence, (ii) a shear-produced turbulence, and (iii) a convective turbulence. As the result of this effect, a non-linear dynamo number decreases with increase of the large-scale magnetic field, so that that the mean-field $\alpha\Omega$, α^2, and $\alpha^2\Omega$ dynamo instabilities are always saturated by the strong large-scale magnetic field.

Key words: dynamo – MHD – turbulence – Sun: interior – activity – galaxies: magnetic fields.

Theory of Differential Rotation

Rogachevskii and Kleeorin (2018), J. Plasma Phys., 84, 735840201
Kleeorin and Rogachevskii (2006), Phys. Rev. E , 73, 046303

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -(\boldsymbol{U}\cdot\nabla)\boldsymbol{u}' - (\boldsymbol{u}'\cdot\nabla)\boldsymbol{U} - \nabla\left(\frac{p'}{\rho_0}\right) - \boldsymbol{g}s' + 2\boldsymbol{u}'\times\boldsymbol{\Omega} + \boldsymbol{U}^N,$$

$$\frac{\partial s'}{\partial t} = -\frac{\Omega_b^2}{g}(\boldsymbol{u}'\cdot\boldsymbol{e}) - (\boldsymbol{U}\cdot\nabla)s' + S^N.$$

$$\boldsymbol{v} = \sqrt{\rho_0}\boldsymbol{u}'$$

$$U^N = \langle(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}'\rangle - (\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}' + \boldsymbol{f}_\nu(\boldsymbol{u}'),$$

$$S^N = \langle(\boldsymbol{u}'\cdot\nabla)s'\rangle - (\boldsymbol{u}'\cdot\nabla)s' - (1/T_0)\nabla\cdot\boldsymbol{F}_\kappa(\boldsymbol{u}', s'),$$

$$s = \sqrt{\rho_0}s'$$

$$\frac{\partial f_{ij}(\boldsymbol{k},\boldsymbol{K})}{\partial t} = (I^U_{ijmn} + L^\Omega_{ijmn})f_{mn} + M^F_{ij} + \hat{\mathcal{N}}\tilde{f}_{ij},$$

$$\frac{\partial F_i(\boldsymbol{k},\boldsymbol{K})}{\partial t} = (J^U_{im} + D^\Omega_{im})F_m + g e_m P_{im}(\boldsymbol{k}_1)\Theta + \hat{\mathcal{N}}\tilde{F}_i,$$

$$\frac{\partial \Theta(\boldsymbol{k},\boldsymbol{K})}{\partial t} = -\mathrm{div}(\boldsymbol{U}\Theta) + \hat{\mathcal{N}}\Theta,$$

$$f_{ij}(\boldsymbol{k},\boldsymbol{K}) = \langle v_i(t,\boldsymbol{k}_1)v_j(t,\boldsymbol{k}_2)\rangle,$$

$$F_i(\boldsymbol{k},\boldsymbol{K}) = \langle s(t,\boldsymbol{k}_1)v_i(t,\boldsymbol{k}_2)\rangle$$

$$\Theta(\boldsymbol{k},\boldsymbol{K}) = \langle s(t,\boldsymbol{k}_1)s(t,\boldsymbol{k}_2)\rangle$$

The spectral τ approximation

$$\hat{\mathcal{N}}f_{ij}(\boldsymbol{k}) - \hat{\mathcal{N}}f^{(0)}_{ij}(\boldsymbol{k}) = -\frac{f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})}{\tau_r(k)},$$

Background Turbulence

$$f_{ij}^{(0)} \equiv \langle v_i(\boldsymbol{k}_1) v_j(\boldsymbol{k}_2)\rangle^{(0)} = \frac{E(k)[1+2k\varepsilon_u\delta(k_z)]}{8\pi k^2(k^2+\tilde{\lambda}^2)(1+\varepsilon_u)} \left[\delta_{ij}(k^2+\tilde{\lambda}^2) - k_i k_j - \tilde{\lambda}_i\tilde{\lambda}_j + \mathrm{i}(\tilde{\lambda}_i k_j - \tilde{\lambda}_j k_i)\right] \rho_0 \langle \boldsymbol{u}'^2\rangle^{(0)},$$

$$F_i^{(0)} \equiv \langle v_i(\boldsymbol{k}_1) s(\boldsymbol{k}_2)\rangle^{(0)} = \frac{3E(k)}{8\pi k^4}[k^2 e_j P_{ij}(\boldsymbol{k}) - \mathrm{i}\tilde{\lambda} k_j P_{ij}(\boldsymbol{e})]\rho_0 \langle s' u'_z\rangle^{(0)},$$

$$\Theta^{(0)} \equiv \langle s(\boldsymbol{k}_1)\, s(\boldsymbol{k}_2)\rangle = \Theta_*\, E(k)/4\pi k^2$$

$$\Theta_* = \rho_0 \langle (s')^2 \rangle$$

$$E(k) = -\mathrm{d}\bar{\tau}(k)/\mathrm{d}k,\ \bar{\tau}(k) = (k/k_0)^{1-q}$$

$$\tilde{\lambda} = (\lambda - \nabla)/2,\ \lambda = -(\nabla \rho_0)/\rho_0.$$

$$\tau(k) = 2\tau_\Omega \bar{\tau}(k)$$

$$\tau_\Omega = \frac{\tau_0}{[1+(C_\Omega^{-1}\Omega\tau_0)^2]^{1/2}}$$

Degree of anisotropy of velocity field:

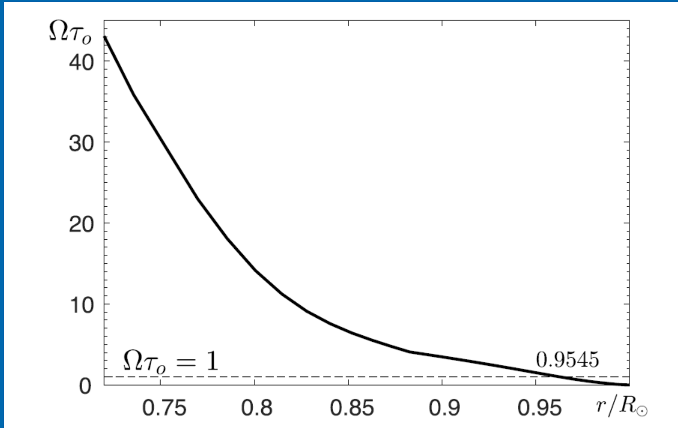
$$\varepsilon_u = \frac{E_K^{(2d)}}{E_K^{(3d)}}$$

Fast rotating turbulence: $\Omega\tau_0 \gg 1$

$$\varepsilon_u \gg 1$$

Reynolds Stress

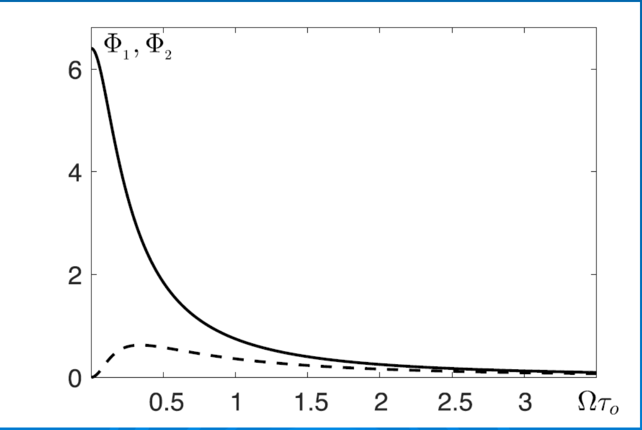
$$\sigma_{r\varphi} \equiv -e_j^{\varphi} e_i^{r} \langle v_i v_j \rangle = \sigma_{r\varphi}^{\nu_T} + \sigma_{r\varphi}^{F} + \sigma_{r\varphi}^{u},$$
$$\sigma_{\theta\varphi} \equiv -e_j^{\varphi} e_i^{\theta} \langle v_i v_j \rangle = \sigma_{\theta\varphi}^{\nu_T} + \sigma_{\theta\varphi}^{F} + \sigma_{\theta\varphi}^{u}.$$



$$\sigma_{r\varphi}^{\nu_T} = \rho_0 \nu_T r \frac{\partial}{\partial r}\left(\frac{U_\varphi}{r}\right),$$
$$\sigma_{\theta\varphi}^{\nu_T} = \rho_0 \nu_T \frac{\sin\theta}{r} \frac{\partial}{\partial\theta}\left(\frac{U_\varphi}{\sin\theta}\right)$$

$$\sigma_{r\varphi}^{u} = -\frac{\lambda^2 \ell_0^2}{20} \rho_0 \langle \boldsymbol{u}'^2 \rangle^{(0)} \tau_\Omega \Omega \sin\theta (1 + \cos^2\theta),$$
$$\sigma_{\theta\varphi}^{u} = \frac{\lambda^2 \ell_0^2}{20} \rho_0 \langle \boldsymbol{u}'^2 \rangle^{(0)} \tau_\Omega \Omega \sin^2\theta \cos\theta.$$

$$\sigma_{r\varphi}^{F} = \tfrac{1}{6} \rho_0 \tau_\Omega^2 g \langle s' u_z' \rangle^{(0)} \Omega \sin\theta [\Phi_1(\omega) + \cos^2\theta \, \Phi_2(\omega)],$$
$$\sigma_{\theta\varphi}^{F} = \tfrac{1}{3} \rho_0 \tau_\Omega^2 g \langle s' u_z' \rangle^{(0)} \Omega \sin^2\theta \cos\theta \, \Phi_2(\omega),$$



$$\rho_0 \frac{\partial U_\varphi}{\partial t} = \frac{1}{r^3} \frac{\partial}{\partial r}(r^3 \sigma_{r\varphi}) + \frac{1}{r \sin^2\theta} \frac{\partial}{\partial\theta}(\sin^2\theta \sigma_{\theta\varphi}) + 2\rho_0 (\boldsymbol{U} \times \boldsymbol{\Omega})_\varphi,$$

Theory of Differential Rotation

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Kleeorin and Rogachevskii (2006), Phys. Rev. E , 73, 046303

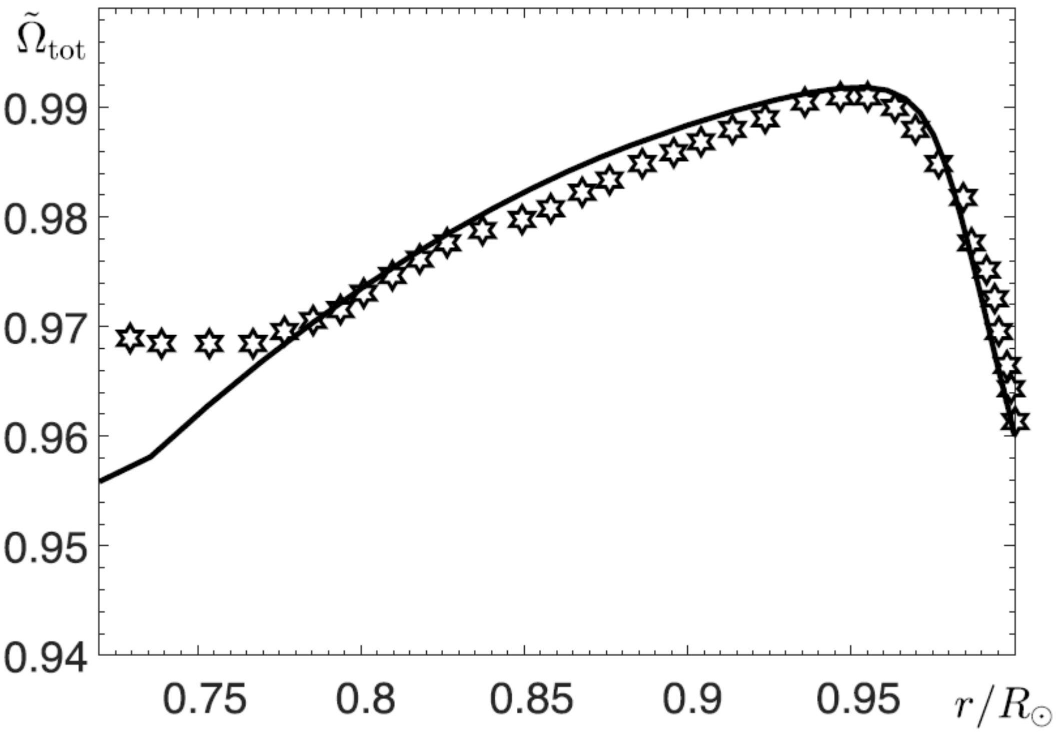

FIGURE 2. The total angular velocity $\tilde{\Omega}_{tot} = \tilde{\Omega}_0 + 1$ that includes the uniform rotation Ω versus the radius $r/R_\odot$ (solid). This theoretical profile is compared with the radial profile of the solar angular velocity obtained from helioseismology observational data (stars) at the latitude $\phi = 30°$ and normalized by the solar rotation frequency $\Omega_\odot(\phi = 0)$ at the equator, where $R_\odot$ is the solar radius.

Previous Theories of Differential Rotation

Kippenhahn, R., Astrophys. J. 137, 664 (1963).

Rüdiger, G., Geophys. Astrophys. Fluid Dyn. 16 (1), 239–261 (1980).

Durney, B. R., Astrophys. J. 297, 787-798 (1985); Astrophys. J. 407, 367-379 (1993).

Kichatinov, L. L. and Rüdiger, G., Astron. Astrophys. 276, 96 (1993);
Astron. Nachr. 326 (6), 379-385 (2005).

An origin of the solar differential rotation is related to an anisotropic eddy viscosity.

The quasi-linear approach has been applied.

Mixing length theory relation is used: $\langle \mathbf{u}'^2 \rangle \propto g \tau_0 \langle u'_z s' \rangle,$

J. Plasma Phys. (2018), *vol.* 84, 735840201
doi:10.1017/S0022377818000272

Mean-field theory of differential rotation in density stratified turbulent convection

I. Rogachevskii[1,2,†] **and N. Kleeorin**[1,2]

A mean-field theory of differential rotation in a density stratified turbulent convection has been developed. This theory is based on the combined effects of the turbulent heat flux and anisotropy of turbulent convection on the Reynolds stress. A coupled system of dynamical budget equations consisting in the equations for the Reynolds stress, the entropy fluctuations and the turbulent heat flux has been solved. To close the system of these equations, the spectral τ approach, which is valid for large Reynolds and Péclet numbers, has been applied. The adopted model of the background turbulent convection takes into account an increase of the turbulence anisotropy and a decrease of the turbulent correlation time with the rotation rate. This theory yields the radial profile of the differential rotation which is in agreement with that for the solar differential rotation.

PHYSICAL REVIEW E **73**, 046303 (2006)

Effect of heat flux on differential rotation in turbulent convection

Nathan Kleeorin* and Igor Rogachevskii†

Generation of Large-Scale Vorticity in Fast Rotating Turbulent Convection

Rogachevskii I. and Kleeorin N., Phys. Rev. E 100, 063101 (2019)

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -(\boldsymbol{U}\cdot\nabla)\boldsymbol{u}' - (\boldsymbol{u}'\cdot\nabla)\boldsymbol{U} - \nabla\left(\frac{p'}{\rho_0}\right) - \boldsymbol{g}s' + 2\boldsymbol{u}'\times\boldsymbol{\Omega} + \boldsymbol{U}^N,$$

$$\frac{\partial s'}{\partial t} = -\frac{\Omega_b^2}{g}(\boldsymbol{u}'\cdot\boldsymbol{e}) - (\boldsymbol{U}\cdot\nabla)s' + S^N.$$

$$\boldsymbol{v} = \sqrt{\rho_0}\,\boldsymbol{u}'$$

$$U^N = \langle(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}'\rangle - (\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}' + \boldsymbol{f}_\nu(\boldsymbol{u}'),$$

$$S^N = \langle(\boldsymbol{u}'\cdot\nabla)s'\rangle - (\boldsymbol{u}'\cdot\nabla)s' - (1/T_0)\nabla\cdot\boldsymbol{F}_\kappa(\boldsymbol{u}', s'),$$

$$s = \sqrt{\rho_0}\,s'$$

$$\frac{\partial f_{ij}(\boldsymbol{k},\boldsymbol{K})}{\partial t} = (I^U_{ijmn} + L^\Omega_{ijmn})f_{mn} + M^F_{ij} + \hat{\mathcal{N}}\tilde{f}_{ij},$$

$$\frac{\partial F_i(\boldsymbol{k},\boldsymbol{K})}{\partial t} = (J^U_{im} + D^\Omega_{im})F_m + g e_m P_{im}(\boldsymbol{k}_1)\Theta + \hat{\mathcal{N}}\tilde{F}_i,$$

$$\frac{\partial \Theta(\boldsymbol{k},\boldsymbol{K})}{\partial t} = -\mathrm{div}(\boldsymbol{U}\Theta) + \hat{\mathcal{N}}\Theta,$$

$$f_{ij}(\boldsymbol{k},\boldsymbol{K}) = \langle v_i(t,\boldsymbol{k}_1)v_j(t,\boldsymbol{k}_2)\rangle,$$

$$F_i(\boldsymbol{k},\boldsymbol{K}) = \langle s(t,\boldsymbol{k}_1)v_i(t,\boldsymbol{k}_2)\rangle$$

$$\Theta(\boldsymbol{k},\boldsymbol{K}) = \langle s(t,\boldsymbol{k}_1)s(t,\boldsymbol{k}_2)\rangle$$

The spectral τ approximation

$$\hat{\mathcal{N}}f_{ij}(\boldsymbol{k}) - \hat{\mathcal{N}}f^{(0)}_{ij}(\boldsymbol{k}) = -\frac{f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})}{\tau_r(k)},$$

Background Turbulence

$$f_{ij}^{(0)} \equiv \langle v_i(\boldsymbol{k}_1)\, v_j(\boldsymbol{k}_2)\rangle = \frac{E(k)\,[1+2k\,\varepsilon_u\,\delta(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\Omega}})]}{8\pi\,k^2\,(k^2+\tilde{\lambda}^2)\,(1+\varepsilon_u)}\Big[\delta_{ij}\,(k^2+\tilde{\lambda}^2)-k_i\,k_j-\tilde{\lambda}_i\,\tilde{\lambda}_j+i\,(\tilde{\lambda}_i\,k_j-\tilde{\lambda}_j\,k_i)\Big]\langle \boldsymbol{v}^2\rangle,$$

$$F_i^{(0)} \equiv \langle v_i(\boldsymbol{k}_1)\, s(\boldsymbol{k}_2)\rangle = \frac{3\,E(k)\,[1+k\,\varepsilon_{_F}\,\delta(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\Omega}})]}{8\pi\,k^2\,(k^2+\tilde{\lambda}^2)}\Big[k^2\,e_j\,P_{ij}(\boldsymbol{k})+i\tilde{\lambda}\,k_j\,P_{ij}(\boldsymbol{e})\Big]F_*,$$

$$\Theta^{(0)} \equiv \langle s(\boldsymbol{k}_1)\, s(\boldsymbol{k}_2)\rangle = \Theta_*\,E(k)/4\pi k^2$$

$$F_* = \rho_0\,\langle u'_z\, s'\rangle$$

$$\Theta_* = \rho_0\,\langle (s')^2\rangle$$

Inhomogeneous and density stratified turbulence:

$$\tilde{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}-\boldsymbol{\nabla})/2, \; \boldsymbol{\lambda} = -(\boldsymbol{\nabla}\rho_0)/\rho_0.$$

$$E(k) = -\mathrm{d}\bar{\tau}(k)/\mathrm{d}k, \; \bar{\tau}(k) = (k/k_0)^{1-q}$$

$$\ell_0 \ll H_\rho; L_x$$

$$\tau(k) = 2\tau_\Omega\,\bar{\tau}(k)$$

$$\tau_\Omega = \frac{\tau_0}{[1+(C_\Omega^{-1}\,\Omega\,\tau_0)^2]^{1/2}}$$

Fast rotating turbulence:

$$\Omega\tau_0 \gg 1$$

$$\varepsilon_u \gg 1 \text{ and } \varepsilon_{_F} \sim 1$$

Degree of anisotropy of velocity field:

$$\varepsilon_u = \frac{E_{\rm K}^{(2{\rm d})}}{E_{\rm K}^{(3{\rm d})}}$$

Generation of Large-Scale Vorticity in Fast Rotating Turbulent Convection

Rogachevskii I. and Kleeorin N., Phys. Rev. E 100, 063101 (2019)

$$\frac{\partial \overline{U}_i}{\partial t} + (\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla})\overline{U}_i = -\nabla_i \left(\frac{\overline{P}}{\rho_0}\right) - g_i \overline{S} + 2(\overline{\boldsymbol{U}} \times \boldsymbol{\Omega})_i - \frac{1}{\rho_0}\nabla_j \left(\rho_0 \langle u_i' \, u_j' \rangle\right),$$

$$\frac{\partial \overline{S}}{\partial t} + (\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla})\overline{S} = -(\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla})S_0 - \frac{1}{\rho_0}\boldsymbol{\nabla} \cdot (\rho_0 \langle \boldsymbol{u}' \, s' \rangle),$$

The Reynolds Stress:

$$f_{ij}^{(u,\Omega)} = -\frac{A_u}{2} \, \rho_0 \, \nu_{_T} \, \Omega \tau_0 \, \frac{\ell_0^2}{H_\rho^2} \Big\{ 4(\overline{W}_i e_j + \overline{W}_j e_i) + 4\left[(\boldsymbol{e}\times\boldsymbol{\nabla})_i \, e_j + (\boldsymbol{e}\times\boldsymbol{\nabla})_j \, e_i\right] \overline{U}_z + 3(q+1)\left[(\boldsymbol{e}\times\boldsymbol{\nabla})_i \overline{U}_j^\perp + (\boldsymbol{e}\times\boldsymbol{\nabla})_j \overline{U}_i^\perp\right] + (3q+7)\left[\nabla_i^\perp \left(\boldsymbol{e}\times\overline{\boldsymbol{U}}\right)_j + \nabla_j^\perp \left(\boldsymbol{e}\times\overline{\boldsymbol{U}}\right)_i\right] \Big\}.$$

$$f_{ij}^{(F,\Omega)} = -A_F \, \rho_0 \, \nu_{_T} \, \Omega \tau_0 \, \frac{\ell_0^2}{H_\rho^2} \Big\{ e_i e_j \overline{W}_z + 2(\overline{W}_i e_j + \overline{W}_j e_i) + 6\left[(\boldsymbol{e}\times\boldsymbol{\nabla})_i \, e_j + (\boldsymbol{e}\times\boldsymbol{\nabla})_j \, e_i\right] \overline{U}_z + (\boldsymbol{e}\times\boldsymbol{\nabla})_i \overline{U}_j^\perp + (\boldsymbol{e}\times\boldsymbol{\nabla})_j \overline{U}_i^\perp + 2\left[\nabla_i^\perp \left(\boldsymbol{e}\times\overline{\boldsymbol{U}}\right)_j + \nabla_j^\perp \left(\boldsymbol{e}\times\overline{\boldsymbol{U}}\right)_i\right] - 4\nabla_z \left[\left(\boldsymbol{e}\times\overline{\boldsymbol{U}}\right)_i e_j + \left(\boldsymbol{e}\times\overline{\boldsymbol{U}}\right)_j e_i\right] \Big\},$$

The Effective Force:

$$\mathcal{F}_i^\Omega = \rho_0 \, \langle v_i \, v_j \rangle^\Omega e_j / H_\rho$$

$$\mathcal{F}_x^\Omega = -2(A_F - A_u) \, \rho_0 \, \nu_{_T} \, \Omega \tau_0 \, \frac{\ell_0^2}{H_\rho^3} \nabla_z \overline{U}_y,$$

$$\mathcal{F}_y^\Omega = -2 \, \rho_0 \, \nu_{_T} \, \Omega \tau_0 \, \frac{\ell_0^2}{H_\rho^3} \left[(A_F + A_u)\nabla_x \overline{U}_z - (A_F - A_u)\overline{W}_y\right],$$

$$\mathcal{F}_z^\Omega = -(5A_F + 4A_u) \, \rho_0 \, \nu_{_T} \, \Omega \tau_0 \, \frac{\ell_0^2}{H_\rho^3} \nabla_x \overline{U}_y,$$

$$A_u = \frac{3(q-1)}{3q-1} \, \frac{\varepsilon_u}{1+\varepsilon_u},$$

$$A_F = \frac{9(q-1)}{2(2q-1)} \, \frac{\varepsilon_{_F} \, \tau_0 \, F_* g}{\rho_0 u_0^2},$$

Generation of Large-Scale Vorticity in Fast Rotating Turbulent Convection: Large-Scale Instability (the mode I)

Rogachevskii I. and Kleeorin N., Phys. Rev. E 100, 063101 (2019)

$$\frac{\partial \overline{U}_y}{\partial t} = -2\,\overline{U}_x \Omega + \frac{\mathcal{F}_y^\Omega}{\rho_0} + \frac{\nu_{_T}}{\rho_0} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\nabla} \overline{U}_y),$$

$$\frac{\partial \overline{W}_y}{\partial t} = 2\Omega\, \nabla_z \overline{U}_y + \left(\boldsymbol{\nabla} \times \frac{\boldsymbol{\mathcal{F}}^\Omega}{\rho_0}\right)_y + \frac{\nu_{_T}}{\rho_0} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\nabla} \overline{W}_y) - g \nabla_z \overline{S}.$$

$$\rho_0 \propto \exp(-\lambda z)$$

$$\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$$

$$\boldsymbol{g} = -g \boldsymbol{e}_z$$

$$\overline{\boldsymbol{W}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$$

$$\rho_0 \overline{\boldsymbol{U}} = [\overline{V}(t,x,z)\rho_0^{1/2}]\boldsymbol{e}_y + \boldsymbol{\nabla} \times [\overline{\Phi}(t,x,z)\rho_0^{1/2}]\boldsymbol{e}_y,$$

$$\overline{V}, \overline{\Phi} \propto \exp(-\lambda z/2)\, \exp(\gamma_{\rm inst} t + i K_x X)$$

The mode I:

$$\nabla_z \overline{U}_y = 0$$

$$\overline{W}_z / \overline{W}_y \sim (H_\rho L_x)/\ell_0^2 \gg 1$$

The growth rate of the instability:

$$\gamma_{\rm inst} = \Omega \frac{\ell_0^2}{H_\rho^2} \left[\frac{3(q-1)}{2(2q-1)} \left(\frac{5\varepsilon_{_F} \tau_0 F_* g}{\rho_0 u_0^2} + \frac{4(2q-1)}{3(3q-1)} \frac{\varepsilon_u}{1+\varepsilon_u}\right)\right]^{1/2} - \nu_{_T} K_x^2.$$

$$L_x = 2\pi / K_x$$

$$\frac{V_*}{\Phi_*} = -\frac{2\Omega}{H_\rho(\gamma_{\rm inst} + \kappa_{_T} K_x^2)}$$

Inside the cyclonic vortices the mean entropy is reduced. $\overline{W}_z > 0$ $\quad \overline{S} < 0$

Inside the anti-cyclonic vortices the mean entropy is increased. $\overline{W}_z < 0$ $\quad \overline{S} > 0$

$$\overline{U}_z = K_x \Phi_* \cos(K_x X + \varphi) \exp(\gamma_{\rm inst} t),$$

$$\overline{W}_z = K_x V_* \cos(K_x X + \varphi) \exp(\gamma_{\rm inst} t)$$

$$\overline{S} = -S_* \cos(K_x X + \varphi) \exp(\gamma_{\rm inst} t)$$

Comparisons with DNS/LES

P. J. Käpylä, M. J. Mantere, and T. Hackman, Astrophys. J. 742, 34 (2011).
M. J. Mantere, P. J. Käpylä, and T. Hackman, Astron. Nachr. 332, 876 (2011).

$$\frac{\mathrm{D}\ln\rho}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{U},$$

$$\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{g} - 2\boldsymbol{\Omega}\times\boldsymbol{U} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot 2\nu\rho\mathbf{S},$$

$$\frac{\mathrm{D}e}{\mathrm{D}t} = -\frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{U} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot K\boldsymbol{\nabla}T + 2\nu\mathbf{S}^2 - \frac{e-e_0}{\tau(z)},$$

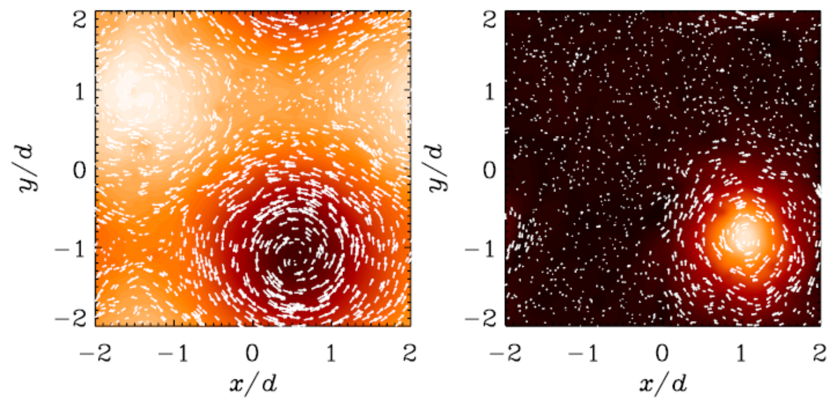

Figure 6. Pressure (colors) and horizontal flows (arrows) from the middle of the convection zone in Runs A9 (left panel) and A1 (right panel).

(A color version of this figure is available in the online journal.)

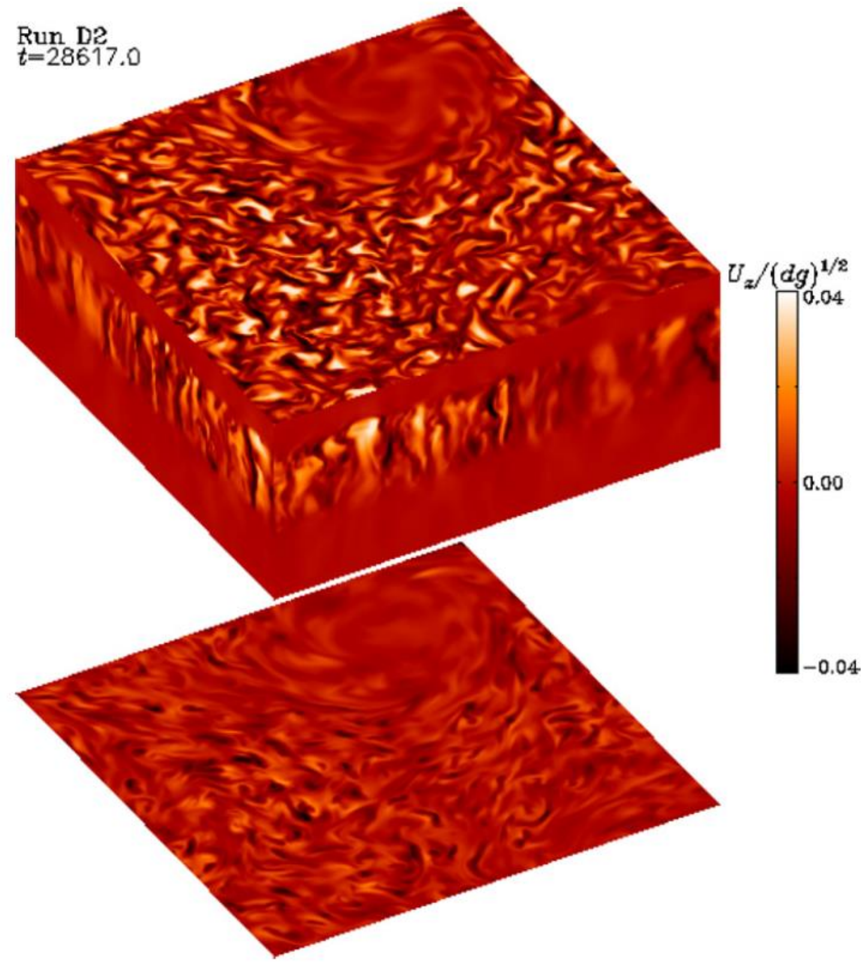

Figure 3. Vertical velocity component U_z at the periphery of the box from Run D2. See also http://www.helsinki.fi/~kapyla/movies.html. The top and bottom panels show slices near the top and bottom of the convectively unstable layer, respectively.

K. L. Chan, Astron. Nachr. 328, 1059 - 1061 (2007)

Generation of Large-Scale Vorticity in Fast Rotating Turbulent Convection: Large-Scale Instability (the mode II)

Rogachevskii I. and Kleeorin N., Phys. Rev. E 100, 063101 (2019)

$$\frac{\partial \overline{U}_y}{\partial t} = -2\overline{U}_x \Omega + \frac{\mathcal{F}_y^\Omega}{\rho_0} + \frac{\nu_{_T}}{\rho_0} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\nabla} \overline{U}_y),$$

$$\frac{\partial \overline{W}_y}{\partial t} = 2\Omega \, \nabla_z \overline{U}_y + \left(\boldsymbol{\nabla} \times \frac{\boldsymbol{\mathcal{F}}^\Omega}{\rho_0}\right)_y + \frac{\nu_{_T}}{\rho_0} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\nabla} \overline{W}_y) - g \nabla_z \overline{S}.$$

$$\rho_0 \propto \exp(-\lambda z)$$

$$\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$$

$$\boldsymbol{g} = -g \boldsymbol{e}_z$$

$$\rho_0 \overline{\boldsymbol{U}} = [\overline{V}(t, x, z) \rho_0^{1/2}] \boldsymbol{e}_y + \boldsymbol{\nabla} \times [\overline{\Phi}(t, x, z) \rho_0^{1/2}] \boldsymbol{e}_y,$$

$$\overline{V}, \overline{\Phi} \propto \exp(\lambda z/2) \, \exp(\gamma_{\rm inst} t + i K_x X).$$

The mode II: $\rho_0 \overline{\boldsymbol{U}}$ **is independent of z**

$$\overline{W}_z / \overline{W}_y \sim \ell_0^2 / (H_\rho L_x) \ll 1$$

$$\overline{\boldsymbol{W}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$$

$$\overline{U}_x = 0$$

$$L_x = 2\pi / K_x$$

The growth rate of the instability:

$$\gamma_{\rm inst} = \Omega \, \frac{\ell_0^2}{H_\rho^2} \left[\frac{6(q-1) \, \varepsilon_{_F} \tau_0 \, F_* \, g}{(2q-1) \, \rho_0 u_0^2}\right]^{1/2} - \nu_{_T} K_x^2.$$

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Generation of a large-scale vorticity in a fast-rotating density-stratified turbulence or turbulent convection

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We find an instability resulting in generation of large-scale vorticity in a fast-rotating small-scale turbulence or turbulent convection with inhomogeneous fluid density along the rotational axis in anelastic approximation. The large-scale instability causes excitation of two modes: (i) the mode with dominant vertical vorticity and with the mean velocity being independent of the vertical coordinate; (ii) the mode with dominant horizontal vorticity and with the mean momentum being independent of the vertical coordinate. The mode with the dominant vertical vorticity can be excited in a fast-rotating density-stratified hydrodynamic turbulence or turbulent convection. For this mode, the mean entropy is depleted inside the cyclonic vortices, while it is enhanced inside the anticyclonic vortices. The mode with the dominant horizontal vorticity can be excited only in a fast-rotating density-stratified turbulent convection. The developed theory may be relevant for explanation of an origin of large spots observed as immense storms in great planets, e.g., the Great Red Spot in Jupiter and large spots in Saturn. It may be also useful for explanation of an origin of high-latitude spots in rapidly rotating late-type stars.

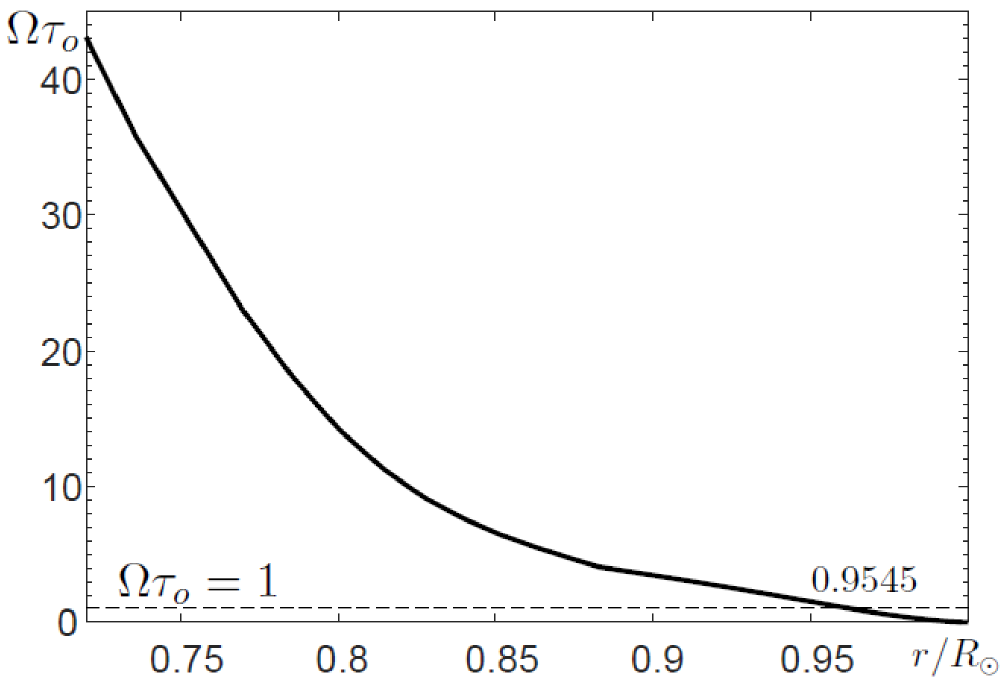

FIGURE 4. The profile of $\Omega\tau_0$ versus $r/R_\odot$ based on the model of the solar convective zone by Spruit (1974).

Conclusions

- The dynamo theories of the algebraic quenching do not take into account the effect of modification of the source of turbulence by the growing large-scale magnetic field. As the result, the nonlinear dynamo number increases with the strong mean magnetic field. This phenomenon is due to the dissipation of the strong large-scale magnetic field resulting in an increase of the total turbulent energy.
- This effect has been studied using the budget equation for the total turbulent energy, and it results in the mean-field dynamo instabilities are always saturated by the strong large-scale magnetic field.
- A mean-field theory of differential rotation in a density stratified turbulent convection has been developed, which is based on a combined effect of the turbulent heat flux and anisotropy of turbulent convection on the Reynolds stress.
- The background turbulent convection takes into account an increase of the turbulence anisotropy and a decrease of the turbulent correlation time with the rotation rate. This theory yields the radial profile of the differential rotation in agreement with the solar differential rotation.
- In similar set-up, we find an instability resulting in generation of large-scale vorticity in a fast rotating small-scale turbulent convection with inhomogeneous fluid density along the rotational axis in anelastic approximation. For this mode, the mean entropy is depleted inside the cyclonic vortices, while it is enhanced inside the anti-cyclonic vortices. The theory may be relevant for explanation of an origin of large spots in great planets, e.g., the Great Red Spot in Jupiter and large spots in Saturn.

THE END