# General relativistic bubble growth in cosmological phase transitions

L. Giombi[1], M. Hindmarsh[1,2]

arXiv:2307.12080v1

1.Department of Physics and Helsinki Institute of Physics, PL 64, FI-00014 University of Helsinki, Finland.

2.Department of Physics and Astronomy, University of Sussex, Brighton BN1 9QH, United Kingdom

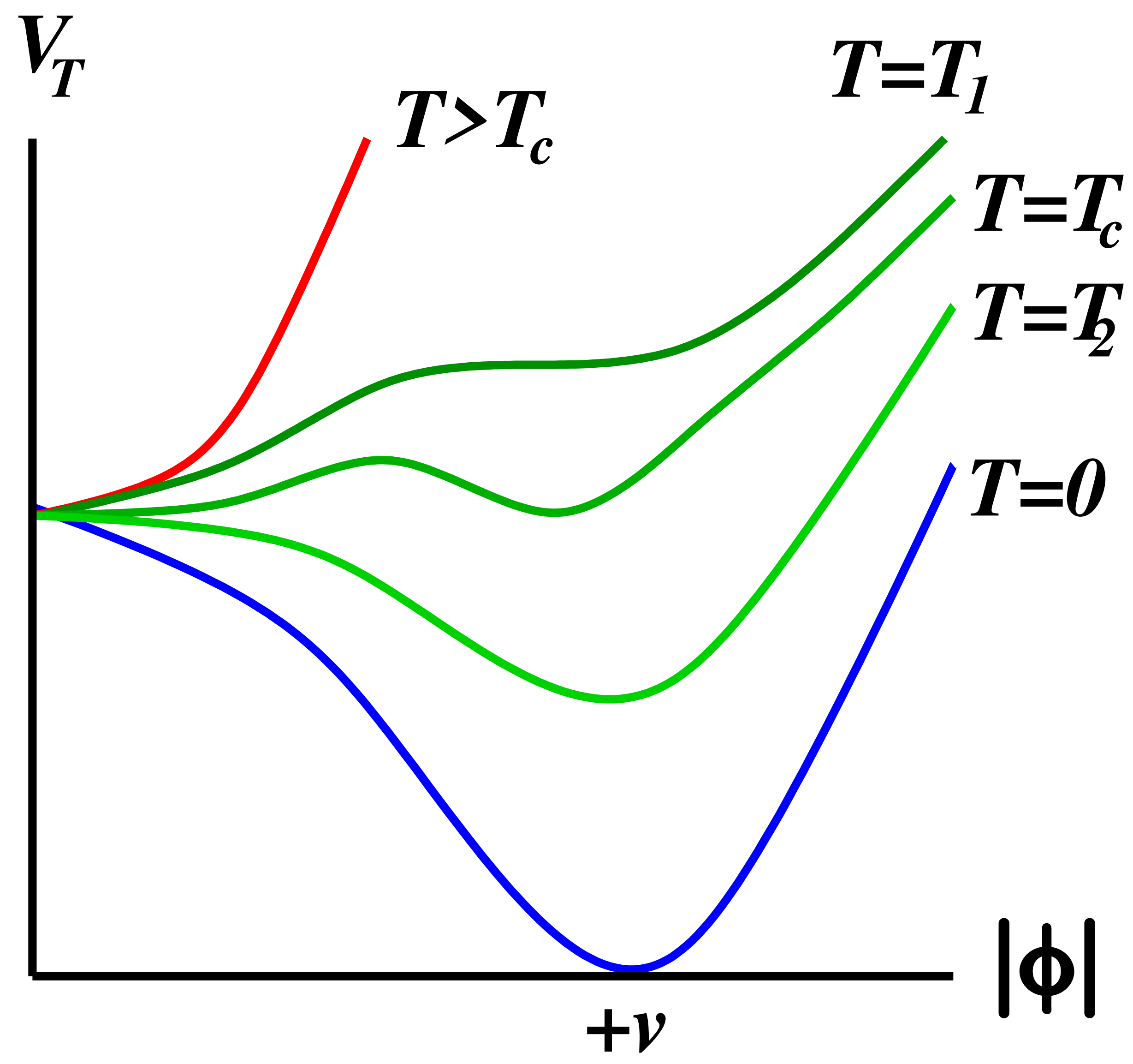
24 October 2023

# First order phase transitions (FOPT) in the early Universe

M.Hindmarsh et al. (2020), arXiv:2008.09136v2

- Phase transitions are a generic feature of many gauge field theories

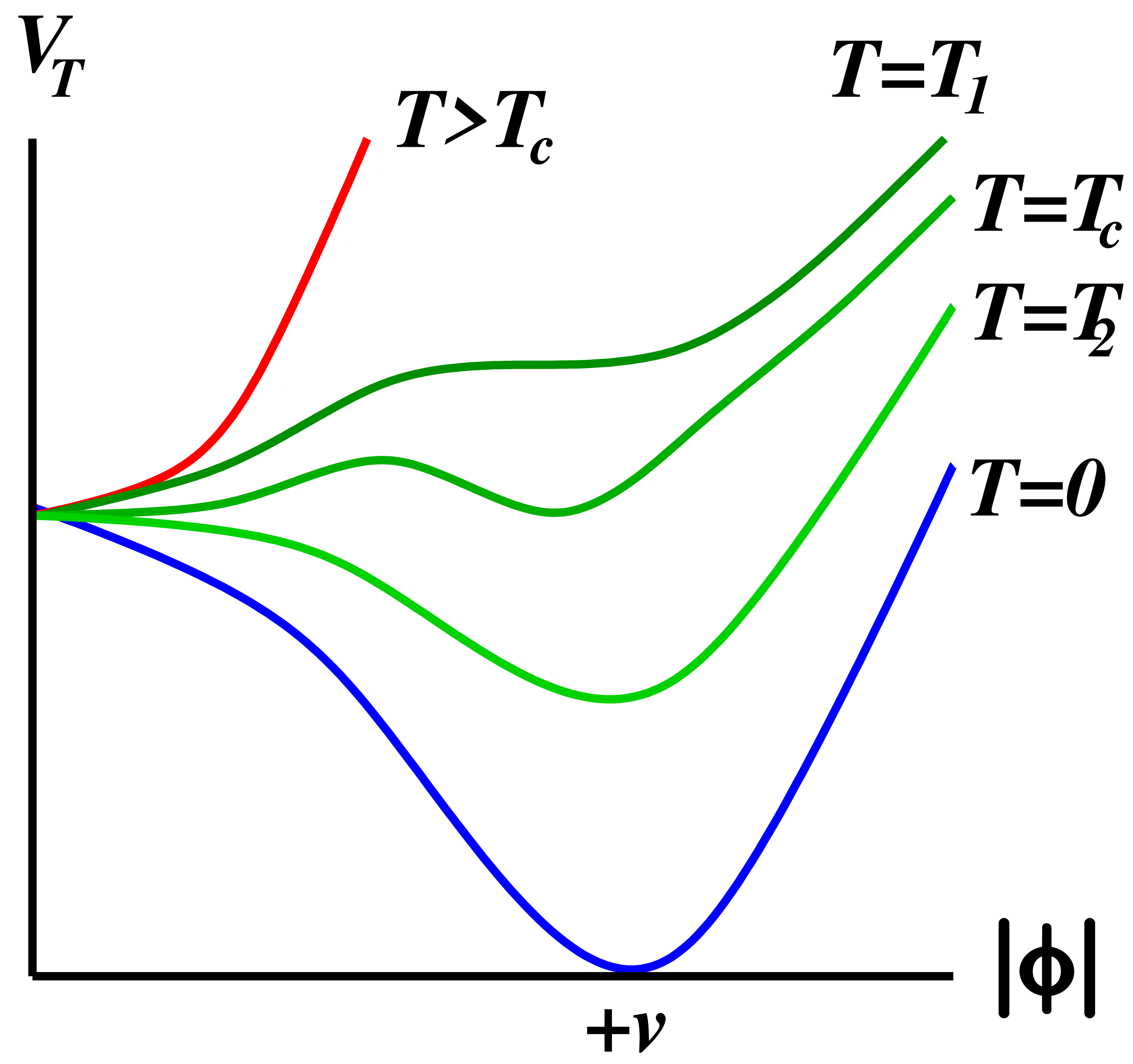
- Usually described by a scalar field $\phi$ with free energy $\mathscr{F}(\phi, T)$



# First order phase transitions (FOPT) in the early Universe

M.Hindmarsh et al. (2020), arXiv:2008.09136v2

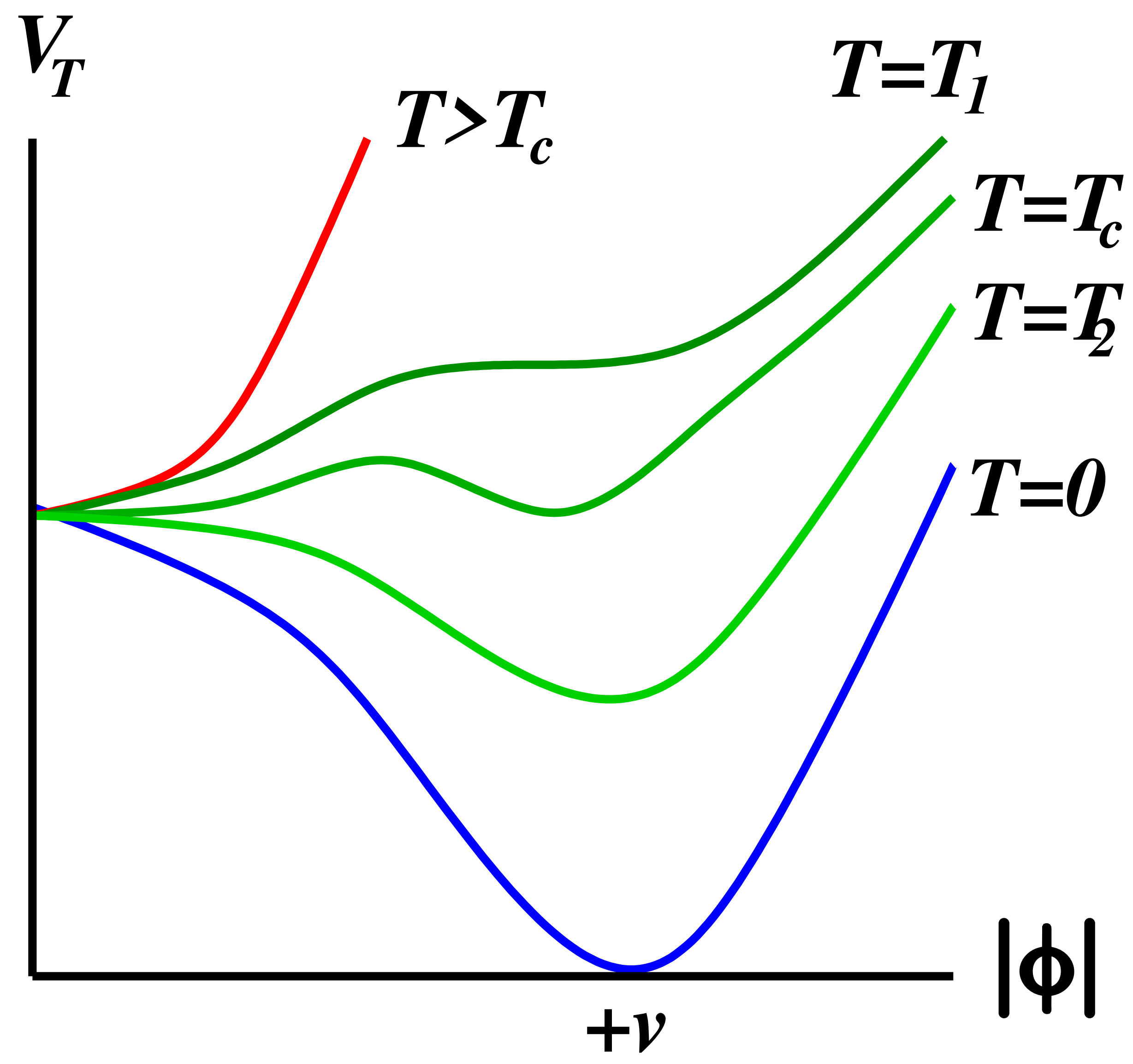
- Phase transitions are a generic feature of many gauge field theories
- Usually described by a scalar field $\phi$ with free energy $\mathscr{F}(\phi, T)$
- FOPT: just below $T_c$ the field $\phi$ is in a metastable phase

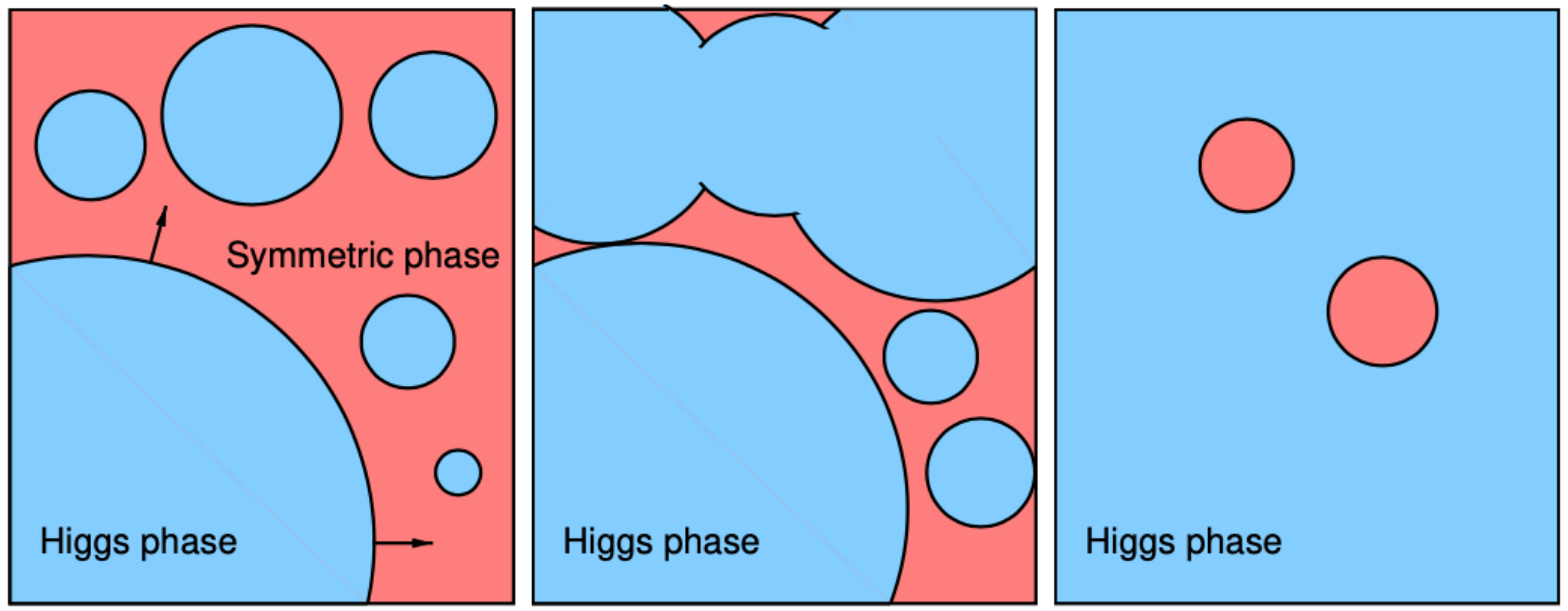


# First order phase transitions (FOPT) in the early Universe

M.Hindmarsh et al. (2020), arXiv:2008.09136v2

- Phase transitions are a generic feature of many gauge field theories
- Usually described by a scalar field $\phi$ with free energy $\mathscr{F}(\phi, T)$
- FOPT: just below $T_c$ the field $\phi$ is in a metastable phase
- Thermal and quantum fluctuations allow the nucleation of bubbles of the stable phase
- Bubbles expand and merge filling up larger and larger portions of the Universe

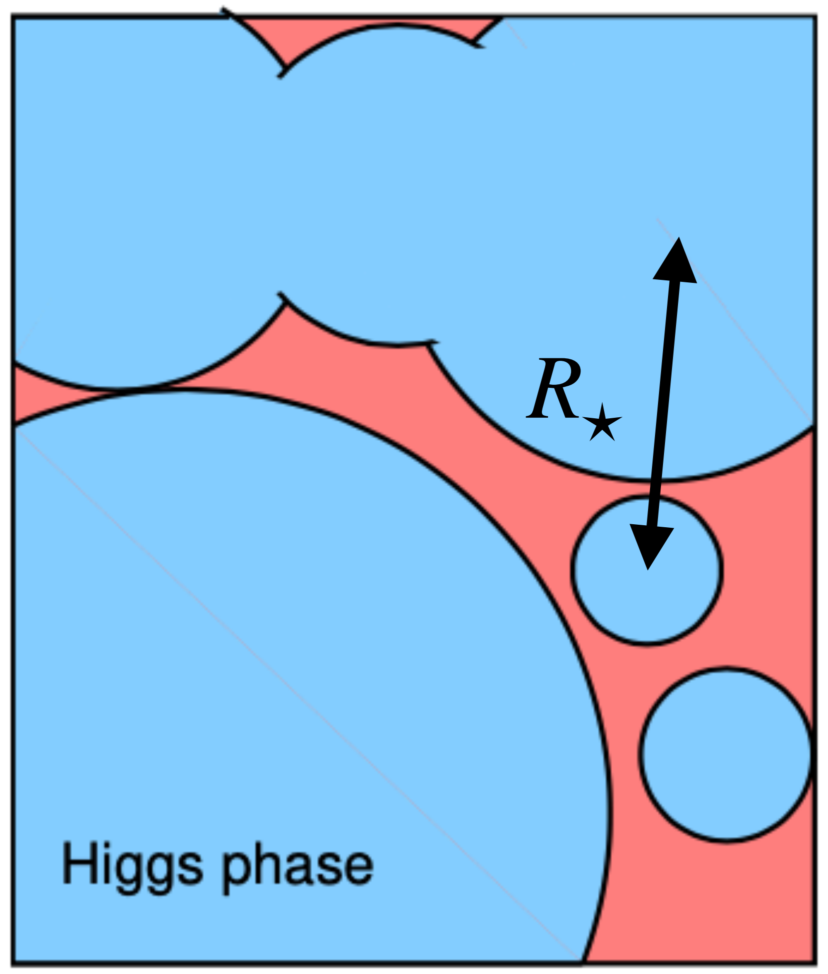


# Objectives

## Hydrodynamic description of a single expanding bubble:

- So far: expansion on a flat Minkowski spacetime ($R_\star \ll H_\star^{-1}$)




M.Hindmarsh et al. (2020), arXiv:2008.09136v2

$R_\star$ : mean bubble spacing after nucleation of all bubbles

$H_\star^{-1}$ : Hubble radius at the time when $1/e$ of metastable phase remains

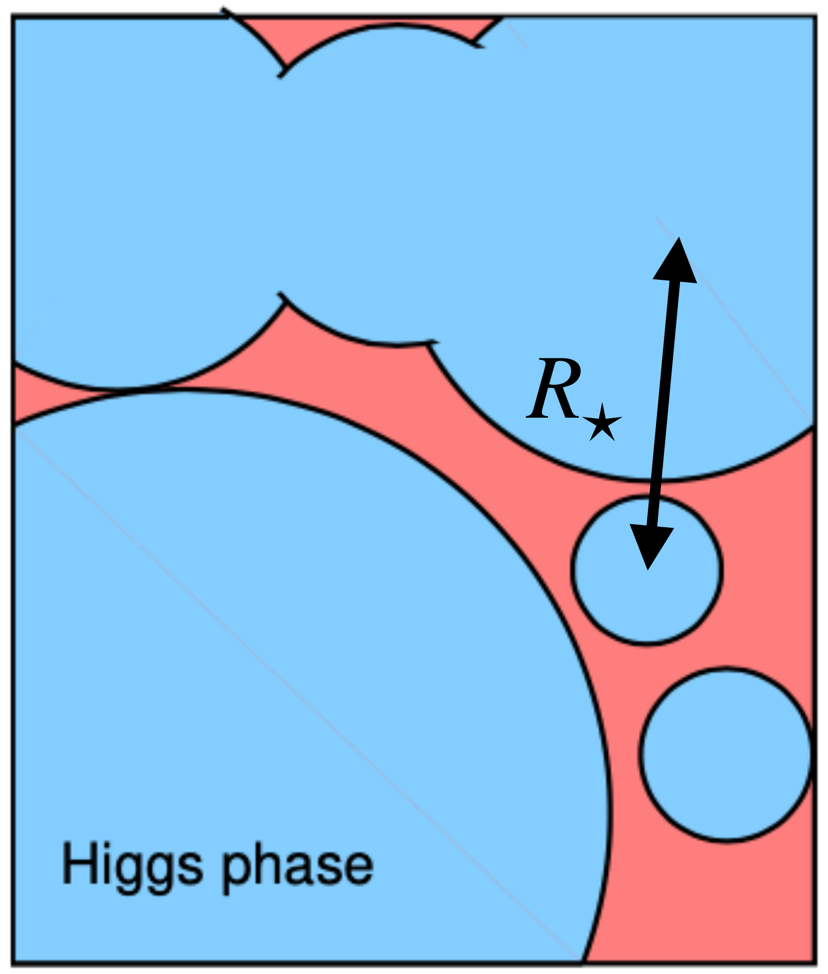
# Objectives

## Hydrodynamic description of a single expanding bubble:

- So far: expansion on a flat Minkowski spacetime ($R_\star \ll H_\star^{-1}$)

- In slow FOPT the timescale of the expansion is of the order of Hubble time ($R_\star \sim H_\star^{-1}$)

→ Need for the full general relativistic treatment




M.Hindmarsh et al. (2020), arXiv:2008.09136v2

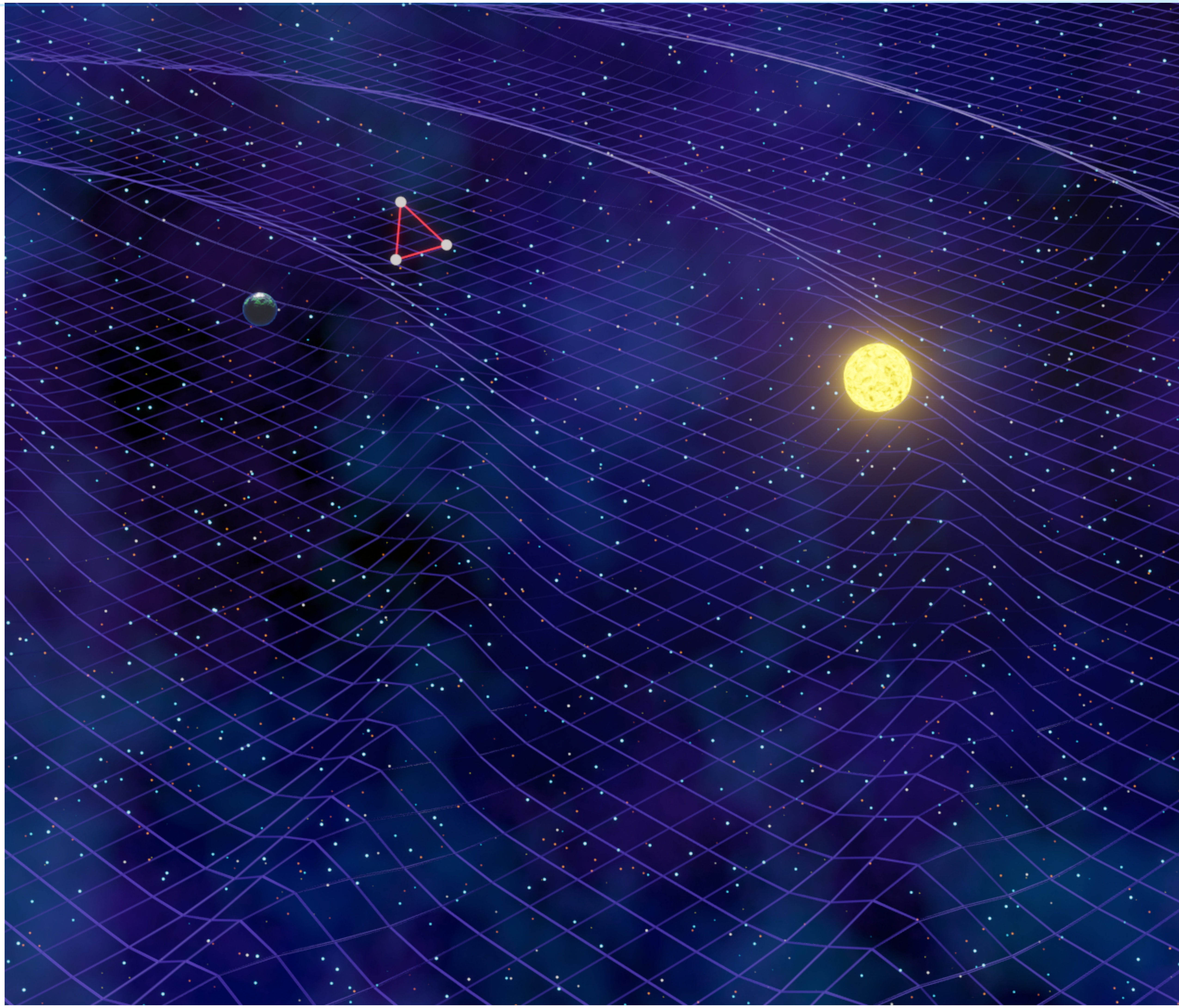
$R_\star$ : mean bubble spacing after nucleation of all bubbles

$H_\star^{-1}$ : Hubble radius at the time when $1/e$ of metastable phase remains

# Motivations

FOPT are a source of the stochastic background of gravitational waves

FOPT at the EW scale ( $\sim 100$ GeV) are experimentally interesting for the LISA mission $\sim 0.1$ mHz - $10$ Hz



Credit: Anna Kormu

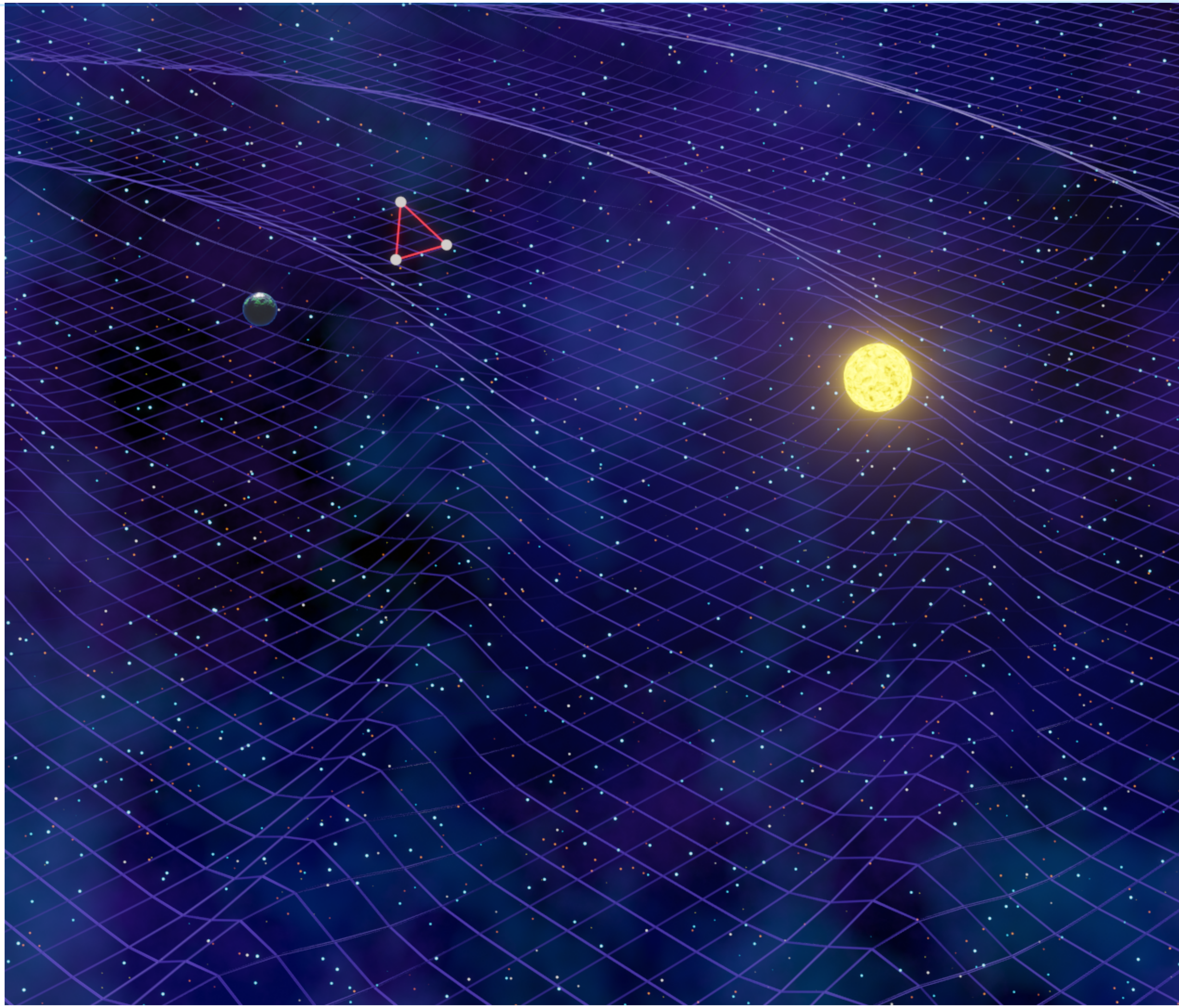
# Motivations

FOPT are a source of the stochastic background of gravitational waves

FOPT at the EW scale ( $\sim 100$ GeV) are experimentally interesting for the LISA mission $\sim 0.1$ mHz - $10$ Hz

- Energy density in gravitational waves sourced by sound waves $\Omega_{sw} \propto (R_\star H_\star)^n$ $\quad 1 \leq n \leq 2$

C. Caprini et al. (2016),arXiv:1512.06239v2



Credit: Anna Kormu

# Motivations

FOPT are a source of the stochastic background of gravitational waves

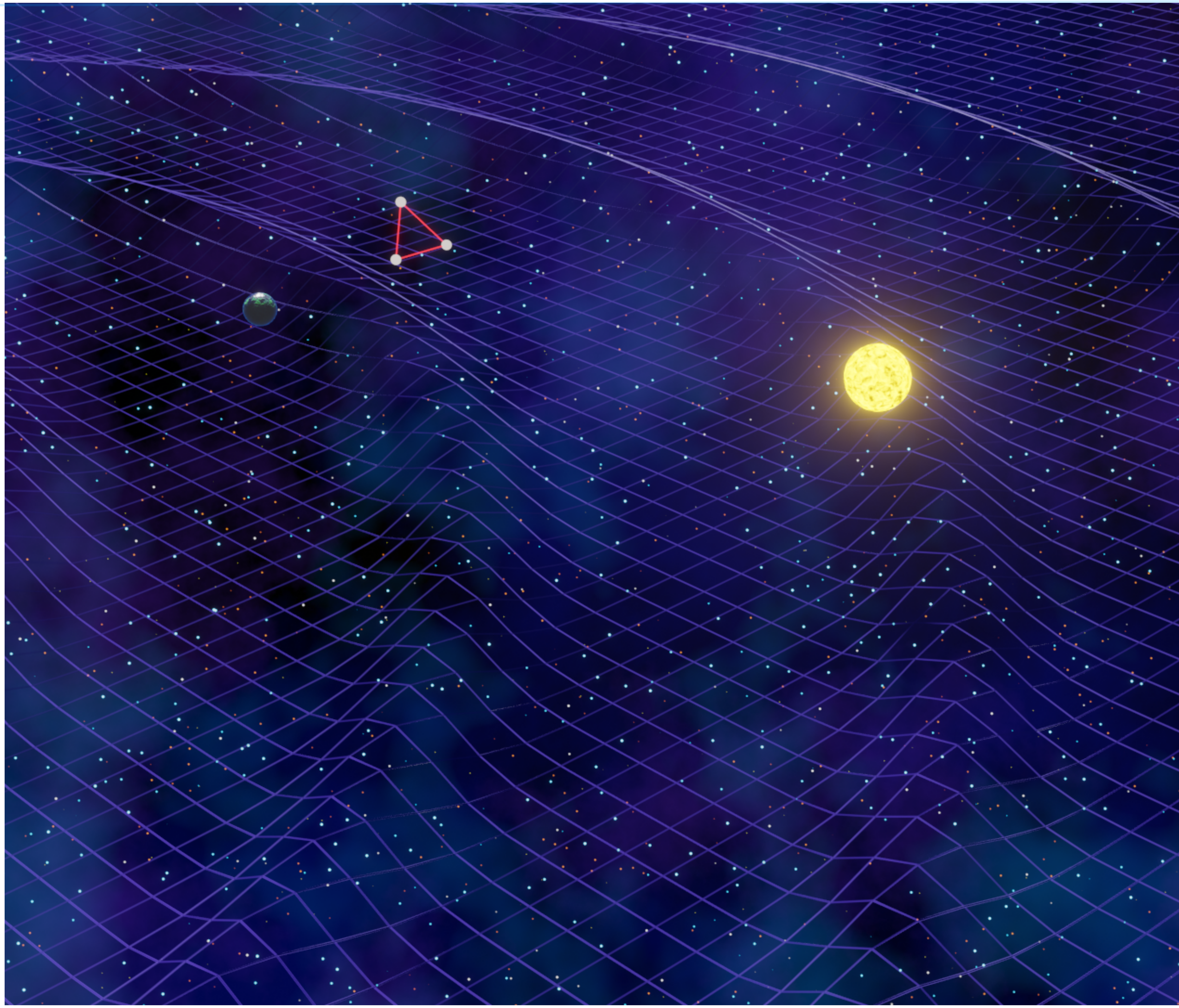
FOPT at the EW scale ( $\sim 100$ GeV) are experimentally interesting for the LISA mission $\sim 0.1$ mHz - $10$ Hz

- Energy density in gravitational waves sourced by sound waves $\Omega_{sw} \propto (R_\star H_\star)^n$ $\quad 1 \leq n \leq 2$

C. Caprini et al. (2016),arXiv:1512.06239v2

- Cosmological scalar perturbations $\Phi$ induce secondary gravitational waves that become important in the limit of large bubbles

$$\frac{\partial_i \Phi \partial_j \Phi}{T_{ij}^{TT}} \sim (HR)^2 \left( \frac{\delta e}{e} \right)$$

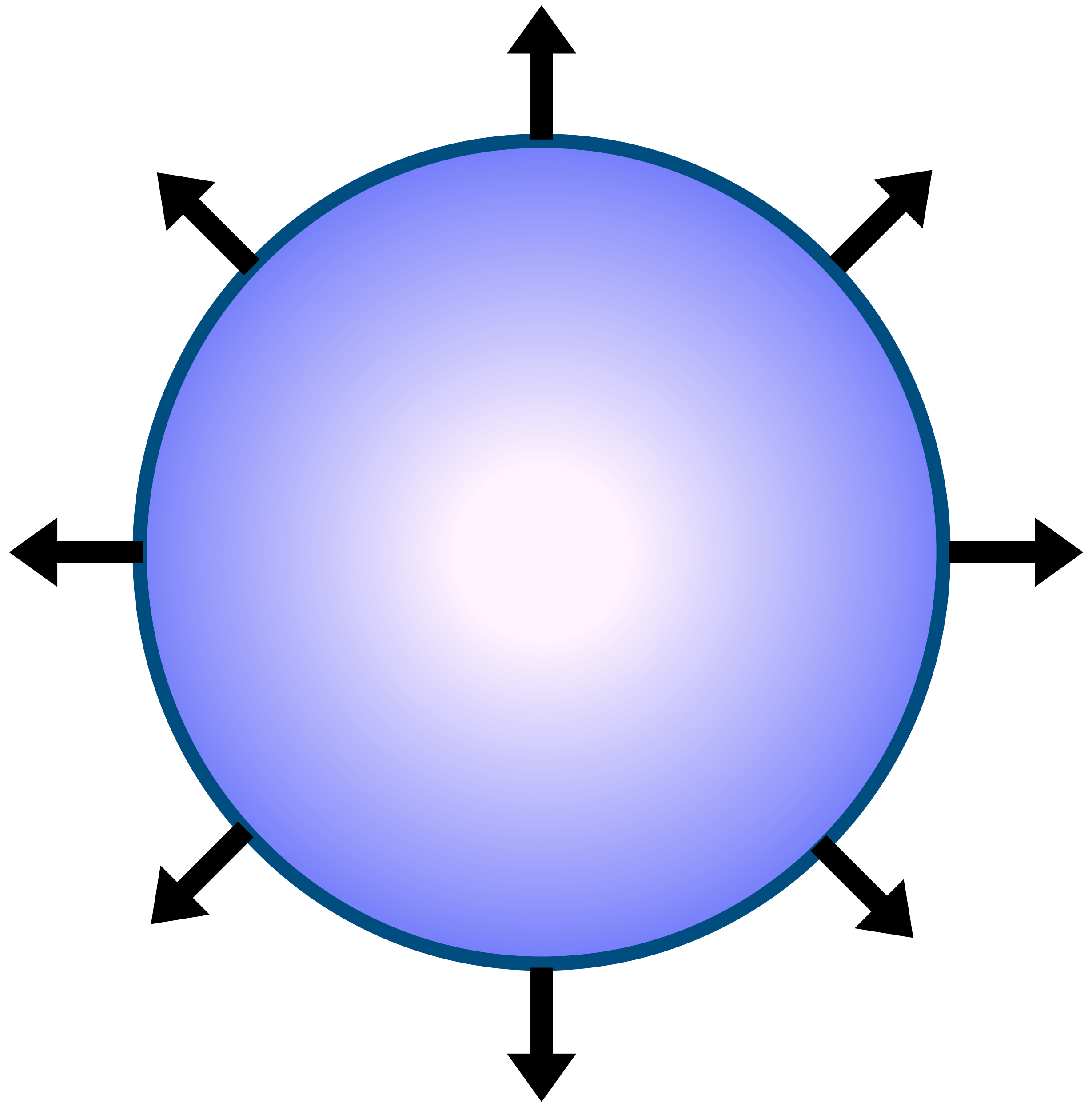


Credit: Anna Kormu

# Hydrodynamic description of a single expanding bubble

- Spherical symmetry: $ds^2 = -a^2 dt^2 + b^2 dr^2 + R^2 \left( d\theta^2 + \sin^2\theta d\varphi^2 \right)$ Misner & Sharp (1964) Phys.Rev 136 B571

$$T^{\mu\nu} = w u^\mu u^\nu + p g^{\mu\nu}, \qquad u^\mu = \frac{1}{a}\delta^{\mu 0}$$

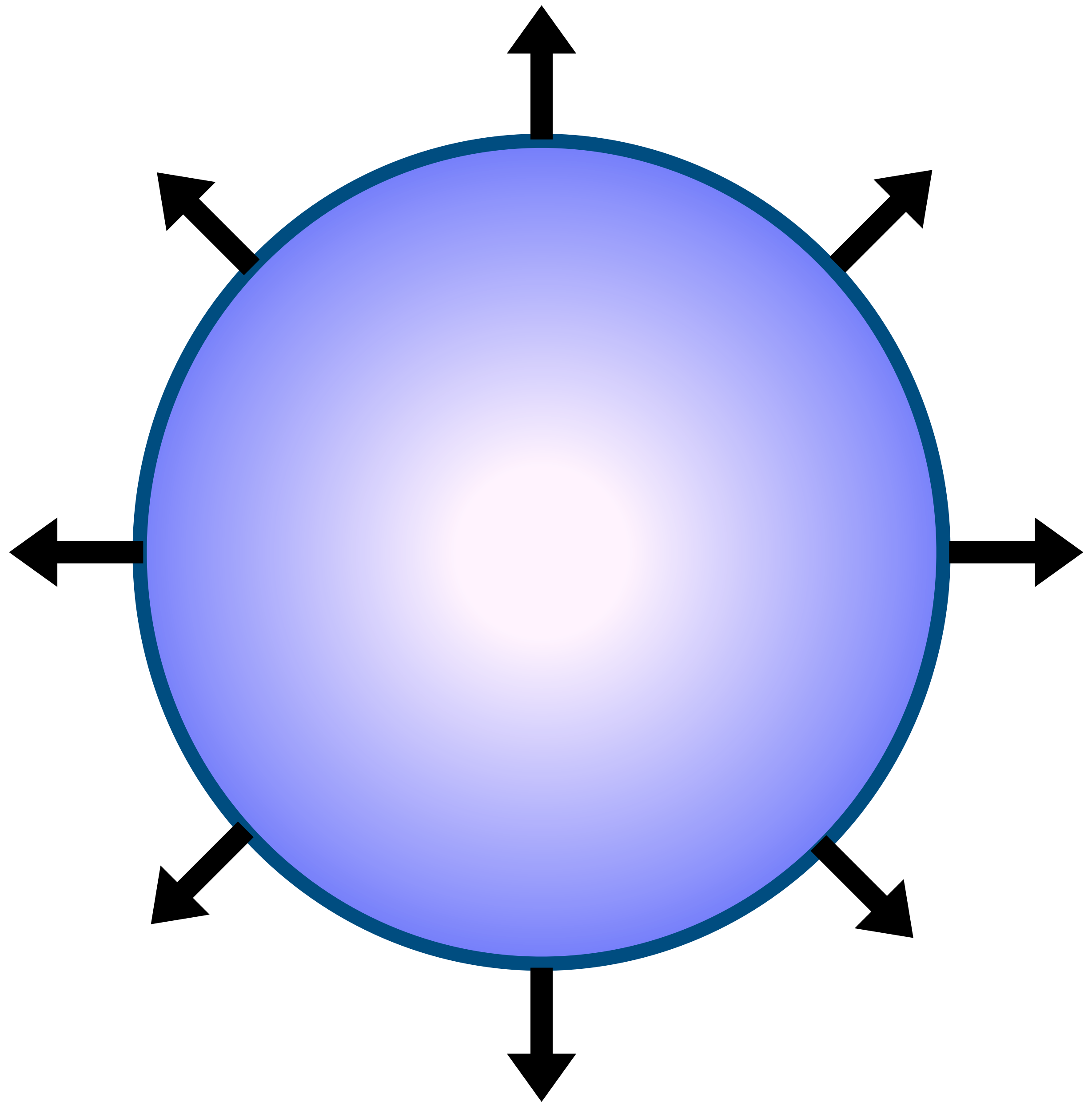


# Hydrodynamic description of a single expanding bubble

- Spherical symmetry: $ds^2 = -a^2 dt^2 + b^2 dr^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)$ Misner & Sharp (1964) Phys.Rev 136 B571

$$T^{\mu\nu} = w u^\mu u^\nu + p g^{\mu\nu}, \qquad u^\mu = \frac{1}{a}\delta^{\mu 0}$$

- Self similarity: $\xi = \frac{R}{t}$ I. Musco et al. (2013) arXiv:1201.2379v3



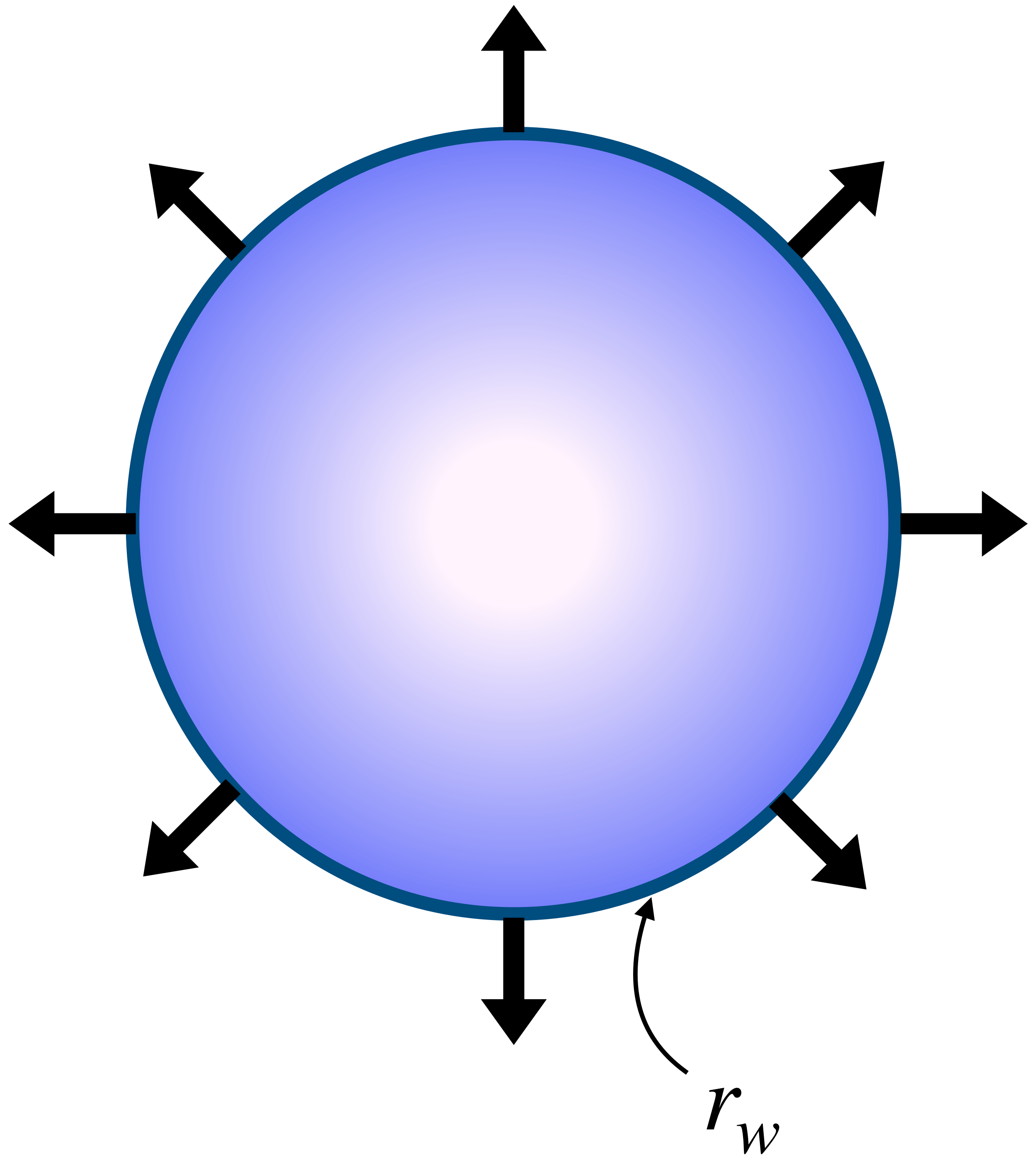
# Hydrodynamic description of a single expanding bubble

- Spherical symmetry: $ds^2 = -a^2 dt^2 + b^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ Misner & Sharp (1964) Phys.Rev 136 B571

$$T^{\mu\nu} = w u^\mu u^\nu + p g^{\mu\nu}, \qquad u^\mu = \frac{1}{a}\delta^{\mu 0}$$

- Self similarity: $\xi = \frac{R}{t}$ I. Musco et al. (2013) arXiv:1201.2379v3

- Steady flow: $R_w = R(t, r_w(t)) = \xi_w t$ Bubble size is a constant fraction of Hubble radius



# Hydrodynamic description of a single expanding bubble

- Spherical symmetry: $ds^2 = -a^2 dt^2 + b^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ Misner & Sharp (1964) Phys.Rev 136 B571

$$T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}, \qquad u^\mu = \frac{1}{a}\delta^{\mu 0}$$

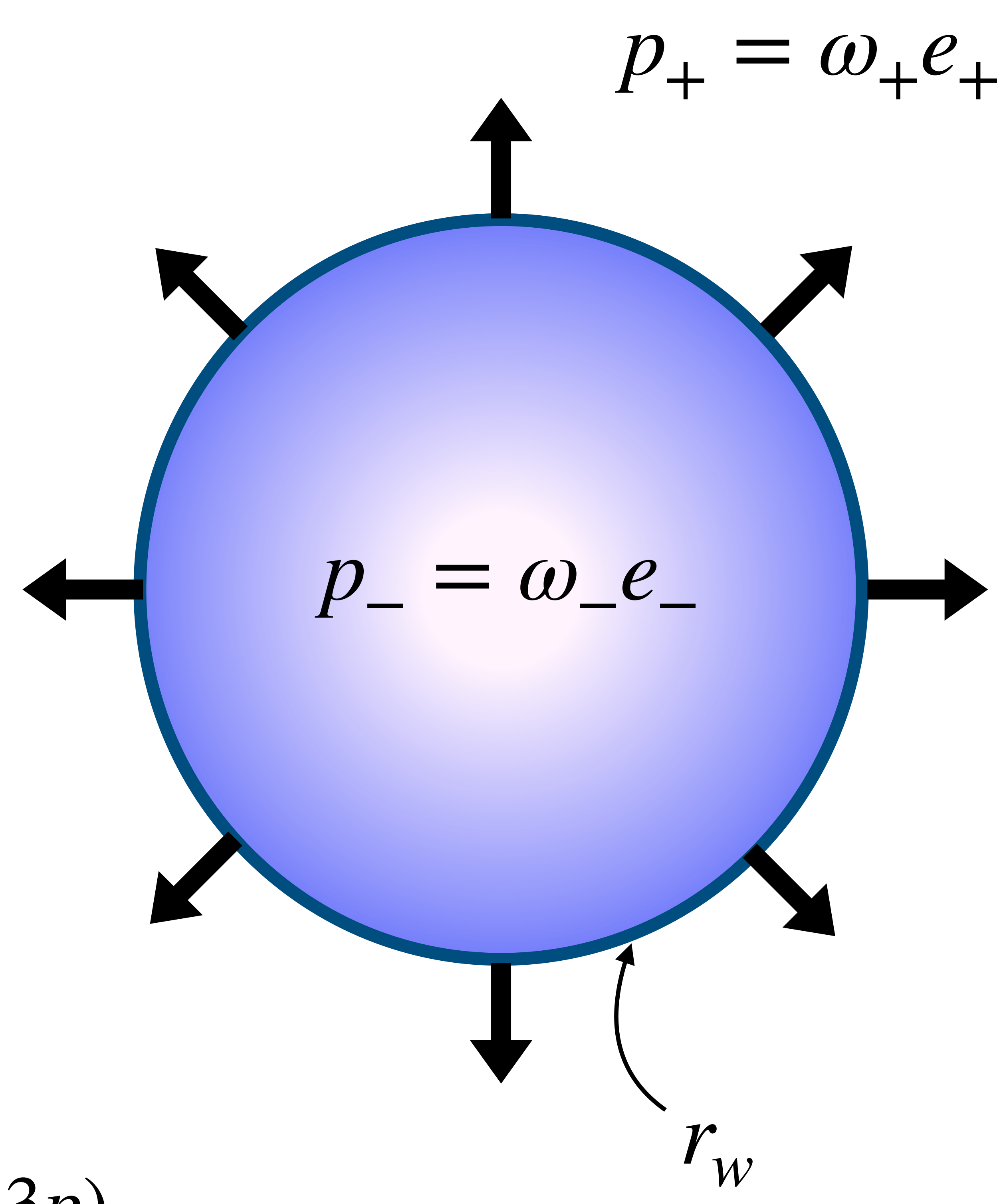
- Self similarity: $\xi = \frac{R}{t}$ I. Musco et al. (2013) arXiv:1201.2379v3

- Steady flow: $R_w = R(t, r_w(t)) = \xi_w t$ Bubble size is a constant fraction of Hubble radius

- Equation of state: $p = \omega e, \quad \omega = \omega_- \Theta(r_w(t) - r) + \omega_+ \Theta(r - r_w(t))$

Strength parameter: at the wall $\alpha_+ = \frac{4}{3}\frac{\theta_+ - \theta_-}{w_+}$

Trace anomaly $\theta = \frac{1}{3}(e + 3p)$



# Hydrodynamic description of a single expanding bubble

The profile of the bubble is given by the solution of the system of Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$

$$\frac{d\ln U}{d\ln\xi} = \left[(\Phi+\omega\Omega)^2 - 2c_s^2\Gamma^2\Phi\right]\left[\frac{\Omega-\Phi}{U^2(\Phi+\omega\Omega)^2 - c_s^2\Gamma^2(\Omega-\Phi)^2}\right],$$

$$\frac{d\ln\Omega}{d\ln\xi} = \frac{\Omega-\Phi}{\Phi+\omega\Omega}\left[2\omega+(1+\omega)\frac{d\ln U}{d\ln\xi}\right],$$

$$\frac{d\ln\Phi}{d\ln\xi} = \frac{1}{\Phi}(\Omega-\Phi)\,.$$

$U \equiv \frac{1}{a}\partial_t R$ — Radial fluid 4-velocity Eulerian observer

$\Gamma \equiv \frac{1}{b}\partial_r R$ — Generalised Lorentz Gamma factor

$\Omega \equiv 4\pi e R^2$ — Energy on a shell of radius $R$

$\Phi \equiv \frac{M}{R}$ — Gravitational potential at radius $R$

$$1 - 2\frac{M}{R} \equiv \partial_\mu R \partial^\mu R = \Gamma^2 - U$$

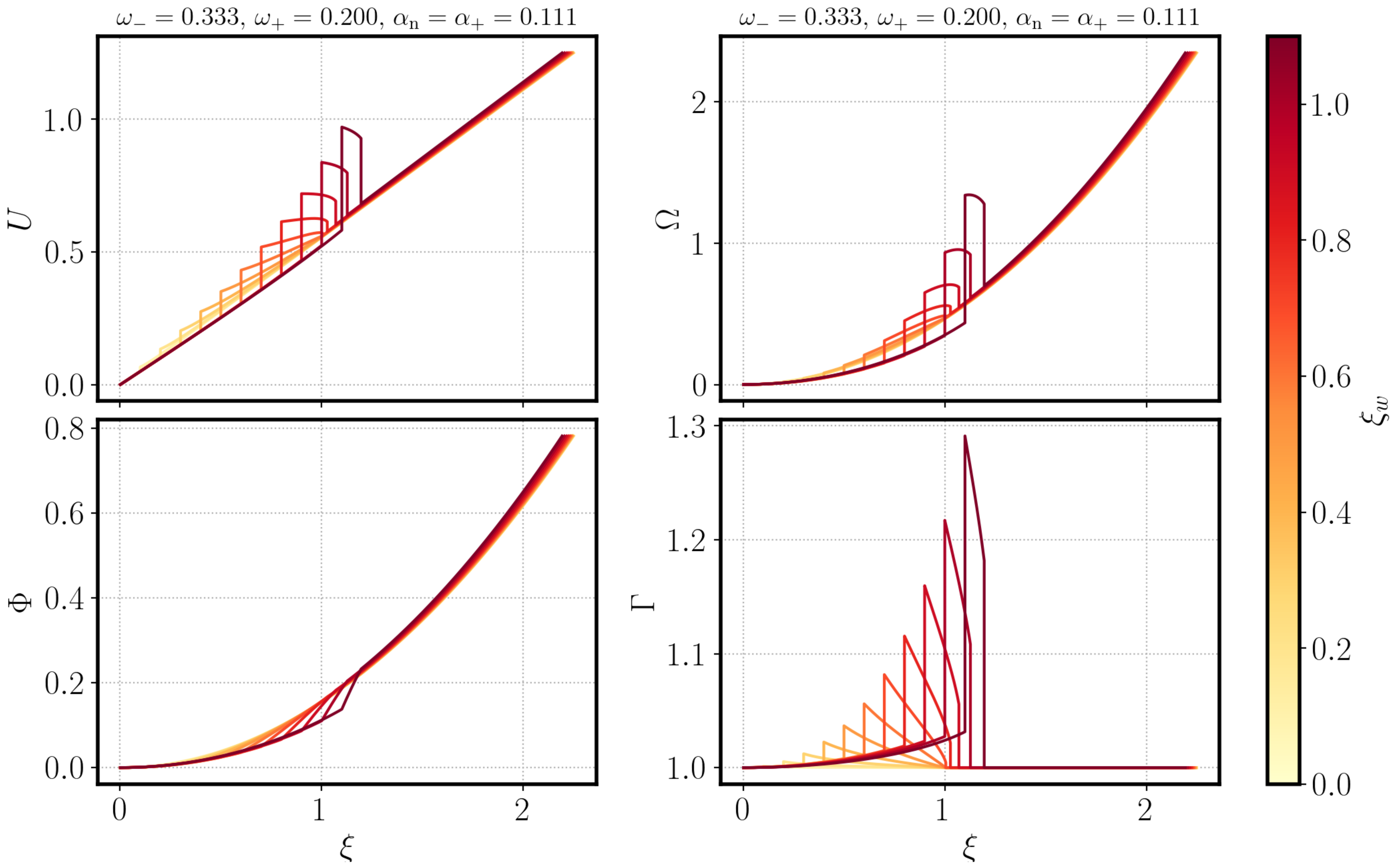
Misner & Sharp (1964)
Phys.Rev 136 B571

# Solving the EoM: deflagration solutions: $v(\xi_w)_- < c_{s\_}$

$v_-$ : fluid entry speed for an observer comoving with the wall

Radial fluid 4-velocity Eulerian observer

Gravitational potential at radius $R$



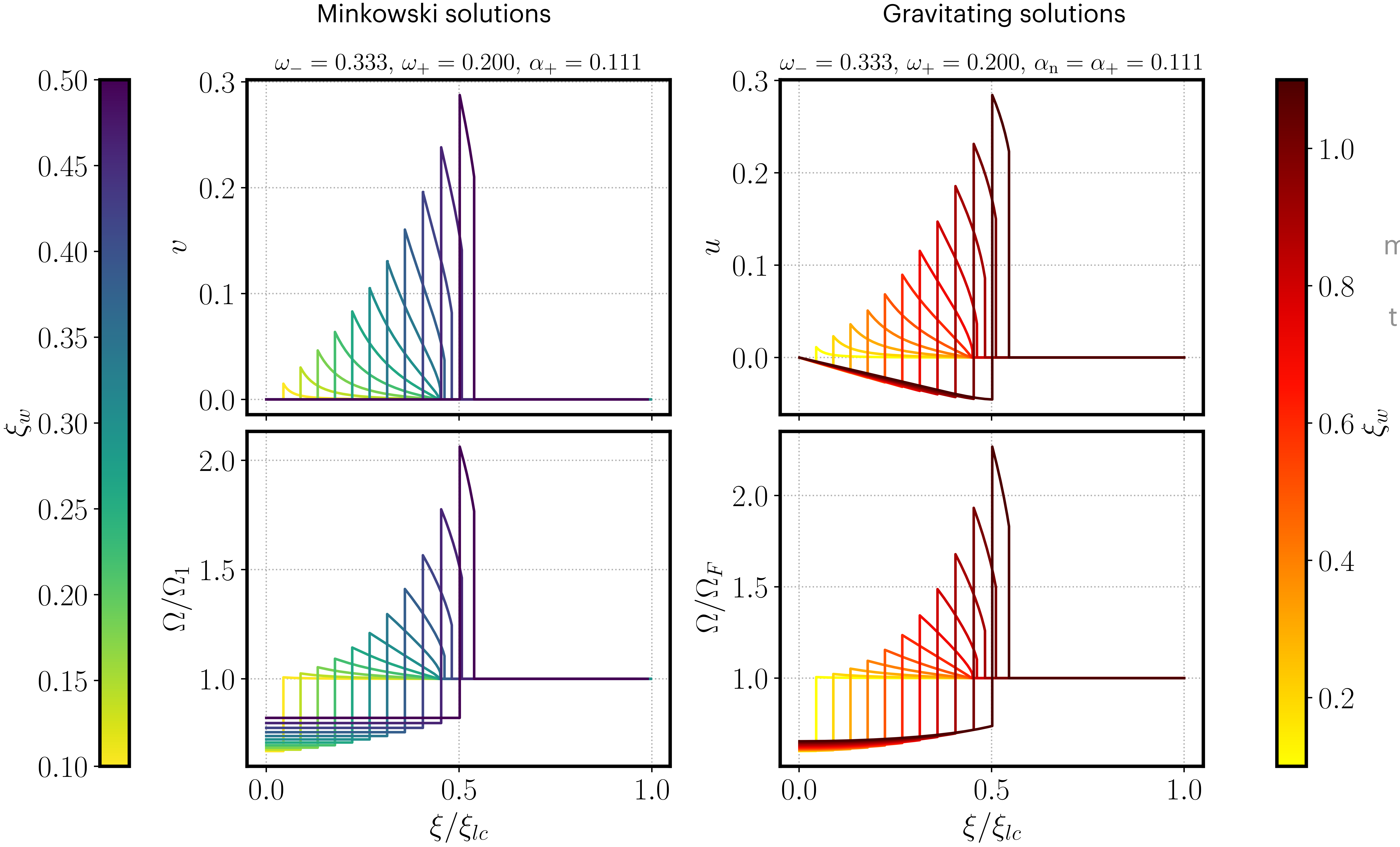

Energy on a shell of radius $R$

Generalised Lorentz Gamma factor

# Deflagration solutions: $v(\xi_w)_- < c_{s\_}$ - Comparison with Minkowski

Constant sound speed model

F. Giese et al., ArXiv:2010.09744

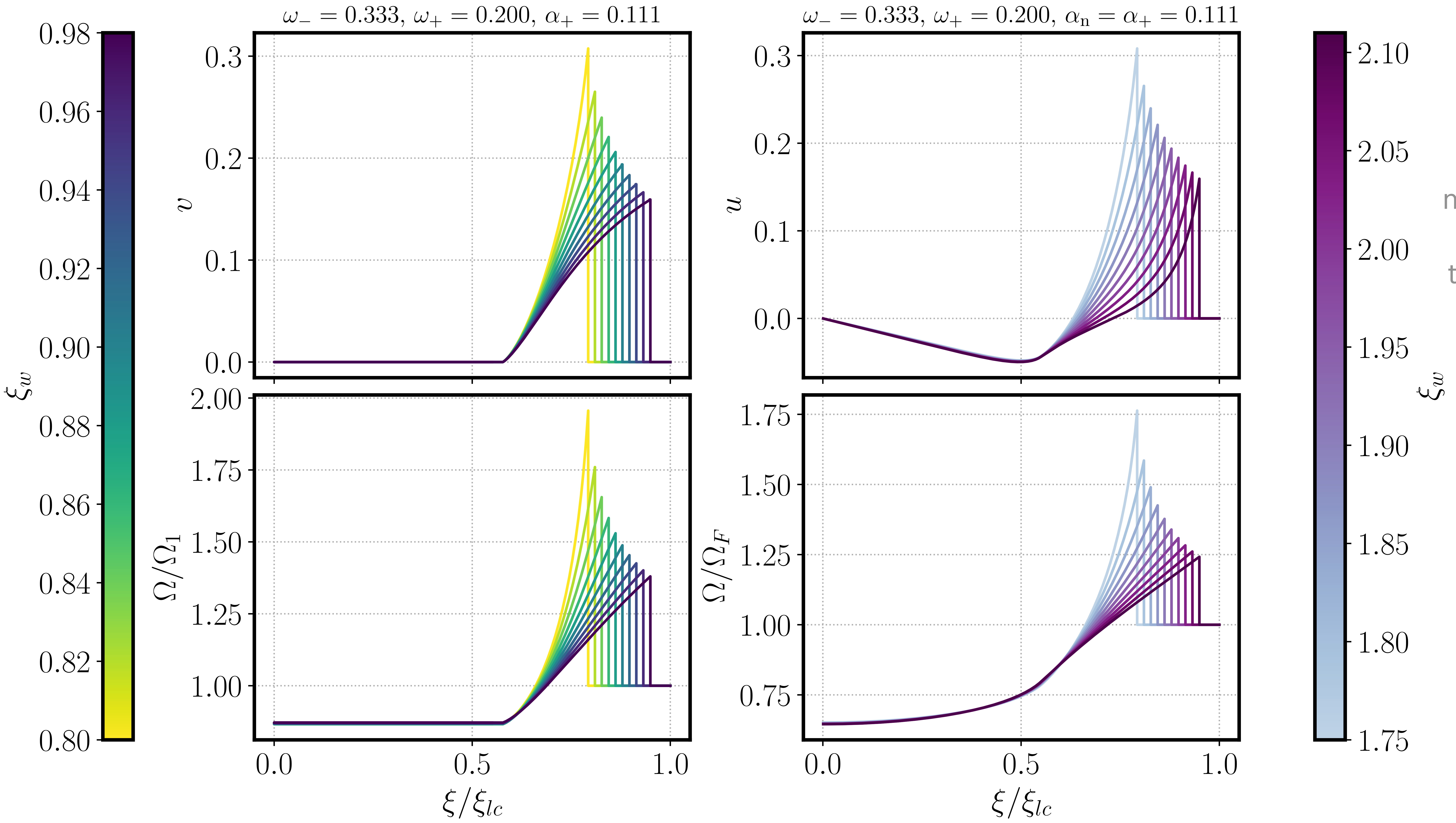



fluid speed measured in the rest frame of the outer FLRW fluid

# Detonation solutions: $v(\xi_w)_+ > c_{s\_}$ - Comparison with Minkowski

Constant sound speed model

F. Giese et al., ArXiv:2010.09744

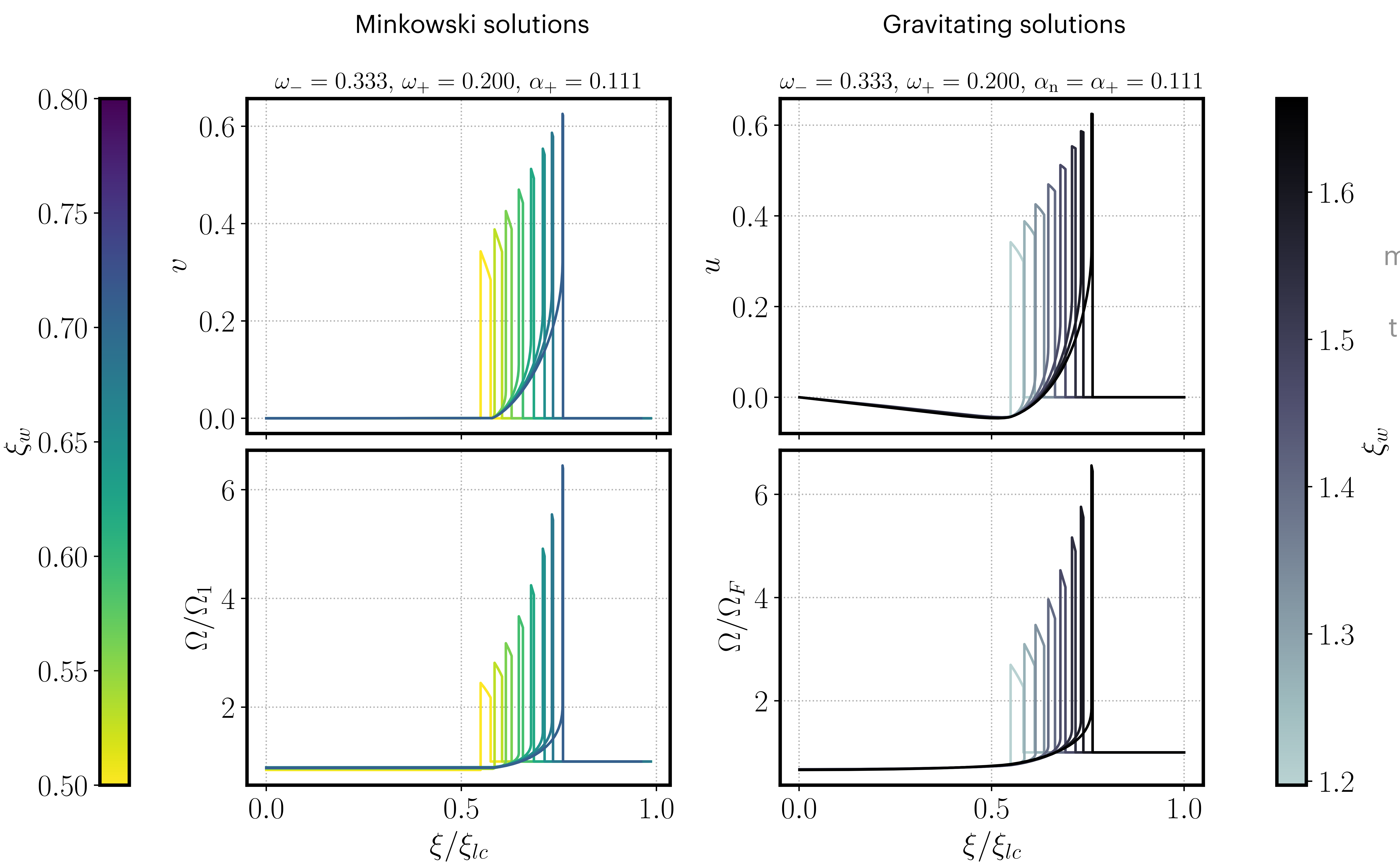



fluid speed measured in the rest frame of the outer FLRW fluid

# Hybrid solutions: $v(\xi_w)_+ = c_{s\_}$ - Comparison with Minkowski

Constant sound speed model

F. Giese et al., ArXiv:2010.09744




fluid speed measured in the rest frame of the outer FLRW fluid

# Curvature of spatial sections around the origin & quasi-Newtonian approximation

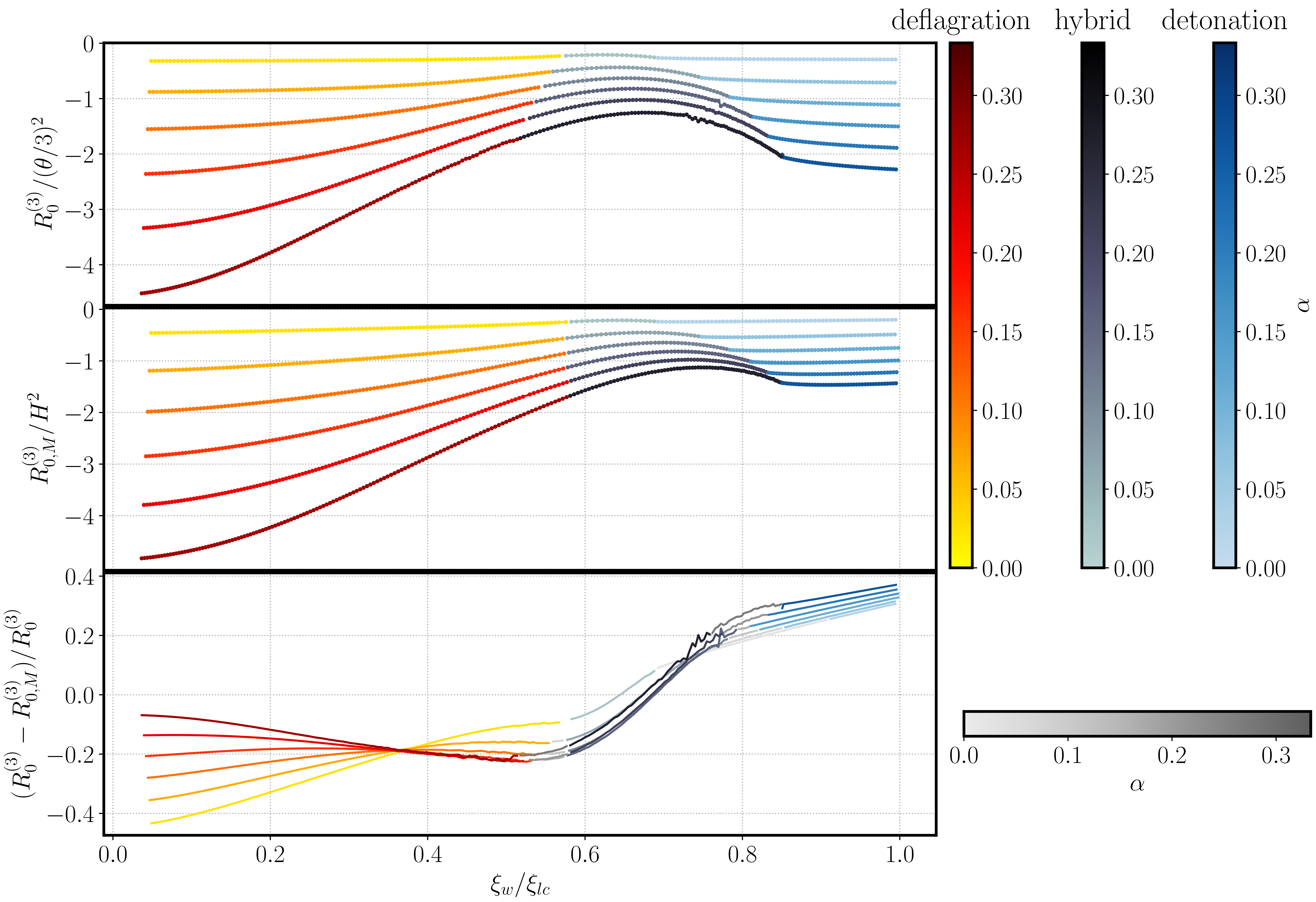
Projection tensor

$$\tilde{h}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

Expansion scalar

$$\theta = \tilde{h}^\alpha{}_\mu \tilde{h}^{\beta\mu} \nabla_\alpha u_\beta$$

$$R_M^{(3)} \simeq 6H^2\delta$$



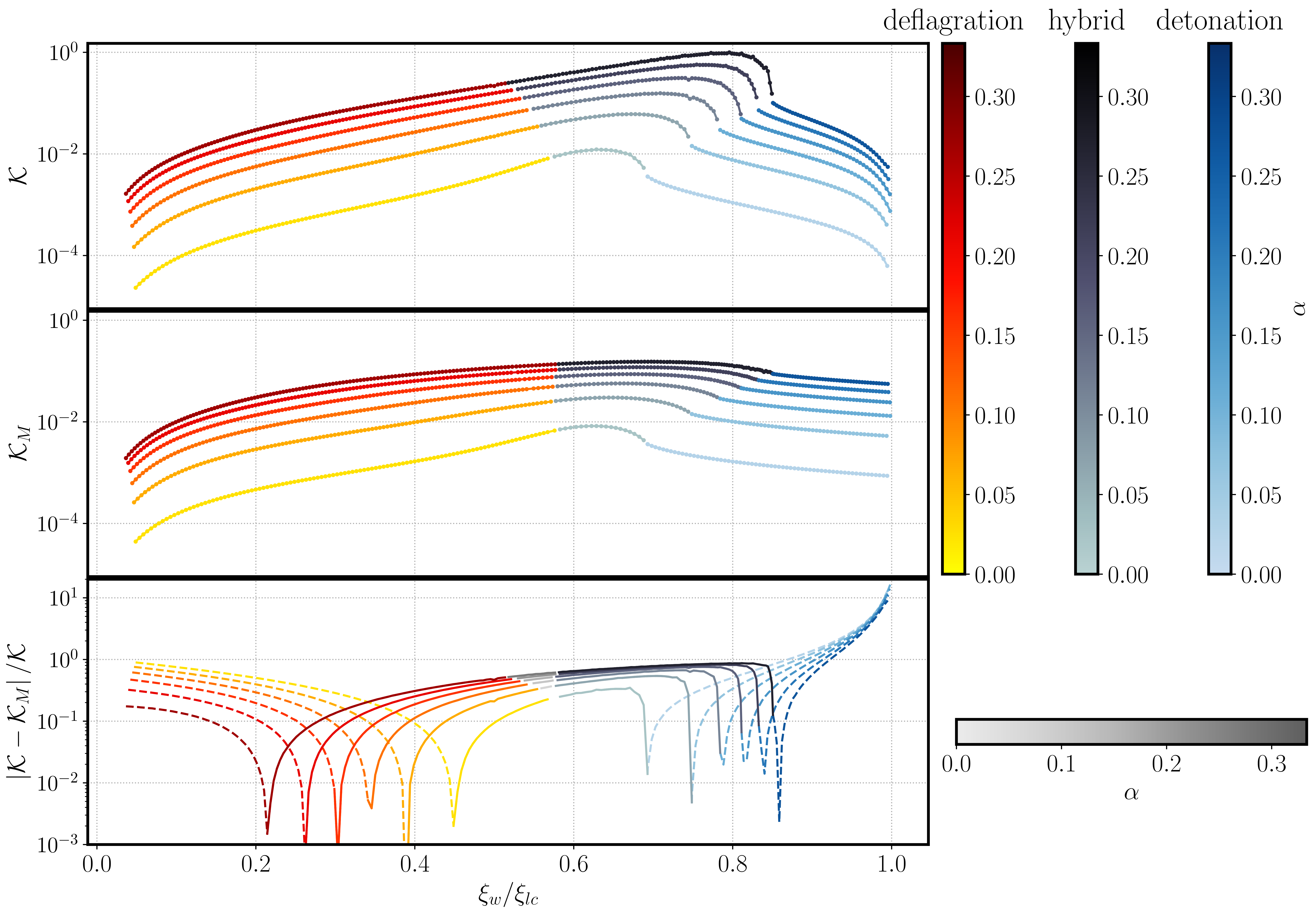
# Kinetic energy fraction

Kinetic energy fraction

$$\mathscr{K} \equiv \frac{E_K}{E_b}$$

$E_K$ : Kinetic energy of the fluid

$E_b$ : Total energy in the bubble



# Conclusions...

- Self-similar gravitating bubble solutions with negative spatial curvature near the origin exist in GR.

- GR effects modify:

    1. Fluid motion and energy around the bubbles

    2. Spatial curvature near the origin

    3. Efficiency of conversion vacuum energy to kinetic energy

# Conclusions...

- Self-similar gravitating bubble solutions with negative spatial curvature near the origin exist in GR.

- GR effects modify:

  1. Fluid motion and energy around the bubbles
  2. Spatial curvature near the origin
  3. Efficiency of conversion vacuum energy to kinetic energy

# ... & future developments

# Conclusions...

- Self-similar gravitating bubble solutions with negative spatial curvature near the origin exist in GR.

- GR effects modify:

  1. Fluid motion and energy around the bubbles → Source primary GW
  2. Spatial curvature near the origin
  3. Efficiency of conversion vacuum energy to kinetic energy

# ... & future developments

# Conclusions...

- Self-similar gravitating bubble solutions with negative spatial curvature near the origin exist in GR.

- GR effects modify:

  1. Fluid motion and energy around the bubbles → Source primary GW
  2. Spatial curvature near the origin
  3. Efficiency of conversion vacuum energy to kinetic energy

# ... & future developments

- Fluid motion generates large scalar perturbations during the phase transition in the limit of large bubbles

# Conclusions...

- Self-similar gravitating bubble solutions with negative spatial curvature near the origin exist in GR.

- GR effects modify:

  1. Fluid motion and energy around the bubbles → Source primary GW
  2. Spatial curvature near the origin
  3. Efficiency of conversion vacuum energy to kinetic energy

## ... & future developments

- Fluid motion generates large scalar perturbations during the phase transition in the limit of large bubbles
  - → Secondary scalar induced GW
  - → Primordial black holes?

# Backup slides

# Asymptotic solutions

$\xi \to 0$

One parameter family of solutions

$$U(\xi \to 0) = \frac{2}{3(1+\omega_-)}\xi$$

$$\Omega(\xi \to 0) = 3k\xi^2$$

$$\Phi(\xi \to 0) = k\xi^2$$

Spatial curvature at the origin

$$R_0^{(3)} = R^{(3)}(\xi \to 0) = \frac{12\xi^2}{R^2}\left[k - \frac{2}{9(1+\omega_-)^2}\right]$$

Since we expect lower energy density in the interior

$$\to \quad 0 < k < \frac{2}{9(1+\omega)^2}$$

# Asymptotic solutions

$\xi \to 0$

One parameter family of solutions

$$U(\xi \to 0) = \frac{2}{3(1+\omega_-)}\xi$$
$$\Omega(\xi \to 0) = 3k\xi^2$$
$$\Phi(\xi \to 0) = k\xi^2$$

Spatial curvature at the origin

$$R_0^{(3)} = R^{(3)}(\xi \to 0) = \frac{12\xi^2}{R^2}\left[k - \frac{2}{9(1+\omega_-)^2}\right]$$

Since we expect lower energy density in the interior

$$0 < k < \frac{2}{9(1+\omega)^2}$$

$\xi \to \infty$

Flat FLRW solution

$$U_F = \frac{2}{3(1+\omega_+)}\frac{\xi}{a_F} \qquad U_F^2 = 2\Phi_F \qquad \Omega_F = 3\Phi_F$$

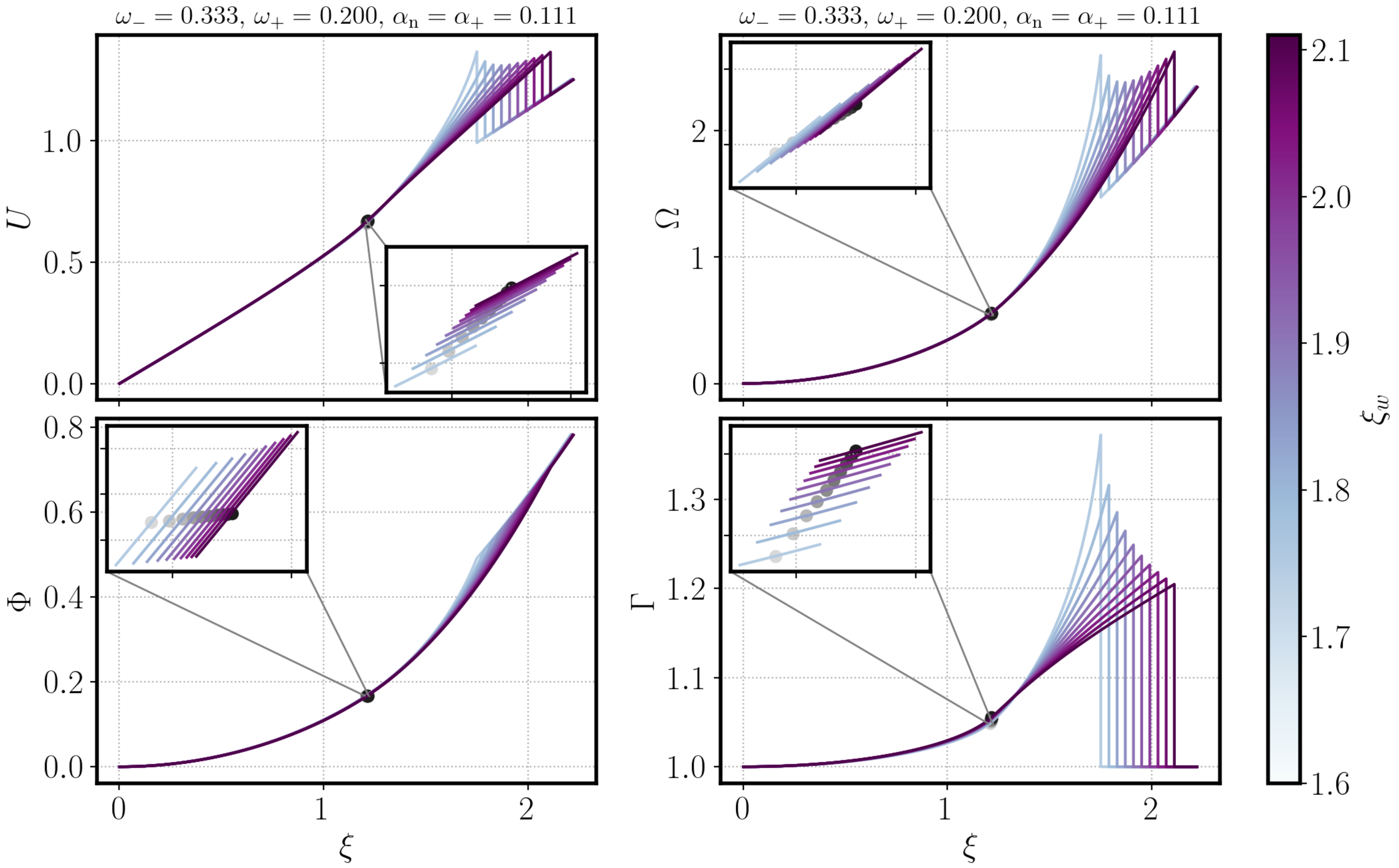
Constant-$\xi$ observers: $V^\mu_\xi = \gamma\left(\frac{1}{a}, \frac{v}{b}, 0, 0\right)$ $\quad v = \frac{\xi - aU}{a\Gamma}$

$$u = \frac{v_F - v}{1 - vv_F}$$

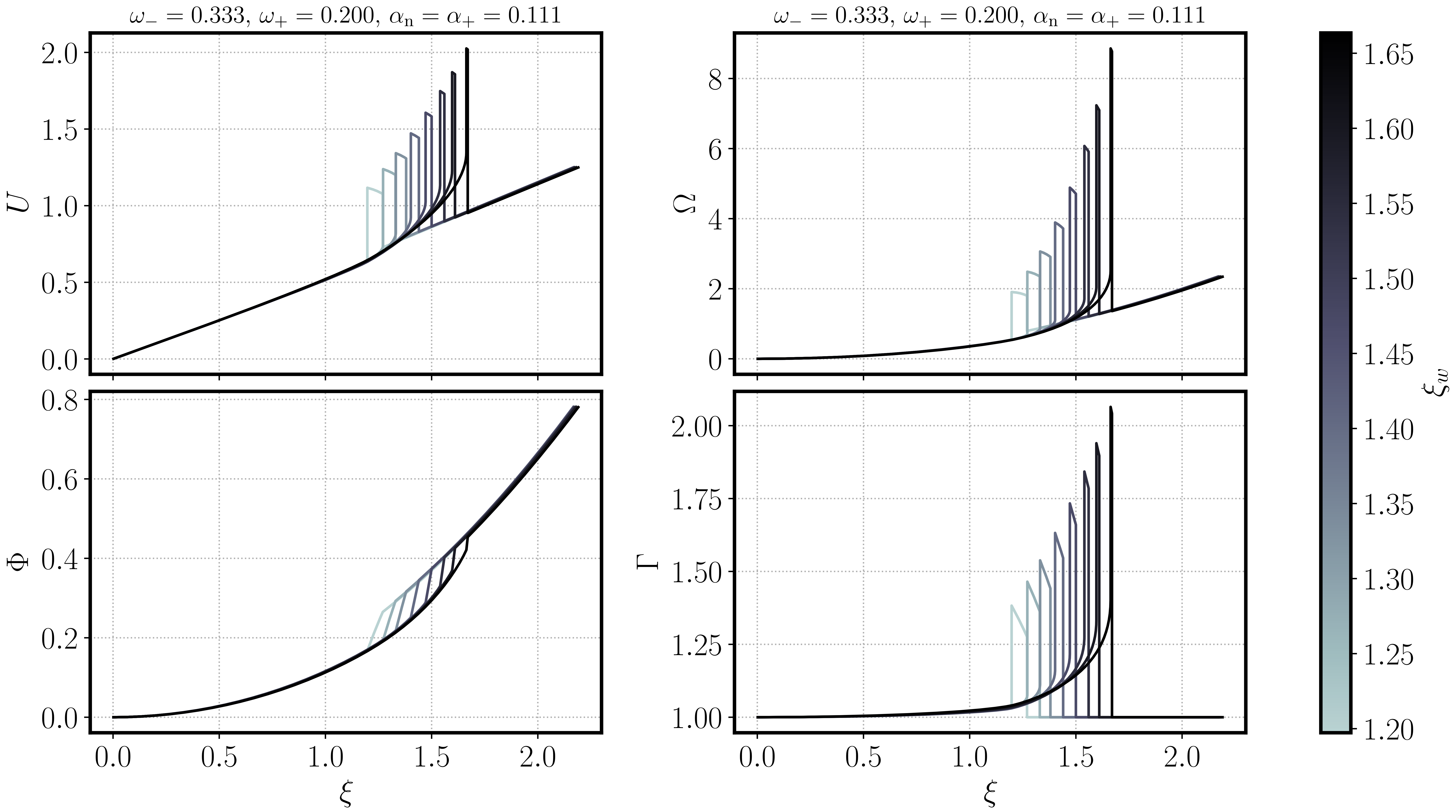
Relative velocity between $V^\mu_\xi$ and another hypothetical constant-$\xi$ observer that lives at the same $\xi$ in FLRW

Measure departure from FLRW

# Detonation solutions: $v(\xi_w)_+ > c_{s\_}$



# Hybrid solutions: $v(\xi_w)_+ = c_{s-}$



# Quasi-Newtonian approximation of the spatial curvature near the origin

Linear scalar perturbations around flat FLRW (Poisson gauge)

$$ds^2 = s^2(\eta)\left[-(1+2\Phi_B)d\eta^2 + (1-2\Psi_B)\eta_{ij}dx^i dx^j\right]$$

Spatial Ricci curvature scalar

$$R^{(3)} = \frac{4}{s^2}\nabla^2\Psi_B$$

Einstein equation

$$\nabla^2\Psi_B \simeq 4\pi s^2 e\left(\frac{\delta e}{e}\right) \simeq \frac{3}{2}s^2H^2\delta$$

Quasi-Newtonian estimate

$$R_M^{(3)} \simeq 6H^2\delta$$

energy density contrast in the Minkowski solutions

F. Giese et al., ArXiv:2010.09744

# Kinetic energy fraction

$$E_b = \int_0^{R_w} 4\pi e_s R^2 dR = \frac{t}{3}\Omega_F(\xi_w)\xi_w,$$

Total amount of energy that was contained in the volume occupied by the bubble before the transition happened

$$e_K \equiv T_{\mu\nu}U^\mu U^\nu - T_{\mu\nu}u^\mu u^\nu = wu^2\gamma_u^2$$

$U^\mu$ Observer moving outward with speed $u$ with respect to the fluid

$$E_K \equiv \int e_K 4\pi R^2 dR = t\int d\xi\,(1+\omega)\Omega u^2\gamma_u^2$$

Amount of the initial energy that has been transferred into kinetic energy of the fluid

Kinetic energy fraction

$$\mathcal{K} \equiv \frac{E_K}{E_b}$$

# Fixed points

i. $(U, \Omega, \Phi) = (0,0,0)$

ii. $U = 0, \quad \Omega = \Phi$

iii. $c_{s_-} = \frac{U_\star}{\Gamma_\star} \frac{\Phi_\star + \omega\Omega_\star}{\Omega_\star - \Phi_\star}, \qquad U_\star = \frac{\Omega_\star - \Phi_\star}{\sqrt{2\Phi_\star}}$

$\overrightarrow{Y}_k = (U_k, \Omega_k, \Phi_k)$ Trajectory of solutions of the Einstein equations with initial condition $k$

The endpoint of $\overrightarrow{Y}_k = (U_k, \Omega_k, \Phi_k)$ move along a line of fixed points $\gamma_\star(k) = (\xi_\star(k), \overrightarrow{Y}_\star(k))$

$\xi_\star(k)$ fixed by the condition $a(\xi \to 0) = 1$