Some Remarks on Quadratic Gravity

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arXiv:2308.11324

OKC@15
18th October 2023

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$(- + + +)$

$c = 1 = \hbar$

$\Lambda \geq 0$

Motivations

Einstein's General Relativity: $$S_{EH} = \frac{M_p^2}{2} \int d^4x \sqrt{-g}(R - 2\Lambda)$$

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2. Observational: cannot explain CMB anisotropies (early times physics)

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1. Theoretical: perturbatively non-renormalizable

2. Observational: cannot explain CMB anisotropies (early times physics)

Simplest model to explain 2. :

$$S = S_{EH} + S_\phi + \cdots\,, \qquad S_\phi \equiv \text{inflaton action}$$

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'Unique' (strictly) renormalizable QFT of gravity in $D = 4$:

[Stelle, PRD (1977)]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(M_p^2(R-2\Lambda)+\frac{\alpha}{6}R^2-\frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

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massive spin-0, $\alpha \sim 10^{10}$

Natural explanation for inflation!

[Starobinsky, 1980+]

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Spin-2 massive ghost
[Salvio & Strumia 2015+;
Anselmi & Piva 2017+;
Donoghue & Menezes 2018+;
Holdom 2015+, etc…]

Task

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(M_p^2(R-2\Lambda)+\frac{\alpha}{6}R^2-\frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

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Question:

$$\lim_{\beta\to\infty} S = \ ?$$

1. Identify the canonically normalized fields
2. Identify masses and interaction couplings
3. Take the limit keeping the canonically normalized fields fixed

Simpler example

$$S = -\frac{1}{4g^2}\int d^4x\, \hat{F}^a_{\mu\nu}\hat{F}^{a\mu\nu}, \qquad \hat{F}^a_{\mu\nu} = \partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu + f^{abc}\hat{A}^b_\mu\, \hat{A}^c_\nu$$

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Canonically normalized field:

$$A^a_\mu = \frac{1}{g}\hat{A}^a_\mu \qquad \Longrightarrow \qquad S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{a\mu\nu},$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu\, A^c_\nu$$

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Limit:

$$\lim_{g\to 0} S \;=\; \frac{1}{2}\int d^4x\, A^a_\mu(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A^a_\nu \qquad (A^a_\mu = \text{fixed})$$

Spectrum

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(M_p^2(R-2\Lambda) + \frac{\alpha}{6}R^2 - \frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

Massless graviton:

$g_{\mu\nu}$ (2 dofs)

Massive spin-0:

ϕ (1 dof)

$$m_0^2 = \frac{M_p^2}{\alpha}$$

Massive spin-2 ghost:

$f_{\mu\nu}$ (5 dofs)

$$m_2^2 = \frac{\widetilde{M}_p^2}{\beta}$$

$$\widetilde{M}_p^2 = M_p^2 + \frac{2}{3}\Lambda(2\alpha + \beta)$$

Spectrum

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$$\widetilde{M}_p^2 = M_p^2 + \frac{2}{3}\Lambda(2\alpha+\beta)$$

$\beta \longrightarrow \infty \quad \Longrightarrow$

$\Lambda = 0$: $\quad m_2^2 \longrightarrow 0$ (massless limit)

$\Lambda \neq 0$: $\quad m_2^2 \longrightarrow \frac{2}{3}\Lambda$ (partially massless limit)

Action in canonical form

$$S[g,\phi,f] = \frac{\widetilde{M}_p^2}{2}\int d^4x\sqrt{-g}(R-2\Lambda) + S_0[g,\phi] + S_2[g,f],$$

$$S_0[g,\phi] = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{m_0^2}{2}\frac{3\overline{M}_p^2}{2}\left(1-e^{\sqrt{2/3}\phi/\overline{M}_p}\right)^2\right]\Bigg|_{g-2f/\widetilde{M}_p}$$

$$S_2[g,f] = -S_{PF}[g,f] - \int d^4x\sqrt{-g}\left[\left(2f_\mu^\rho f_{\rho\nu} - f f_{\mu\nu}\right)R^{\mu\nu} + \left(\Lambda - \frac{R}{2}\right)\left(f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}f^2\right)\right]$$

$$-\frac{1}{2}\frac{m_2^2}{\widetilde{M}_p}\int d^4x\sqrt{-g}\left[5f_{\mu\nu}f^{\mu\nu}f - 4f^{\mu\nu}f_\mu^\rho f_{\rho\nu} - f^3\right]$$

$$+\frac{8}{3}\frac{1}{M_p^2}\frac{1}{\widetilde{M}_p}\int d^4x d^4y d^4z \frac{\delta^{(3)}S_{EH}}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)\delta g_{\alpha\beta}(z)} f_{\mu\nu}(x)f_{\rho\sigma}(y)f_{\alpha\beta}(z) + O(f^4)$$

$$\overline{M}_p^2 \equiv M_p^2 + \frac{4}{3}\alpha\Lambda$$

$$\widetilde{M}_p^2 \equiv M_p^2 + \frac{2}{3}(2\alpha+\beta)\Lambda$$

Couplings

spin-2 sector coupling:

$$\frac{1}{\widetilde{M}_p} = \frac{1}{M_p}\left(\frac{1}{1 + 2\Lambda(2\alpha + \beta)/3M_p^2}\right)^{\frac{1}{2}}$$

spin-0 sector coupling:

$$\frac{1}{\overline{M}_p} = \frac{1}{M_p}\left(\frac{1}{1 + 4\Lambda\alpha/3M_p^2}\right)^{\frac{1}{2}}$$

Couplings dependence on Λ $\Longrightarrow$ additional dependence on α, β !

Features of the limit $\beta \longrightarrow \infty$

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(M_p^2(R-2\Lambda) + \frac{\alpha}{6}R^2 - \frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

$$\lim_{\beta\to\infty} S = \ ?$$

- The limit distinguishes $\Lambda = 0$ & $\Lambda \neq 0$
  (couplings, particle spectrum, enhanced gauge symmetry)
- Limit is regular only in $D = 4$
- When $\Lambda \neq 0$ the resulting theory is much simpler

Result of the limit $\beta \longrightarrow \infty$ $(\Lambda > 0)$

- Massless spin-2 and $\pm 2, \pm 1$ helicities of massive spin-2 ghost decouple
- Massive spin-0 $(\tilde{\phi})$ & helicity-0 $(\tilde{\chi})$ of spin-2 ghost survive

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$$S_{\phi\chi}[g, \tilde{\phi}, \tilde{\chi}] = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(\partial_\mu \tilde{\chi} \partial^\mu \tilde{\chi} - \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right) - V(\tilde{\phi}, \tilde{\chi}) \right]$$

$$V(\tilde{\phi}, \tilde{\chi}) = \frac{\Lambda}{36 \bar{M}_p^2} \left(\tilde{\chi}^2 - \tilde{\phi}^2 - 6 \bar{M}_p^2 \right)^2 + \frac{m_0^2}{12 \bar{M}_p^2} \tilde{\phi}^2 \left(\tilde{\chi} + \tilde{\phi} \right)^2$$

$$\bar{M}_p^2 = M_p^2 + \frac{4}{3} \alpha \Lambda$$

Constraint:

$$T^{(\phi\chi)} = 0, \qquad T_{\mu\nu}^{(\phi\chi)} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\phi\chi}}{\delta g^{\mu\nu}}$$

Ghost instability?

An exercise: Hamiltonian for <u>purely time-dependent</u> fields in de Sitter

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \qquad a(t) = e^{\sqrt{\frac{\Lambda}{3}}t}$$

$$H = a(t)^3 \left[\frac{1}{2}\left(\partial_0 \tilde{\phi}\right)^2 - \frac{1}{2}(\partial_0 \tilde{\chi})^2 + V(\tilde{\phi}, \tilde{\chi})\right]$$

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$$H = a(t)^3 \left[\frac{1}{2}\left(\partial_0\tilde{\phi}\right)^2 - \frac{1}{2}(\partial_0\tilde{\chi})^2 + V(\tilde{\phi},\tilde{\chi})\right]$$

Constraint:

$$T^{(\phi\chi)} = 0 \quad \Leftrightarrow \quad \frac{1}{2}\left(\partial_0\tilde{\phi}\right)^2 - \frac{1}{2}(\partial_0\tilde{\chi})^2 = 2V(\tilde{\phi},\tilde{\chi})$$

$$\Longrightarrow \quad H = 3a(t)^3 V(\tilde{\phi},\tilde{\chi}) \geq 0$$

Summary

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(M_p^2(R-2\Lambda)+\frac{\alpha}{6}R^2-\frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

- Renormalizability + ghost-like nature of spin-2 $\Longrightarrow$ limit $\beta\to\infty$ is regular
- The limit $\beta\to\infty$ depends non-trivially on Λ
- When $\Lambda\neq 0$: structure of degrees of freedom is different (but same number); spin-2 sector decouples; the resulting theory is given by two interacting scalar fields

Implications?

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(M_p^2(R - 2\Lambda) + \frac{\alpha}{6}R^2 - \frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

- Can the limit $\beta \to \infty$ help understand the high-energy behavior of the spin-2 ghost?
- Current quantization approaches to Quadratic Gravity are formulated in flat spacetime. Are they still valid when $\Lambda \neq 0$?
- Role of the cosmological constant in quantum gravity?

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- Can the limit $\beta \to \infty$ help understand the high-energy behavior of the spin-2 ghost?
- Current quantization approaches to Quadratic Gravity are formulated in flat spacetime. Are they still valid when $\Lambda \neq 0$?
- Role of the cosmological constant in quantum gravity?

A nice formula:

$$\Lambda = \frac{3}{2}m_2^2\frac{\beta - M_p^2/m_2^2}{\beta + 2M_p^2/m_0^2}$$

…Extra Slides…

Quadratic Gravity as Quantum Gravity

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(M_p^2(R-2\Lambda)+\frac{\alpha}{6}R^2-\frac{\beta}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)$$

Cosmological constant: $\Lambda \sim 10^{-122} M_p^2$

Natural candidate for inflaton: $\alpha \sim 10^{10}$

[Starobinsky, 1980+]

In my opinion, if we accept these facts very <u>important implications</u> follow:

1. The framework of perturbative QFT and the criterion of renormalizability (as a tool to select theories) are quite successful also when applied to gravity!

2. CMB observations have provided for the first time a <u>test</u> of higher-curvature gravity and an 'indirect' proof of quantized gravity (the scalar field is a gravitational dof)!!

3. Contrary to some beliefs, Starobinsky inflation is <u>not</u> just a model!

Motivations

Obvious question: What about the spin-2 massive ghost?

1. Throw the entire theory away just because maybe we don't know how to deal with the spin-2 ghost?

2. Or, instead, after appreciating the achievements described before, should we feel very motivated to understand the role of the ghost at a deeper level?

I opt for the 2nd option!

Recent proposals to recover unitarity with ghost

S-matrix unitarity and optical theorem:

$$S^{+}S = 1, \qquad S = 1 + iT,$$
$$1 = \sum_{\{n\}} c_n \, |n\rangle\langle n| \qquad \Longrightarrow \qquad 2Im\{\langle a|T|a\rangle\} = \sum_{\{n\}} c_n |\langle n|T|a\rangle|^2$$

Interesting approaches

- Quantize the ghost with negative norms ($c_n < 0$ for ghost states but positive energies) [Salvio, Strumia (2014+); Holdom (2021+); etc]
- Loop corrections make the ghost decay after times of order $\tau \sim M_p^2/m_2^3$: treat the ghost as an unstable particle, unitarity restored for $t > \tau$ [Donoghue, Menezes (2018+)]
- Replace the Feynman $i\epsilon$ with the *Fakeon* prescription and convert the ghost into a purely virtual particle (LHS=0 for ghost cuts and $c_n = 0$ for ghost states) [Anselmi & Piva 2017+]

Additional spin-0 field

$$S[g,\phi] = \frac{\bar{M}_p^2}{2}\int d^4x\,\sqrt{-g}(R-2\Lambda) - \frac{\beta}{4}\int d^4x\,\sqrt{-g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + S_0[g,\phi],$$

$$S_0[g,\phi] = \int d^4x\,\sqrt{-g}\left[-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{m_0^2}{2}\frac{3\bar{M}_p^2}{2}\left(1-e^{\sqrt{2/3}\phi/\bar{M}_p}\right)^2\right]$$

$$\bar{M}_p^2 \equiv M_p^2 + \frac{4}{3}\alpha\Lambda$$

Shifted Planck Mass when $\Lambda \neq 0$

$$m_0^2 \equiv \frac{M_p^2}{\alpha}$$

Mass of the spin-0 field

Additional spin-2 field

Spin-2 field $f_{\mu\nu}$: [Kaku et al. (1977); Hindawi et al. (1996); Tekin (2016); Anselmi & Piva (2018)]

$$S[g,\phi,f] = \frac{\widetilde{M}_p^2}{2}\int d^4x\sqrt{-g}(R-2\Lambda) + S_0[g,\phi]$$
$$-\int d^4x\sqrt{-g}\left[\widetilde{M}_p\left(G_{\mu\nu}+\Lambda g_{\mu\nu}\right)f^{\mu\nu} - \frac{m_2^2}{2}\left(f_{\mu\nu}f^{\mu\nu}-f^2\right)\right]$$

Diagonalizaton: $g_{\mu\nu} \to g_{\mu\nu} - \frac{2}{\widetilde{M}_p} f_{\mu\nu}$

$$S[g,\phi,f] = \frac{\widetilde{M}_p^2}{2}\int d^4x\sqrt{-g}(R-2\Lambda) + S_0\left[g - 2f/\widetilde{M}_p,\phi\right] + S_2[g,f],$$

$$\widetilde{M}_p^2 \equiv M_p^2 + \frac{2}{3}(2\alpha+\beta)\Lambda$$

Shifted Planck Mass when $\Lambda \neq 0$

Additional spin-2 field

$$S_2[g,f] = -S_{PF}[g,f] - \int d^4x\sqrt{-g}\left[\left(2f_\mu^{\ \rho}f_{\rho\nu} - f f_{\mu\nu}\right)R^{\mu\nu} + \left(\Lambda - \frac{R}{2}\right)\left(f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}f^2\right)\right]$$

$$-\frac{1}{2}\frac{m_2^2}{\widetilde{M}_p}\int d^4x\sqrt{-g}\left[5f_{\mu\nu}f^{\mu\nu}f - 4f^{\mu\nu}f_\mu^{\ \rho}f_{\rho\nu} - f^3\right]$$

$$+\frac{8}{3}\frac{1}{M_p^2}\frac{1}{\widetilde{M}_p}\int d^4x d^4y d^4z \frac{\delta^{(3)}S_{EH}}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)\delta g_{\alpha\beta}(z)} f_{\mu\nu}(x)f_{\rho\sigma}(y)f_{\alpha\beta}(z)$$

$$+O(f^4)$$

$S_{PF}[g,f]$ is the covariant Fierz-Pauli action for $f_{\mu\nu}$ with mass m_2^2

$$m_2^2 = \frac{\widetilde{M}_p^2}{\beta} = \frac{M_p^2}{\beta} + \frac{2}{3}\left(2\frac{\alpha}{\beta} + 1\right)\Lambda$$

spin-2 ghost mass depends on Λ !

$$m_2^2 \geq \frac{2}{3}\Lambda$$

$$(\Lambda \geq 0, \beta > 0)$$

Action in canonical form

$$S[g,\phi,f] = \frac{\widetilde{M}_p^2}{2}\int d^4x\sqrt{-g}(R-2\Lambda) + S_0[g,\phi] + S_2[g,f],$$

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$$\overline{M}_p^2 \equiv M_p^2 + \frac{4}{3}\alpha\Lambda$$

$$\widetilde{M}_p^2 \equiv M_p^2 + \frac{2}{3}(2\alpha+\beta)\Lambda$$

Couplings

n-point interaction couplings for the spin-2 sector:

$$\sim \left(\frac{1}{\widetilde{M}_p}\right)^{n-2} = \left(\frac{1}{M_p}\right)^{n-2} \left(\frac{1}{1+2\Lambda(2\alpha+\beta)/3M_p^2}\right)^{\frac{n-2}{2}}$$

Couplings dependence on Λ $\Longrightarrow$ additional dependence on β !

Degrees of freedom

Linear analysis:

$$\frac{\delta S}{\delta f^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \widetilde{M}_p\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right) = m_2^2\left(f_{\mu\nu} - g_{\mu\nu} f\right)$$

4 Constraints: $\boxed{\nabla_\mu f^\mu_\nu = \nabla_\nu f}$

$$g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \widetilde{M}_p\left(m_2^2 - \frac{2}{3}\Lambda\right) f = 0$$

1 trace constraint: $\boxed{f = 0}$ $\quad (m_2^2 - 2\Lambda/3 \neq 0)$

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1 trace constraint: $\boxed{f = 0}$ $\quad (m_2^2 - 2\Lambda/3 \neq 0)$

dof $f_{\mu\nu}$: $\quad 10 - 4 - 1 = 5 \quad \Longrightarrow$ massive spin-2 with 5 helicities

Degrees of freedom: remark

$$\tilde{M}_p \left(m_2^2 - \frac{2}{3}\Lambda \right) f = 0, \qquad\qquad m_2^2 - \frac{2}{3}\Lambda = 0 \quad ?$$

In this case the linear trace equation vanishes identically

Two possibilities:

1. $\Lambda = 0$: $\quad m_2^2 = 0 \quad$ (massless)
2. $\Lambda \neq 0$: $\quad m_2^2 = \frac{2}{3}\Lambda \quad$ (partially massless)

These cases need a separate discussion!

NB:

$$\beta \longrightarrow \infty \qquad \Longrightarrow \qquad m_2^2 - \frac{2\Lambda}{3} = \frac{\bar{M}_p^2}{\beta} \longrightarrow 0$$

Case $\Lambda = 0$

It is a massless limit:

$$\beta \to \infty \quad \Longrightarrow \quad m_2^2 = \frac{M_p^2}{\beta} \to 0$$

NB: typically, the massless limit in theories of Massive Gravity can lead to strong coupling even below M_p. [Reviews by Hinterbichler (2011) and de Rham (2014)]

Digression on Massive Gravity with $\Lambda = 0$

$$S_{MG} = \int d^4x \sqrt{-g}\left[-\frac{1}{2}\nabla_\rho f_{\mu\nu}\nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu}\nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2}\nabla_\rho f \nabla^\rho f - \frac{m_2^2}{2}\left(f_{\mu\nu}f^{\mu\nu} - f^2\right) + O(f^3)\right]$$

Naively, the limit $m_2^2 \to 0$ seems to give a *massless* spin-2 with 2 dofs

$$S_{MG}^{(2)} = \int d^4x \sqrt{-g}\left[-\frac{1}{2}\nabla_\rho f_{\mu\nu}\nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu}\nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2}\nabla_\rho f \nabla^\rho f\right]$$

Gauge symmetry gives 2 dofs

$$\delta f_{\mu\nu}(x) = \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x),$$

Digression on Massive Gravity with $\Lambda = 0$

$$S_{MG} = \int d^4x\,\sqrt{-g}\Big[-\frac{1}{2}\nabla_\rho f_{\mu\nu}\nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu}\nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2}\nabla_\rho f \nabla^\rho f - \frac{m_2^2}{2}\left(f_{\mu\nu}f^{\mu\nu} - f^2\right) + O(f^3)\Big]$$

Stückelberg formalism:

$$f_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{m_2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right) + \frac{2}{m_2^2}\nabla_\mu\nabla_\nu\chi,$$

Gauge symmetries:

$$\delta\varphi_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu,$$
$$\delta A_\mu = -m_2\xi_\mu + \nabla_\mu\xi,$$
$$\delta\chi = -m_2\,\xi,$$

Massless limit (2+2+1 = 5 dofs):

$$S_{MG}[\varphi, A, \chi] = S_{FP}^{(m_2=0)}[\varphi] + \int d^4x\,\sqrt{-g}\left(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - 3\nabla_\rho\chi\nabla^\rho\chi\right) + O(f^3)$$

Possible strong coupling from helicity-0 interactions: $O(f^3) \sim \frac{1}{m_2} O(\chi^3) \;\rightarrow\; \infty$

[Reviews by Hinterbichler (2011) and de Rham (2014)]

Case $\Lambda = 0$

Stückelberg formalism:

$$f_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{m_2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right) + \frac{2}{m_2^2}\nabla_\mu\nabla_\nu\chi$$

The limit is regular in $D = 4$:

$$\lim_{\beta\to\infty} S = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\,R - M_p\int d^4x\sqrt{-g}G_{\mu\nu}\varphi^{\mu\nu} + \frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu} + S_{\phi\chi}[g,\phi,\chi]$$

$$S_{\phi\chi}[g,\phi,\chi] = \int d^4x\sqrt{-g}\left[\frac{1}{2}e^{-\sqrt{2/3}\chi/M_p}\left(\nabla_\mu\chi\nabla^\mu\chi - \nabla_\mu\phi\nabla^\mu\phi\right) - \frac{m_0^2}{2}\frac{3M_p^2}{2}e^{-2\sqrt{2/3}\chi/M_p}\left(1 - e^{\sqrt{2/3}\phi/M_p}\right)^2\right]$$

Case $\Lambda = 0$

Stückelberg formalism:

$$f_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{m_2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right) + \frac{2}{m_2^2}\nabla_\mu \nabla_\nu \chi$$

The limit is regular in $D = 4$:

$$\lim_{\beta\to\infty} S = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\,R - M_p\int d^4x\sqrt{-g}G_{\mu\nu}\varphi^{\mu\nu} + \frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu} + S_{\phi\chi}[g,\phi,\chi]$$

$$S_{\phi\chi}[g,\phi,\chi] = \int d^4x\sqrt{-g}\left[\frac{1}{2}e^{-\sqrt{2/3}\chi/M_p}\left(\nabla_\mu\chi\nabla^\mu\chi - \nabla_\mu\phi\nabla^\mu\phi\right) - \frac{m_0^2}{2}\frac{3M_p^2}{2}e^{-2\sqrt{2/3}\chi/M_p}\left(1 - e^{\sqrt{2/3}\phi/M_p}\right)^2\right]$$

$f_{\mu\nu}$ splits into 5 massless ghost-like dofs (± 2, ± 1, 0)

[Saravani & Hinterbichler (2015)]

Limit $\beta \to \infty$ with $\Lambda = 0$: massless limit

Does a strong coupling (below M_p) arise in quadratic gravity?

[first asked by Hinterbichler & Saravani (2016)]

A strong coupling in the limit $m_2^2 \to 0$ (i.e., $\beta \to \infty$) can be avoided *only* in $D = 4$!

Stückelberg decomposition for $\Lambda = 0$ and in D dimensions:

$$f_{\mu\nu} = \tilde{f}_{\mu\nu} + \frac{1}{m_2}(\nabla_\mu \tilde{A}_\nu + \nabla_\nu \tilde{A}_\mu), \qquad \tilde{A}_\mu = A_\mu + \frac{1}{m_2}\nabla_\mu \chi$$

$$\Rightarrow S_2'[g,f] = \frac{M_p^{D-2}}{2}\int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g}\left[-M_p^{\frac{D-2}{2}} G_{\mu\nu} f^{\mu\nu} + \frac{m_2^2}{2}(f_{\mu\nu} f^{\mu\nu} - f^2)\right]$$

$$= \frac{M_p^{D-2}}{2}\int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g}\left[-M_p^{\frac{D-2}{2}} G_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{m_2^2}{2}(\tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu} - \tilde{f}^2)\right.$$

$$\left. + \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + 2 m_2 \tilde{f}^{\mu\nu}(\nabla_\mu \tilde{A}_\nu - g_{\mu\nu} \nabla^\rho \tilde{A}_\rho) - 2 R^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu \right],$$

Possible strong coupling from $R^{\mu\nu}\tilde{A}_\mu \tilde{A}_\nu \sim \frac{1}{m_2^2} R^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi$???

Limit $\beta \to \infty$ with $\Lambda = 0$: massless limit

Make a field redefinition:

$$\tilde{f}_{\mu\nu} \to \tilde{f}_{\mu\nu} + a\, \tilde{A}_\mu \tilde{A}_\nu + b\, g_{\mu\nu}\, \tilde{A}_\rho \tilde{A}^\rho$$

In the massless limit $m_2^2 \to 0$ $(\beta \to \infty,\ \Lambda = 0)$ we get

$$\Longrightarrow \quad S_2'[g, \tilde{f}, \tilde{A}] = \frac{M_p^{D-2}}{2} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \Big[-M_p^{\frac{D-2}{2}} G_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{m_2^2}{2} (\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu} - \tilde{f}^2) + \frac{1}{2} F^{\mu\nu} F_{\mu\nu}$$

$$+2 m_2 \tilde{f}^{\mu\nu} \left(\nabla_\mu \tilde{A}_\nu - g_{\mu\nu} \nabla^\rho \tilde{A}_\rho \right) + m_2^2 a\, \tilde{f}^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu + m_2^2 [b(1-D) - a] \tilde{f} \tilde{A}_\rho \tilde{A}^\rho$$

$$-\left(a M_p^{\frac{D-2}{2}} + 2 \right) R^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu + M_p^{\frac{D-2}{2}} \left(\left(1 - \tfrac{D}{2}\right) b - \tfrac{a}{2} \right) R \tilde{A}_\rho \tilde{A}^\rho$$

$$-m_2 (2b(1-D) - 3a) \tilde{A}_\mu \tilde{A}_\nu \nabla^\mu \tilde{A}^\nu - \frac{m_2^2}{2} (b^2 D(1-D) + 2ab(1-D)) \left(\tilde{A}_\rho \tilde{A}^\rho \right)^2 \Big]$$

4 conditions to avoid strong coupling in the massless limit:

$$a M_p^{\frac{D-2}{2}} + 2 = 0, \qquad 2b(1-D) - 3a = 0,$$

$$\left(1 - \tfrac{D}{2}\right) b - \tfrac{a}{2} = 0, \qquad b^2 D(1-D) + 2ab(1-D) = 0$$

can be simultaneously satisfied *only* in $D = 4$!!!

$$a = -\frac{2}{M_p} = -2b,$$

Case $\Lambda > 0$

It is NOT a massless limit:

$$\beta \to \infty \qquad \Longrightarrow \qquad m_2^2 = \frac{\bar{M}_p^2}{\beta} + \frac{2}{3}\Lambda \longrightarrow \frac{2}{3}\Lambda$$

NB: In Massive Gravity theories this limit is known as *partially massless limit* and in general may lead to strong coupling! [de Rham et al. (2018)]

Digression on Massive Gravity with $\Lambda > 0$

$$S_{MG} = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}\nabla_\rho f_{\mu\nu}\nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu}\nabla^\mu f^{\rho\nu} - \nabla_\mu f\nabla_\nu f^{\mu\nu} + \frac{1}{2}\nabla_\rho f\nabla^\rho f$$
$$+\Lambda\left(f_{\mu\nu}f^{\mu\nu} - \tfrac{1}{2}f^2\right) - \tfrac{m_2^2}{2}\left(f_{\mu\nu}f^{\mu\nu} - f^2\right) + O(f^3)\Big]$$

Naively, the limit $m_2^2 \to \frac{2}{3}\Lambda$ seems to give a *partially massless* spin-2 with 4 dofs

$$S_{MG}^{(2)} = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}\nabla_\rho f_{\mu\nu}\nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu}\nabla^\mu f^{\rho\nu} - \nabla_\mu f\nabla_\nu f^{\mu\nu} + \frac{1}{2}\nabla_\rho f\nabla^\rho f$$
$$+\Lambda\left(f_{\mu\nu}f^{\mu\nu} - \tfrac{1}{2}f^2\right) - \tfrac{\Lambda}{3}\left(f_{\mu\nu}f^{\mu\nu} - f^2\right)\Big]$$

Scalar gauge symmetry (10-4-2=4 dofs)

$$\delta f_{\mu\nu}(x) = \nabla_\mu\nabla_\nu\zeta(x) + \frac{\Lambda}{3}g_{\mu\nu}\zeta(x),$$

[Deser, Nepomechie (1984); Deser, Waldron (2001+); etc]

Digression on Massive Gravity with $\Lambda > 0$

$$S_{MG} = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}\nabla_\rho f_{\mu\nu}\nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu}\nabla^\mu f^{\rho\nu} - \nabla_\mu f\nabla_\nu f^{\mu\nu} + \frac{1}{2}\nabla_\rho f\nabla^\rho f$$
$$+\Lambda\left(f_{\mu\nu}f^{\mu\nu} - \tfrac{1}{2}f^2\right) - \frac{m_2^2}{2}\left(f_{\mu\nu}f^{\mu\nu} - f^2\right) + O(f^3)\Big]$$

Stückelberg trick

$$f_{\mu\nu} = \varphi_{\mu\nu} + \sqrt{\frac{3}{\Lambda}}\frac{1}{\Delta}\left(\nabla_\mu\nabla_\nu\chi + g_{\mu\nu}\frac{\Lambda}{3}\chi\right)$$

$$\Delta \equiv m_2^2 - \frac{2}{3}\Lambda$$

Gauge symmetries:

$$\delta\varphi_{\mu\nu} = \nabla_\mu\nabla_\nu\zeta + \frac{\Lambda}{3}g_{\mu\nu}\zeta,$$

$$\delta\chi = -\sqrt{\frac{\Lambda}{3}}\,\Delta\,\zeta,$$

Partially massless limit $\Delta \to 0$ (4+1=5 dofs):

$$S_{MG}[\varphi,\chi] = S_{FP}^{(\Delta=0)}[\varphi] + 3\int d^4x\sqrt{-g}\left(-\frac{1}{2}\nabla_\rho\chi\nabla^\rho\chi - \frac{m_\chi^2}{2}\chi^2\right) + O(f^3), \qquad m_\chi^2 \equiv -\frac{4}{3}\Lambda$$

Possible strong coupling from χ interactions: $O(f^3) \sim \frac{1}{\Delta}O(\chi^3) \;\to\; \infty$

[de Rham, Hinterbichler, Johnson (2018)]

Case $\Lambda > 0$

Stückelberg formalism:

$$f_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{\sqrt{\Lambda}\sqrt{m_2^2 - 2\Lambda/3}}\left(\nabla_\mu\nabla_\nu\chi + \frac{\Lambda}{3}g_{\mu\nu}\chi\right)$$

$$\lim_{\beta\to\infty} S = \frac{\widetilde{M}_p^2}{2}\int d^4x\,\sqrt{-g}(R - 2\Lambda)$$
$$+\int d^4x\,\sqrt{-g}\left[-\widetilde{M}_p\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)\varphi^{\mu\nu} + \frac{\Lambda}{3}\left(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2\right)\right] + S_{\phi\chi}$$

$$S_{\phi\chi}[g,\phi,\chi] = \int d^4x\,\sqrt{-g}\left[\frac{1}{2}e^{-\sqrt{2/3}\chi/\bar{M}_p}\left(\nabla_\mu\chi\nabla^\mu\chi - \nabla_\mu\phi\nabla^\mu\phi\right)\right.$$
$$\left. -\Lambda\,\bar{M}_p^2\left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p}\right)^2 - \frac{m_0^2}{2}\frac{3\bar{M}_p^2}{2}e^{-2\sqrt{2/3}\chi/\bar{M}_p}\left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p}\right)^2\right]$$

$f_{\mu\nu}$ splits into 1 partially massless graviton (4 dof) + 1 scalar dof

Case $\Lambda > 0$

Interaction couplings: $\frac{1}{\tilde{M}_p} \sim \frac{1}{\sqrt{\Lambda\beta}}$ for spin-2 sector & $\frac{1}{\bar{M}_p}$ for spin-0 sector

- Expand in $\varphi_{\mu\nu}$ and $g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{2}{\tilde{M}_p} h_{\mu\nu}$
- Diagonalize kinetic term for $h_{\mu\nu}$ and $\varphi_{\mu\nu}$

$$\lim_{\beta\to\infty} S = S^{(2)}_{EH}[\bar{g}, h] + S^{(2)}_{2}[\bar{g}, \varphi]\,|_{m_2^2 = \frac{2}{3}\Lambda} + S_{\phi\chi}[\bar{g}, \phi, \chi]$$

Spin-2 sector completely decouples!

The only compatible metric background is

$$\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$$

Case $\Lambda > 0$

Resulting interacting theory:

$$S_{\phi\chi}[g,\phi,\chi] = \int d^4x\,\sqrt{-g}\left[\frac{1}{2}e^{-\sqrt{2/3}\chi/\bar{M}_p}\left(\nabla_\mu\chi\nabla^\mu\chi - \nabla_\mu\phi\nabla^\mu\phi\right)\right.$$

$$\left.-\Lambda\,\bar{M}_p^2\left(1-e^{\sqrt{2/3}\phi/\bar{M}_p}\right)^2 - \frac{m_0^2}{2}\frac{3\bar{M}_p^2}{2}e^{-2\sqrt{2/3}\chi/\bar{M}_p}\left(1-e^{\sqrt{2/3}\phi/\bar{M}_p}\right)^2\right]$$

$$\bar{M}_p^2 = M_p^2 + \frac{4}{3}\alpha\Lambda$$

Constraint:

$$g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \tilde{M}_p\left(m_2^2 - \frac{2}{3}\Lambda\right)\varphi = -3T^{(\phi\chi)}, \qquad \varphi = g^{\mu\nu}\varphi_{\mu\nu}, \qquad T_{\mu\nu}^{(\phi\chi)} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\phi\chi}}{\delta g^{\mu\nu}}$$

$$\beta \longrightarrow \infty \quad \Longrightarrow \quad m_2^2 \longrightarrow \frac{2}{3}\Lambda \quad \Longrightarrow \quad T^{(\phi\chi)} = 0$$

Physical implications?

$$g_2^2 \equiv \frac{2}{\beta}$$

$$\frac{dg_2^2}{d\tau} = -\frac{133}{10} g_2^4, \qquad g_2^2(\tau) = \frac{g_2^2(0)}{1 + \frac{133}{10} g_2^2(0)\tau}, \qquad \tau = \frac{1}{(4\pi)^2} \log \mu/\mu_0$$

g_2^2 (β) decreases (grows) with the energy

[Avramidi & Barvinsky (1985); etc]

Physical implications?

$$g_2^2 \equiv \frac{2}{\beta}$$

$$\frac{dg_2^2}{d\tau} = -\frac{133}{10} g_2^4, \qquad g_2^2(\tau) = \frac{g_2^2(0)}{1 + \frac{133}{10} g_2^2(0)\tau}, \qquad \tau = \frac{1}{(4\pi)^2} \log \mu/\mu_0$$

g_2^2 (β) decreases (grows) with the energy

[Avramidi & Barvinsky (1985); etc]

Limit $\beta \longrightarrow \infty$ as high (infinite)-energy limit ?