Problem sheet / winter school 2024 / Intro cosmology

Problem 1:

geodesic equation: $\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$

(a) $ds^2 = -dt^2 + a^2\delta_{ij}dx^i dx^j \rightsquigarrow g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & a^2 & & \\ & & a^2 & \\ 0 & & & a^2 \end{pmatrix} \rightsquigarrow g^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & \frac{1}{a^2} & & \\ & & \frac{1}{a^2} & \\ 0 & & & \frac{1}{a^2} \end{pmatrix}$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right)$$

$$\Gamma^0_{\beta 0} = \Gamma^0_{0\beta} = \frac{1}{2}g^{00}\left(\underbrace{\partial_0 g_{\beta 0}}_{=0} + \underbrace{\partial_\beta g_{00}}_{=0} - \underbrace{\partial_0 g_{0\beta}}_{=0}\right) = 0$$

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\left(\underbrace{\partial_0 g_{0\lambda}}_{=0} + \underbrace{\partial_0 g_{0\lambda}}_{=0} - \underbrace{\partial_\lambda g_{00}}_{=0}\right) = 0$$

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\underbrace{\partial_i g_{jl}}_{=0} + \underbrace{\partial_j g_{il}}_{=0} - \underbrace{\partial_l g_{ij}}_{=0}\right) = 0$$

$$\Gamma^0_{ij} = \frac{1}{2}g^{00}\left(\underbrace{\partial_i g_{0j}}_{=0} + \underbrace{\partial_j g_{0i}}_{=0} - \partial_0 g_{ij}\right) = +\frac{1}{2}\frac{1}{a^2}(a^2)^{\cdot} g_{ij} = +\frac{\dot a}{a}g_{ij} = H g_{ij}$$

$$\Gamma^i_{0j} = \frac{1}{2}g^{il}\left(\partial_0 g_{jl} + \underbrace{\partial_j g_{0l}}_{=0} - \underbrace{\partial_l g_{0j}}_{=0}\right) = \frac{1}{2}\frac{1}{a^2}(a^2)^{\cdot} g^{il}g_{jl} = \frac{\dot a}{a}\delta^i_l = H\delta^i_l$$

(b) $P^\mu = \frac{dx^\mu}{d\lambda} = (E, p^i)$

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \rightsquigarrow \frac{d}{d\lambda}P^\mu = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta$$

$\mu = 0$: $\frac{dE}{d\lambda} = -a\dot a\,\delta_{il}P^i P^l$

use $P_\mu P^\mu = g_{\mu\nu}P^\mu P^\nu = 0 \Rightarrow -(P^0)^2 + a^2\delta_{ij}P^iP^l \Rightarrow E^2 = a^2\delta_{ij}P^iP^l$

and $\frac{dE}{d\lambda} = \frac{dt}{d\lambda}\frac{dE}{dt} = E\frac{dE}{dt}$

$\rightsquigarrow E\frac{dE}{dt} = -\frac{\dot a}{a}E^2 \Rightarrow \frac{1}{E}\frac{dE}{dt} = -\frac{1}{a}\frac{da}{dt} \Rightarrow \frac{d}{dt}\ln(aE) = 0$

(c) $\frac{d}{dt}\ln(aE) = 0 \Rightarrow E \propto a^{-1} \Rightarrow \lambda \propto a \Rightarrow z = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a_0 - a_1}{a_1}$

w.l.o.g. $a_0 \equiv 1 \Rightarrow z = \frac{1 - a(t)}{a(t)}$

Problem 2:

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{pmatrix}$$

$$\nabla_\mu T^{\mu}{}_{\nu} = \partial_\mu T^{\mu}{}_{\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda}{}_{\nu} - \Gamma^{\lambda}_{\nu\mu} T^{\mu}{}_{\lambda} \overset{!}{=} 0$$

$$\nu = 0 \Rightarrow \partial_0 T^0{}_0 + \Gamma^{\mu}_{\mu 0} T^0{}_0 - \Gamma^{\lambda}_{0\mu} T^{\mu}{}_{\lambda} = 0$$

$$\Rightarrow -\dot{\rho} + \Gamma^{i}_{i0}(-\rho) - \Gamma^{i}_{0j} T^{j}{}_{i} = 0$$

$$\Rightarrow -\dot{\rho} - 3H\rho - 3HP = 0$$

$$\Rightarrow \underline{\underline{\dot{\rho} + 3H(\rho + P) = 0}}$$

(b) $\dot{\rho} + 3H(\rho + P) = 0$

$$\Rightarrow \frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}P = 0$$

$$\overset{a\neq 0}{\Longrightarrow} \frac{d}{dt}(a^3\rho) + P\frac{d}{dt}(a^3) = 0$$

$$\Rightarrow \frac{d}{dt}(V\rho) + P\frac{d}{dt}V = 0$$

$$\Rightarrow \underline{\underline{dU + PdV = 0}}$$

(c) $P(t) = w\,\rho(t)$

$$\Rightarrow \dot{\rho} + 3H(1+w)\rho = 0$$

$$\Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0$$

$$\Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

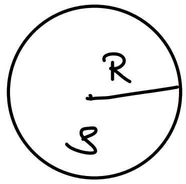
$$\Rightarrow d\ln\rho = -3(1+w)\,d\ln a$$

$$\Rightarrow \ln\rho = -3(1+w)\ln a + C_0$$

$$\Rightarrow \rho = \exp(C_0)\exp(-3(1+w)\ln a)$$

$$\Rightarrow \rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Problem 3:



(a)

grav. force

$$\ddot{R} = -\frac{GM}{R^2} \qquad \text{where} \quad M = \frac{4\pi}{3}R^3\rho = \text{const}$$

$$\Rightarrow \dot{R}\ddot{R} = -GM\frac{\dot{R}}{R^2} \Rightarrow \frac{1}{2}\dot{R}^2 = +GM\frac{1}{R} + E$$

E is energy per unit mass of particle.

$$\Rightarrow \dot{R}^2 = +\frac{8\pi}{3}G\rho\, R^2 + 2E$$

identify: $R = R_0 a(t)$ & $E = -\frac{K}{2}$ $\Rightarrow$ $\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2R_0^2}}$

Friedmann

(b) transformation $a\,d\eta = dt$

$$\Rightarrow \dot{a} = \frac{da}{dt} = \frac{da}{d\eta}\frac{d\eta}{dt} = a'\frac{1}{a}\left(=\mathcal{H}\right)$$

$$\Rightarrow \ddot{a} = \frac{d}{dt}\left(\frac{a'}{a}\right) = \frac{d}{d\eta}\left(\frac{a'}{a}\right)\frac{d\eta}{dt} = \left(\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)\frac{1}{a}$$

1. Friedmann: $\boxed{\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho}$

2 Friedmann: $\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = -\frac{4\pi G}{3}a^2(\rho + 3P)$

$$\Rightarrow \boxed{\frac{a''}{a} = -\frac{4\pi G}{3}a^2(-\rho + 3P)}$$

(c) ansatz: $a(t) = \left(\frac{t}{t_0}\right)^\alpha \Rightarrow \dot{a} = \alpha\left(\frac{t}{t_0}\right)^{\alpha-1}\frac{1}{t_0} \Rightarrow \ddot{a} = (\alpha-1)\alpha\left(\frac{t}{t_0}\right)^{\alpha-2}\frac{1}{t_0^2}$

$H(t) = \frac{\dot{a}}{a} = \frac{\alpha}{t} \Rightarrow H_0 = \frac{\alpha}{t_0}$; moreover $\rho(t) = \rho_0 a^{-3(1+w)}$

Friedmann 1: $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)} \Rightarrow \alpha^2\frac{1}{t^2} = \frac{8\pi G}{3}\rho_0\left(\frac{t}{t_0}\right)^{-3(1+w)}$

$\to H_0^2 = \frac{8\pi G}{3}\rho_0$ and $-2 = -3(1+w)\alpha$ $\Rightarrow \boxed{\alpha = \frac{2}{3(1+w)}}$

Friedmann 2: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_0 a^{-3(1+w)}(1+3w) \Rightarrow \frac{\alpha(\alpha-1)}{t^2} = -\frac{4\pi G}{3}\rho_0\left(\frac{t}{t_0}\right)^{-3(1+w)\frac{2}{3}\frac{1}{1+w}}(1+3w)$

$$\Rightarrow \quad \alpha(\alpha-1) = -\frac{4\pi G}{3}\rho_0 t_0^2 (1+3w) \Rightarrow \alpha(\alpha-1) = -\frac{1}{2}H_0^2 t_0^2 (1+3w)$$

$$\Rightarrow \quad \alpha(\alpha-1) = -\frac{1}{2}\alpha^2(1+3w) \Rightarrow \alpha - 1 = -\frac{1}{2}\alpha(1+3w) \Rightarrow \frac{3}{2}\alpha(1+w) = 1 \quad \checkmark$$

<u>Conformal time</u>: $a(\eta) = \left(\frac{\eta}{\eta_0}\right)^\beta \Rightarrow a' = \beta\left(\frac{\eta}{\eta_0}\right)^{\beta-1}\frac{1}{\eta_0} \Rightarrow a'' = (\beta-1)\beta\left(\frac{\eta}{\eta_0}\right)^{\beta-2}\frac{1}{\eta_0^2}$

Friedmann 1: $\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-1-3w} \Rightarrow \frac{\beta^2}{\eta^2} = \frac{8\pi G}{3}\rho_0 \left(\frac{\eta}{\eta_0}\right)^{(-1-3w)\beta}$

$$\Rightarrow \frac{\beta^2}{\eta_0^2} = \mathcal{H}_0^2 = \frac{8\pi G}{3}\rho_0 \quad \text{and} \quad -2 = (-1-3w)\beta \Rightarrow \boxed{\beta = \frac{2}{1+3w}}$$

Friedmann 2: $\frac{a''}{a} = +\frac{4\pi G}{3}(1-3w)\rho_0 a^{-3(1+w)+2}$

$$\Rightarrow \beta(\beta-1)\frac{1}{\eta^2} = \frac{4\pi G}{3}(1-3w)\rho_0\left(\frac{\eta}{\eta_0}\right)^{[-3(1+w)+2]\beta}$$

$$\Rightarrow \frac{\beta(\beta-1)}{\eta_0^2} = \frac{4\pi G}{3}(1-3w)\rho_0 \quad \text{and} \quad [-3(1+w)+2]\beta = -2 \quad \checkmark$$

$$\Rightarrow \frac{\beta-1}{\beta} = \frac{1}{2}(1-3w) \Rightarrow 2(\beta-1) = \beta(1-3w) \Rightarrow \beta(1+3w) = 2 \quad \checkmark$$

Problem 4

$$ds^2 = a^2(\eta)\left\{-(1+2A)d\eta^2 + 2\partial_i B\, dx^i d\eta + \left[(1+2C - \tfrac{2}{3}\nabla^2 E)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j\right.$$

small coordinate trafo $\tilde{x}^\mu(x) = x^\mu + \xi^\mu(x)$, where $\xi^\mu = (\xi^0, \partial^i \xi) \ll x^\mu$ and $x^\mu = (\eta, x^i)$

$$\Rightarrow \quad g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}(\tilde{x})$$

$$= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}(x^\mu + \xi^\mu)$$

$$= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\left(\tilde{g}_{\alpha\beta}(x) + \xi^\gamma \frac{\partial}{\partial x^\gamma}\tilde{g}_{\alpha\beta}\right)$$

$$= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\left(\tilde{g}_{\alpha\beta} + 2\mathcal{H}\xi^0\tilde{g}_{\alpha\beta}\right)$$

$\boxed{00}$: $$g_{00}(x) = \left(1 + \frac{d}{d\eta}\xi^0\right)\left(1 + \frac{d}{d\eta}\xi^0\right)\tilde{g}_{00}\left(1 + 2\mathcal{H}\xi^0\right) + \underbrace{\frac{d}{d\eta}\xi^i\frac{d}{d\eta}\xi^j\left(\tilde{g}_{ij} + 2\mathcal{H}\xi^0\tilde{g}_{ij}\right)}_{=\mathcal{O}(\xi^2)\simeq 0}$$

$$= \left(1 + 2(\xi^0)'\right)\tilde{g}_{00}\left(1 + 2\mathcal{H}\xi^0\right)$$

$$\Rightarrow 1 + 2A = \left(1 + 2(\xi^0)'\right)(1 + 2\tilde{A})\left(1 + 2\mathcal{H}\xi^0\right) \quad \Rightarrow 1 + 2A = 1 + 2(\xi^0)' + 2\tilde{A} + 2\mathcal{H}\xi^0$$

$$\Rightarrow \tilde{A} = \underline{A - (\xi^0)'} - \mathcal{H}\xi^0$$

Newtonian gauge: $\tilde{B} = \tilde{E} = 0 \Rightarrow B + \xi^0 - \xi' = 0$ & $E - \xi = 0$

$\Rightarrow$ choose $\underline{\xi = E}$ and $\underline{\xi^0 = E' - B}$

Enough freedom because $\xi(\eta, \vec{x})$ and $\xi^0(\eta, \vec{x})$ are independent.

Newtonian $\rightarrow$ spatially flat gauge ($\tilde{C} = \tilde{E} = 0$)

start with $B = E = 0$ & $A = \Psi$ & $C = -\Phi$ (Newtonian gauge)

now demand $\tilde{C} = 0$ & $\tilde{E} = 0 \Rightarrow C - \mathcal{H}\xi^0 - \frac{1}{3}\nabla^2\xi = 0$ & $\underbrace{E}_{=0} - \xi = 0$

$\Rightarrow -\Phi - \mathcal{H}\xi^0 = 0$ & $\xi = 0 \Rightarrow \xi^0 = -\frac{1}{\mathcal{H}}\Phi$ & $\xi = 0$

We see that we can go to spatially flat gauge by shifting the time coordinate.

Problem 5

(a) $\mathcal{R} = -C - \frac{1}{3}K^2 E - \mathcal{H}(v+B)$

$\Rightarrow \tilde{\mathcal{R}} = -C - \frac{1}{3}K^2 E - \mathcal{H}(v+B) - \mathcal{H}\xi^0 + \frac{1}{3}K^2\xi - \frac{1}{3}K^2\xi + \mathcal{H}(+\xi' + \xi^0 - \xi')$

$= \mathcal{R}$

$\mathcal{R} = \Phi - \mathcal{H}v$ (in Newtonian gauge: $B=E=0$)

(b) $\mathcal{R}' = \Phi' - \mathcal{H}'v - \mathcal{H}v'$

$= \underline{\Phi'} - \mathcal{H}'v + \left(\mathcal{H}^2 v + \frac{\mathcal{H}\bar{P}'}{\bar{\rho}+\bar{P}} v\right) + \frac{\mathcal{H}}{\bar{\rho}+\bar{P}}\delta P + \underline{\mathcal{H}\Phi}$

from (16b) and (16c):

$4\pi G a^2 \delta\rho = -K^2\Phi + 3\mathcal{H}\, 4\pi G a^2(\bar{\rho}+\bar{P})v$

$\Rightarrow v = \frac{1}{3\mathcal{H}} \frac{\delta\rho}{(\bar{\rho}+\bar{P})} + \frac{K^2\Phi}{12\mathcal{H}\pi G a^2(\bar{\rho}+\bar{P})}$

$\Rightarrow \mathcal{R}' = \Phi' + \mathcal{H}\Phi + (-\mathcal{H}' + \mathcal{H}^2)v + \frac{\mathcal{H}}{\bar{\rho}+\bar{P}}\delta P + \frac{1}{3}\frac{\bar{P}'}{(\bar{\rho}+\bar{P})^2}\delta\rho + \frac{\bar{P}'K^2\Phi}{(\bar{\rho}+\bar{P})^2 12\pi G a^2}$

from (11)

$\mathcal{H}^2 - \mathcal{H}' = \left(\frac{a'}{a}\right)^2 - \frac{a''}{a} + \left(\frac{a'}{a}\right)^2 = 2\mathcal{H}^2 - \frac{a''}{a} = \frac{8\pi G a^2}{3}\left(2\bar{\rho} - \frac{\bar{\rho}}{2} + \frac{3}{2}\bar{P}\right)$

$= 4\pi G a^2(\bar{\rho}+\bar{P})$

$\Rightarrow \Phi' + \mathcal{H}\Phi + (-\mathcal{H}' + \mathcal{H}^2)v = -4\pi G a^2(\bar{\rho}+\bar{P})v + (\mathcal{H}^2 - \mathcal{H}')v$

$= 0$

$\Rightarrow (\bar{\rho}+\bar{P})\frac{\mathcal{R}'}{\mathcal{H}} = \left(\delta P + \frac{1}{3}\frac{\bar{P}'}{\mathcal{H}(\bar{\rho}+\bar{P})}\delta\rho\right) + \frac{\bar{P}'K^2\Phi}{(\bar{\rho}+\bar{P})3\mathcal{H}\,4\pi G a^2}$

$\Rightarrow (\bar{\rho}+\bar{P})\frac{\mathcal{R}'}{\mathcal{H}} = \left(\delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho\right) - \frac{\bar{P}'K^2\Phi}{\bar{\rho}'4\pi G a^2}$

We have seen that $\Phi \simeq \text{const}$ for $k \ll \mathcal{H}$

$$\frac{\mathcal{R}'}{\mathcal{H}} = \frac{1}{(\bar{\rho}+\bar{P})}\left(\delta P - \delta P\right) - \frac{\bar{P}'}{\bar{\rho}'}\left(\frac{k^2}{\mathcal{H}^2 - \mathcal{H}'}\right)\Phi$$

$$= \mathcal{O}\left(\frac{k^2}{\mathcal{H}^2}\right) \simeq 0$$

(c) $\mathcal{R} = \Phi - \mathcal{H} v \overset{(16c)}{=} \Phi + \mathcal{H}\dfrac{\Phi' + \mathcal{H}\Phi}{4\pi G a^2 (\bar{\rho}+\bar{P})}$

$$= \Phi + \frac{\mathcal{H}\Phi' + \mathcal{H}^2\Phi}{(1+w)\,\mathcal{H}^2\,\frac{3}{2}}$$

$$\overset{\Phi' \simeq 0}{=} \Phi\left(1 + \frac{2}{3}\,\frac{1}{1+w}\right) = \frac{\frac{5}{3}+w}{1+w}\,\Phi = \frac{5+3w}{3+3w}\,\Phi$$

transition from radiation to matter era:

$$\frac{5+3\frac{1}{3}}{3+3\frac{1}{3}}\,\Phi_{rad} = \frac{5}{3}\,\Phi_{matter} \Rightarrow \underline{\underline{\Phi_{matter} = \frac{9}{10}\,\Phi_{rad}}}$$