Winter School 2024
Problem Sheet / Introduction to cosmology

Units: $\hbar = c = 1$ and the metric signature is $(-,+,+,+)$.
Comment: Please focus on the problems you find most informative.

Problem 1 [Redshift]

Here, we derive the redshift of light from the geodesic equation, which describes the trajectory $x^\mu(\lambda)$ of a freely falling particle on a curved background. It reads

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\,, \tag{1}$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right) \tag{2}$$

is the Christoffel symbol (also recall that $g^{\alpha\lambda}g_{\lambda\beta} = \delta^\alpha_\beta$).

(a) Assuming a flat FRW geometry with spacetime interval $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, show that the only non-vanishing Christoffel symbols are those with one time index; explicitly

$$\Gamma^0_{ij} = \frac{\dot{a}}{a}g_{ij}\,, \quad \text{and} \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j\,. \tag{3}$$

(b) Evaluate the $\mu = 0$ component of (1) in terms of the four momentum $P^\mu = dx^\mu/d\lambda = (E, p^i)$. Using $P^\mu P_\mu = g_{\mu\nu}P^\mu P^\nu = 0$, show that it can be written as

$$\frac{d}{dt}\ln(aE) = 0\,. \tag{4}$$

(c) Using $E = h/\lambda$, derive from this the photon redshift $z = (\lambda(t_0) - \lambda(t))/\lambda(t)$ (with $t < t_0$) in terms of the scale factor $a(t)$ (w.l.o.g. you can assume that $a(t_0) = 1$).

Problem 2 [EMT]

The FRW ansatz for the energy momentum tensor (EMT) is $T^\mu_\nu = \mathrm{diag}(-\rho, P, P, P)$ where $\rho(t)$ and $P(t)$ are the energy and pressure densities, respectively. Its covariant conservation equation is

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\lambda\mu}T^\lambda_\nu - \Gamma^\lambda_{\nu\mu}T^\mu_\lambda = 0\,. \tag{5}$$

(a) By using (2), derive from the $\nu = 0$ component

$$\dot{\rho} + 3H(\rho + P) = 0\,. \tag{6}$$

(b) Show that (6) recovers the thermodynamic relation $dU = -PdV$ with $U = \rho V$ and $V \propto a^3$.

(c) Solve the conservation equation for a fluid with general (but constant) equation of state, i.e., $P = w\rho$.

Problem 3 [Friedmann equation]

To gain intuition, we follow a heuristic derivation of the Friedmann equation based on a non-relativistic Newtonian analysis. Consider a sphere of mass density $\rho(t)$ and radius $R(t)$.

(a) Argue that a non-relativistic test particle on the surface of the sphere experiences the acceleration

$$\ddot{R} = -\frac{4\pi G}{3} R\rho \,. \tag{7}$$

Derive from this, assuming the constancy of the total mass enclosed in the ball ball of radius $R(t)$,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{R^2} \,, \tag{8}$$

where E is an integration constant. What is its interpretation? The Friedmann equation follows when we identify $R = aR_0$ and $E = -k/2$.

(b) For vanishing spatial curvature k, the Friedmann equations in physical time read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \,. \tag{9}$$

Express them in terms of conformal time η.

(c) Solve the Friedmann equations for a universe dominated by a fluid with constant equation of state parameter $-1/3 < w < 1$ using a simple power-law ansatz, both in physical time t and conformal time η. Impose $a(t_0) = 1$.

Problem 4 [Gauge freedom]

Consider the perturbed FRW geometry (in conformal time η),

$$ds^2 = a^2(\eta)\Big\{ -(1+2A)d\eta^2 + 2\partial_i B dx^i d\eta + \left[(1 + 2C - \frac{2}{3}\nabla^2 E)\delta_{ij} + 2\partial_i\partial_j E\right] dx^i dx^j \Big\} \tag{10}$$

where $A(\eta, \mathbf{x})$, $B(\eta, \mathbf{x})$, $C(\eta, \mathbf{x})$ and $E(\eta, \mathbf{x})$ are small scalar perturbations that break the background isotropy and inhomogeneity (we drop tensor and vector perturbations for simplicity). The metric transforms under a general coordinate transformation $\tilde{x}^\alpha(x)$ as

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}(\tilde{x})\,. \tag{11}$$

(a) Next, we consider small transformations $\tilde{x}^\mu = x^\mu + \xi^\mu$ ($\xi^\mu \ll x^\mu$), which we decompose as $\xi^\mu = (\xi^0, \partial^i\xi)$. Use (11), to derive the transformation law of $A(\eta, \mathbf{x})$ valid at linear order in ξ^μ. The other transformations can be derived in the same way. Overall, they are:

$$\begin{aligned} \tilde{A} &= A - (\xi^0)' - \mathcal{H}\xi^0\,, & \tilde{B} &= B + \xi^0 - \xi'\,, \\ \tilde{C} &= C - \mathcal{H}\xi^0 - \frac{1}{3}\nabla^2\xi & \tilde{E} &= E - \xi\,, \end{aligned} \tag{12}$$

where $\mathcal{H} = a'(\eta)/a(\eta)$.

(b) Argue that ξ^μ offers enough freedom to realize the Newtonian gauge, i.e. $\tilde{B} = \tilde{E} = 0$ and $\tilde{A} \equiv \Psi$ and $\tilde{C} \equiv -\Phi$. Work out the transformation that takes you from the Newtonian to the spatially flat gauge, defined through $\tilde{C} = \tilde{E} = 0$ (a convenient gauge for calculating inflationary perturbations).

Problem 5 [Comoving curvature perturbation]

An important quantity in cosmological perturbation theory is the comoving curvature perturbation. Using the ansatz in (10), it is given by (in momentum space)

$$\mathcal{R} = -C - \frac{1}{3}k^2 E - \mathcal{H}(v + B)\,, \tag{13}$$

where $T^i_0 \equiv -(\bar{\rho} + \bar{P})ik_i v$.

(a) Using that the bulk velocity transforms as $\tilde{v} = v + \xi'$ alongside the transformation in (12), show that $\mathcal{R}$ is gauge-invariant. Evaluate it in terms of the Newtonian gauge variable Φ.

(b) Recall the following perturbation equations from the lecture (in Newtonian gauge, assuming $\Psi = \Phi$):

$$v' = -(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho} + \bar{P}})v - \frac{1}{\bar{\rho} + \bar{P}}\delta P - \Phi \tag{14a}$$

$$4\pi G a^2 \delta\rho = -k^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) \tag{14b}$$

$$4\pi G a^2 (\bar{\rho} + \bar{P})v = -\left(\Phi' + \mathcal{H}\Phi\right) \tag{14c}$$

Use the above equations (alongside the background Friedmann equation) to show that

$$(\bar{\rho} + \bar{P})\frac{\mathcal{R}'}{\mathcal{H}} = \left(\delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho\right) - \frac{\bar{P}'}{\bar{\rho}'}\frac{k^2\Phi}{4\pi G a^2}\,. \tag{15}$$

Conclude that for $k \ll \mathcal{H}$, we have $\mathcal{R}'/\mathcal{H} = \mathcal{O}(k^2/\mathcal{H}^2) \simeq 0$, provided the adiabatic condition $\delta P/\delta\rho = P'/\rho'$ is fulfilled. In other words, $\mathcal{R}$ is conserved on superhorizon scales.

(c) Show that for a background fluid with equation of state $w = const$, we have (for $k \ll \mathcal{H}$)

$$\mathcal{R} = \frac{5+3w}{3(1+w)}\Phi + \mathcal{O}(k^2/\mathcal{H}^2)\,. \tag{16}$$

Using the conservation of $\mathcal{R}$, we conclude that Φ is not conserved as we go from radiation ($w = 1/3$) to matter domination ($w = 0$).