Units: $\hbar = c = 1$ and the metric signature is $(-, +, +, +)$.

Comment: Please focus on the problems you find most informative.

**Problem 1** [Redshift]

Here, we derive the redshift of light from the geodesic equation, which describes the trajectory $x^\mu(\lambda)$ of a freely falling particle on a curved background. It reads

\[
\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda},
\]

where

\[
\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})
\]

is the Christoffel symbol (also recall that $g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$).

(a) Assuming a flat FRW geometry with spacetime interval $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, show that the only non-vanishing Christoffel symbols are those with one time index; explicitly

\[
\Gamma^t_{ij} = \frac{\dot{a}}{a}g^{ij}, \quad \text{and} \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a} \delta^i_j.
\]

(b) Evaluate the $\mu = 0$ component of (1) in terms of the four momentum $P^\mu = dx^\mu/d\lambda = (E, p^i)$. Using $P^\mu P_\mu = g_{\mu\nu}P^\mu P^\nu = 0$, show that it can be written as

\[
\frac{d}{dt} \ln (aE) = 0.
\]

(c) Using $E = h/\lambda$, derive from this the photon redshift $z = (\lambda(t_0) - \lambda(t))/\lambda(t)$ (with $t < t_0$) in terms of the scale factor $a(t)$ (w.l.o.g. you can assume that $a(t_0) = 1$).

**Problem 2** [EMT]

The FRW ansatz for the energy momentum tensor (EMT) is $T^\mu_\nu = \text{diag}(-\rho, P, P, P)$ where $\rho(t)$ and $P(t)$ are the energy and pressure densities, respectively. Its covariant conservation equation is

\[
\nabla_\mu T^\mu_\nu = 0.
\]

(a) By using (2), derive from the $\nu = 0$ component

\[
\ddot{\rho} + 3H(\rho + P) = 0.
\]
(b) Show that (6) recovers the thermodynamic relation $dU = -PdV$ with $U = \rho V$ and $V \propto a^3$.

(c) Solve the conservation equation for a fluid with general (but constant) equation of state, i.e., $P = w\rho$.

**Problem 3** [Friedmann equation]

To gain intuition, we follow a heuristic derivation of the Friedmann equation based on a non-relativistic Newtonian analysis. Consider a sphere of mass density $\rho(t)$ and radius $R(t)$.

(a) Argue that a non-relativistic test particle on the surface of the sphere experiences the acceleration

$$\ddot{R} = -\frac{4\pi G}{3} R \rho.$$  \hspace{1cm}(7)

Derive from this, assuming the constancy of the total mass enclosed in the ball of radius $R(t)$,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3} \rho + \frac{2E}{R^2},$$  \hspace{1cm}(8)

where $E$ is an integration constant. What is its interpretation? The Friedmann equation follows when we identify $R = aR_0$ and $E = -k/2$.

(b) For vanishing spatial curvature $k$, the Friedmann equations in physical time read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho,$$  \hspace{1cm}(9)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P).$$

Express them in terms of conformal time $\eta$.

(c) Solve the Friedmann equations for a universe dominated by a fluid with constant equation of state parameter $-1/3 < w < 1$ using a simple power-law ansatz, both in physical time $t$ and conformal time $\eta$. Impose $a(t_0) = 1$.

**Problem 4** [Gauge freedom]

Consider the perturbed FRW geometry (in conformal time $\eta$),

$$ds^2 = a^2(\eta) \left\{ - (1 + 2A) d\eta^2 + 2\partial_i B dx^i d\eta + \\
\left[ (1 + 2C - \frac{2}{3} \nabla^2 E)\delta_{ij} + 2\partial_i \partial_j E \right] dx^i dx^j \right\}$$  \hspace{1cm}(10)

where $A(\eta, \mathbf{x}), B(\eta, \mathbf{x}), C(\eta, \mathbf{x})$ and $E(\eta, \mathbf{x})$ are small scalar perturbations that break the background isotropy and inhomogeneity (we drop tensor
and vector perturbations for simplicity). The metric transforms under a general coordinate transformation \( \tilde{x}^\alpha(x) \) as
\[
g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x}).
\] (11)

(a) Next, we consider small transformations \( \tilde{x}^\mu = x^\mu + \xi^\mu \) (\( \xi^\mu \ll x^\mu \)), which we decompose as \( \xi^\mu = (\xi^0, \partial^i \xi^i) \). Use (11), to derive the transformation law of \( A(\eta, x) \) valid at linear order in \( \xi^\mu \). The other transformations can be derived in the same way. Overall, they are:
\[
\tilde{A} = A - (\xi^0)' - H \xi^0, \quad \tilde{B} = B + \xi^0 - \xi',
\]
\[
\tilde{C} = C - \mathcal{H} \xi^0 - \frac{1}{3} \nabla^2 \xi, \quad \tilde{E} = E - \xi,
\] (12)
where \( \mathcal{H} = a'(\eta)/a(\eta) \).

(b) Argue that \( \xi^\mu \) offers enough freedom to realize the Newtonian gauge, i.e. \( \tilde{B} = \tilde{E} = 0 \) and \( \tilde{A} \equiv \Psi \) and \( \tilde{C} \equiv -\Phi \). Work out the transformation that takes you from the Newtonian to the spatially flat gauge, defined through \( \tilde{C} = \tilde{E} = 0 \) (a convenient gauge for calculating inflationary perturbations).

**Problem 5** [Comoving curvature perturbation]

An important quantity in cosmological perturbation theory is the comoving curvature perturbation. Using the ansatz in (10), it is given by (in momentum space)
\[
\mathcal{R} = -C - \frac{1}{3} k^2 E - \mathcal{H}(v + B),
\] (13)
where \( T^i_0 \equiv -(\bar{\rho} + \bar{P})i k_i v \).

(a) Using that the bulk velocity transforms as \( \tilde{v} = v + \xi' \) alongside the transformation in (12), show that \( \mathcal{R} \) is gauge-invariant. Evaluate it in terms of the Newtonian gauge variable \( \Phi \).

(b) Recall the following perturbation equations from the lecture (in Newtonian gauge, assuming \( \Psi = \Phi \)):
\[
v' = -(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho} + \bar{P}})v - \frac{1}{\bar{\rho} + \bar{P}} \delta \bar{P} - \Phi \quad (14a)
\]
\[
4\pi G a^2 \delta \bar{\rho} = -k^2 \Phi - 3 \mathcal{H} (\Phi' + \mathcal{H} \Phi) \quad (14b)
\]
\[
4\pi G a^2 (\bar{\rho} + \bar{P}) v = -(\Phi' + \mathcal{H} \Phi) \quad (14c)
\]
Use the above equations (alongside the background Friedmann equation) to show that
\[
(\bar{\rho} + \bar{P}) \frac{\mathcal{R}'}{\mathcal{H}} = \left( \delta \bar{P} - \frac{\bar{P}'}{\bar{\rho}} \delta \bar{\rho} \right) - \frac{\bar{P}'}{\bar{\rho}} \frac{k^2 \Phi}{4\pi G a^2}.
\] (15)
Conclude that for $k \ll H$, we have $R' / \mathcal{H} = O(k^2 / H^2) \simeq 0$, provided the adiabatic condition $\delta P / \delta \rho = P' / \rho'$ is fulfilled. In other words, $R$ is conserved on superhorizon scales.

(c) Show that for a background fluid with equation of state $w = const$, we have (for $k \ll H$)

$$R = \frac{5 + 3w}{3(1 + w)} \Phi + O(k^2 / H^2) . \quad (16)$$

Using the conservation of $R$, we conclude that $\Phi$ is not conserved as we go from radiation ($w = 1/3$) to matter domination ($w = 0$).