# The physics of the $\theta$ angle

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**Abstract**

In this seminar we discuss how the physical mesonic and baryonic amplitudes depend on the $\theta$ angle and we compute the couplings that violate strong CP invariance.

# Contents



# 1 Introduction

The Lagrangian of Yang-Mills theory contains, in addition to the usual term, also a topological term:

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \theta q(x) \tag{1}$$

where $q(x)$ is the topological charge density given by:

$$q(x) = \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad ; \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{2}$$

The additional term violates the invariance under $CP$. This is called strong $CP$ violation to distinguish it from the $CP$ violation present in the weak sector of the Standard Model. Experiments, however, do not show any violation of strong $CP$ and require a very small value for $\theta < 10^{-9}$.

In this seminar we will determine the dependence of physical quantities on $\theta$ and study the processes that violate strong $CP$. The most efficient way of doing this is to use the low energy effective Lagrangian of QCD that contains the fields of the pseudoscalar mesons and baryons instead of the original quarks and gluons. This is due to the fact that in the effective Lagrangian the effect of the axial $U(1)$ anomaly is explicitly displayed and because of this the amplitudes for the hadronic processes can be easily computed. This Lagrangian cannot be explicitly derived from the fundamental QCD Lagrangian as in the $CP^{N-1}$ model [1], but can only be constructed requiring that it has the same anomalous and non-anomalous symmetries of the fundamental QCD Lagrangian.

The logic for constructing such an effective Lagrangian is the following. If we neglect the quark mass matrix the QCD Lagrangian with $N_f$ quark flavours has a $U(N_f) \times U(N_f)$ chiral symmetry that is spontaneously broken to the diagonal vectorial $U(N_f)_V$. The pseudoscalar bosons are the Goldstone bosons corresponding to the spontaneous breaking of the chiral symmetry and are exactly massless in the chiral limit when the quark masses are put to zero. In the realistic world, however, the light quarks are not massless, but have a mass that is small with respect to the scale $\Lambda_{QCD}$. At low energy the pseudosclar bosons are described by the following chiral Lagrangian:

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + \frac{F_\pi}{2\sqrt{2}}\,\mathrm{Tr}\left(M\left(U + U^\dagger\right)\right) \tag{3}$$

where $U$ contains the fields of the pseudoscalar mesons, that are composite states of a quark and an antiquark:

$$U_{ij} = -\frac{2\sqrt{2}m_i}{\mu_i^2 F_\pi}\bar{\Psi}_{R;i}\cdot\Psi_{L;j} \quad ; \quad \Psi_{R,L} = \frac{1 \pm \gamma_5}{2}\Psi \tag{4}$$

$F_\pi = 95$MeV is the pion decay constant The central dot in the first equation means that there is a sum over colour indices. We take the mass matrices of both the quarks and mesons to be diagonal and real:

$$m_{ij} = m_i \delta_{ij} \quad ; \quad M_{ij} = \mu_i^2 \delta_{ij} \tag{5}$$

They are related by the Gell-Mann, Oakes and Renner relation:

$$\mu_i^2 F_\pi^2 = -2m_i < \bar{\Psi}_i \cdot \Psi_i > \tag{6}$$

implying that the ratio $\frac{m_i}{\mu_i^2}$ is independent of $i$. Notice that Eq. (4) is a consequence of Eq. (5) and of the following equation:

$$\frac{U_{ij}}{< U_{ij} >} = 2\frac{\bar{\Psi}_{R;i}\cdot\Psi_{L;j}}{< \bar{\Psi}_i \cdot \Psi_j >} \tag{7}$$

It can be easily checked that the first term in the Lagrangian in Eq. (3) is invariant, as the QCD Lagrangian without the term involving the masses of the quarks, under the chiral $U(N_f) \times U(N_f)$ group that acts on $U$ as follows:

$$U \to AUB^\dagger; U^\dagger \to BU^\dagger A^\dagger; A^{-1} = A^\dagger \quad ; \quad B^{-1} = B^\dagger \tag{8}$$

[1] See for instance Ref. [1] and References therein

while the mass term breaks explicitly this symmetry precisely as the quark mass matrix does in QCD. The chiral symmetry is spontaneously broken by imposing that the meson field satisfies the constraint:

$$UU^\dagger = \frac{F_\pi^2}{2} \tag{9}$$

that implies:

$$U(x) = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \quad \Phi(x) = \Pi^a \tau^a + \frac{S}{\sqrt{N_f}} \tag{10}$$

where $\tau^a$ are the generators of $SU(N_f)$ in the fundamental representation normalized as

$$\mathrm{Tr}\left[\tau^a \tau^b\right] = \delta^{ab} \tag{11}$$

In the case of a $U(3)$ flavour symmetry $\Pi^a(x)$ corresponds the the fields of the octet of the pseudoscalar mesons, while $S$ is a $SU(3)$ singlet. In this case we get:

$$\Pi^a \tau^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^0 + \eta_8/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}k^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta_8/\sqrt{3} & \sqrt{2}k^0 \\ \sqrt{2}k^- & \sqrt{2}\bar{k}^0 & -2\eta_8/\sqrt{3} \end{pmatrix} \tag{12}$$

Lagrangian in Eq. (3) does not contain, however, the effect of the $U(1)$ axial anomaly because, apart from the mass term, it is invariant under the axial $U(1)$, while this is not the case for QCD. This effect can be included by adding a term containing the topological charge density :

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + \frac{F_\pi}{2\sqrt{2}}\,\mathrm{Tr}\left(M\left(U + U^\dagger\right)\right) + \frac{i}{2} q(x)\,\mathrm{Tr}\left(\log U - \log U^\dagger\right) \tag{13}$$

Once we have introduced the extra field $q(x)$ we could also include an arbitrary power of it. However it turns out that in the large number $N_c$ of colours we need to introduce only a quadratic term because higher powers of $q$ are negligible when $N_c \to \infty$. In this way we arrive at the following Lagrangian:

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + \frac{F_\pi}{2\sqrt{2}}\,\mathrm{Tr}\left(M\left(U + U^\dagger\right)\right) + \frac{i}{2} q(x)\,\mathrm{Tr}\left(\log U - \log U^\dagger\right) + \frac{q^2}{aF_\pi^2} \tag{14}$$

In the next sections we add to this Lagrangian also a term with the $\theta$ angle and the baryons, we study the dependence of the physical quantities on $\theta$ and we compute processes which violate strong $CP$. The results that we review here have been originally found in Refs. [2, 3, 4, 5, 6, 7, 8] and appeared in the review in Ref. [1].

Finally in the last section of this seminar we include in the effective action the field of the axion and we use it to determine in a clean way its mass. These results have been obtained together with Gabriele Veneziano [9].

## 2 Adding the $\theta$ angle

In this section we start from the effective Lagrangian in Eq. (14) with the addition of the term with the $\theta$ angle:

$$L = \frac{1}{2}\operatorname{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + \frac{F_\pi}{2\sqrt{2}}\operatorname{Tr}\left(M\left(U + U^\dagger\right)\right) + \frac{i}{2}q(x)\operatorname{Tr}\left(\log U - \log U^\dagger\right)$$
$$+ \frac{q^2}{aF_\pi^2} - \theta q \tag{15}$$

We can eliminate $q$ through its equation of motion:

$$q(x) = \frac{aF_\pi^2}{2}\left[\theta - \frac{i}{2}q(x)\operatorname{Tr}\left(\log U - \log U^\dagger\right)\right] \tag{16}$$

and we get:

$$L = \frac{1}{2}\operatorname{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + \frac{F_\pi}{2\sqrt{2}}\operatorname{Tr}\left(M\left(U + U^\dagger\right)\right) - \frac{aF_\pi^2}{4}\left[\theta - \frac{i}{2}\operatorname{Tr}\left(\log U - \log U^\dagger\right)\right]^2 \tag{17}$$

Since $UU^\dagger$ is proportional to the unit matrix and the mass matrix is diagonal the vacuum expectation value of $U$ must be of the type:

$$< U_{ij} >= e^{-i\phi_i}\delta_{ij}\frac{F_\pi}{\sqrt{2}} \tag{18}$$

where $\phi_i$ are quantities that are determined by minimizing the energy as we will see soon. It is convenient to introduce the matrix $V$ that has a vacuum expectation value proportional to the unit matrix:

$$U_{ij} = V_{ij}e^{-i\phi_i}; \quad < V_{ij} >= \frac{F_\pi}{\sqrt{2}}\delta_{ij} \tag{19}$$

and rewrite Eq. (17) in terms of the field $V$. We get $(M_{ij}(\theta) = \mu_i^2 \cos\phi_i \delta_{ij})$ :

$$
\begin{aligned}
L = &\frac{1}{2}\,\mathrm{Tr}\left(\partial_\mu V \partial_\mu V^\dagger\right) + \frac{aF_\pi^2}{16}\left[\mathrm{Tr}\left(\log V - \log V^\dagger\right)\right]^2 \\
&+ \frac{F_\pi}{2\sqrt{2}}\,\mathrm{Tr}\left( M(\theta)\left(V + V^\dagger\right) - \frac{2F_\pi}{\sqrt{2}}\right) \\
&+ \frac{F_\pi^2}{2}\sum_{i=1}^{N_f}\mu_i^2\cos\phi_i - \frac{aF_\pi^2}{4}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)^2 \\
&+ i\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)\frac{F_\pi}{\sqrt{2}}\left[\frac{aF_\pi}{2\sqrt{2}}\,\mathrm{Tr}\left(\log V - \log V^\dagger\right) - \left(V - V^\dagger\right)\right]
\end{aligned}
\tag{20}
$$

The angles $\phi_i$ are determined by minimizing the energy that follows from the previous Lagrangian, namely:

$$
E = \frac{F_\pi^2}{2}\left[\frac{a}{2}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)^2 - \sum_{i=1}^{N_f}\mu_i^2\cos\phi_i\right]
\tag{21}
$$

that implies the following set of equations:

$$
\mu_i^2 \sin\phi_i = a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right) \quad ; \quad i = 1\ldots N_f
\tag{22}
$$

Inserting for $V$ the expressions given in Eq. (10) for $U$ we get:

$$
\begin{aligned}
L = \frac{1}{2}\,\mathrm{Tr}\left(\partial_\mu V \partial_\mu V^\dagger\right) - \frac{aN_f}{2}S^2 + \frac{F_\pi^2}{2}\,\mathrm{Tr}\left[M(\theta)\left(\cos\frac{\sqrt{2}\Phi}{F_\pi} - 1\right)\right] + \\
+\frac{aF_\pi}{\sqrt{2}}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)\mathrm{Tr}\left[\frac{F_\pi}{\sqrt{2}}\sin\frac{\sqrt{2}\Phi}{F_\pi} - \Phi\right]
\end{aligned}
\tag{23}
$$

where $\Phi$ is given in Eq. (10).

The way to proceed is the following. First we have to solve Eq.s (22) that determine $\phi_i$ as a function of $\theta, a$ and $\mu_i^2$. Then insert them in the effective Lagrangian in Eq. (23) that will depend on $\theta, a$ and $\mu_i^2$. Before we proceed it is useful to show that the quantities that we will extract from the previous effective Lagrangian will be invariant under the shift $\theta \to \theta + 2\pi$. This follows from the fact that, if we have found a solution $\phi_i(\theta)$ of Eq.s (22) then it is easy to show that also the following will be a solution:

$$\phi_1(\theta + 2\pi) = \phi_1(\theta) + 2\pi \quad ; \quad \phi_i(\theta + 2\pi) = \phi_i(\theta) \quad ; \quad i = 2 \ldots N_f \tag{24}$$

But the physical quantities depend only on $e^{i\phi_i}$ and therefore are invariant under a shift of $2\pi$ of the $\theta$ angle.

It is also clear that strong CP is conserved if $\theta - \sum_{i=1}^{N_f} \phi_i = 0$. This happens when:

1. $\theta = 0$ that implies that $\phi_i = 0$,

2. the mass of a quark flavour is zero

3. and also sometimes if $\theta = \pi$.

# 3 The Witten-Veneziano relation

In order to get the Witten-Veneziano relation we have to consider the theory without fermions. In this case the original effective Lagrangian in Eq. (15) becomes:

$$L^{\text{noferm.}} = \frac{q^2}{aF_\pi^2} - \theta q - iqJ \tag{25}$$

where we have added an external source that is coupled to the topological charge density $q$. From the previous expression one can compute the partition function:

$$Z(J, \theta) \equiv e^{-iW(J,\theta)} = e^{-iV_4 aF_\pi^2(\theta + iJ)^2/4} \tag{26}$$

The vacuum energy is equal to:

$$E(\theta) \equiv \frac{W(0, \theta)}{V_4} = \frac{aF_\pi^2}{4}\theta^2 \tag{27}$$

From it we get:

$$\left.\frac{d^2 E(\theta)}{d\theta^2}\right|_{\theta=0} = \frac{aF_\pi^2}{2} \tag{28}$$

On the other hand the mass of the singlet field can be obtained from the effective Lagrangian in Eq. (23) and it is equal to:

$$M_S^2 = aN_f \tag{29}$$

Putting together Eq.s (28) and (29) we get the Witten-Veneziano relation:

$$M_S^2 = \frac{2N_f}{F_\pi} \frac{d^2E(\theta)}{d\theta^2}\bigg|_{\theta=0} \tag{30}$$

# 4 Strong $CP$ violating mesonic amplitudes

We start this section by solving the minimization equations in Eq. (21) in the case of two flavours and in the limit where $a >> \mu_1^2, \mu_2^2$. In this case we must impose that $\theta = \phi_1 + \phi_2$ and the minimization equations become:

$$\mu_1^2 \sin\phi_1 = \mu_2^2 \sin(\theta - \phi_1) \tag{31}$$

that can be easily solved giving:

$$\sin\phi_1 = \frac{\mu_2^2 \sin\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}} \quad ; \quad \sin\phi_2 = \frac{\mu_1^2 \sin\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}} \tag{32}$$

and

$$\cos\phi_1 = \frac{\mu_1^2 + \mu_2^2\cos\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}} \quad ; \quad \cos\phi_2 = \frac{\mu_2^2 + \mu_1^2\cos\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}} \tag{33}$$

Computing the corresponding energy in Eq. (21) we get:

$$E(\theta) = -\frac{F_\pi^2}{2}\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta} \tag{34}$$

For equal masses ($\mu_1 = \mu_2 = \mu$) we get:

$$E(\theta) = -F_\pi^2\mu^2 \left|\cos\frac{\theta}{2}\right| \tag{35}$$

Notice that both Eq.s (34) and (35) are periodic of period $2\pi$ in $\theta$. We have solved the minimization equation in the limit in which $a >> \mu_1^2, \mu_2^2$. Let us add the first correction. We have to solve the following equations:

$$\mu_1^2 \sin\phi_1 = \mu_2^2 \sin\phi_2 = a(\theta - \phi_1 - \phi_2) \tag{36}$$

and let us insert in it the following expansion:

$$\phi_{1,2} = \bar{\phi}_{1,2} + \epsilon\delta\phi_{1,2} \quad ; \quad \epsilon = \frac{\mu_1\mu_2}{a} \tag{37}$$

One gets:

$$\phi_1 = \bar{\phi}_1 - \epsilon\frac{\sin\theta}{R^3}\frac{\mu_2^2 + \mu_1^2\cos\theta}{\mu_1^2}; \phi_2 = \bar{\phi}_2 - \epsilon\frac{\sin\theta}{R^3}\frac{\mu_1^2 + \mu_2^2\cos\theta}{\mu_1^2} \tag{38}$$

where $\bar{\phi}_{1,2}$ is the previous solution:

$$\bar{\phi}_1 + \bar{\phi}_1 = \theta \quad ; \quad R = \sqrt{\frac{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}{\mu_1^2\mu_2^2}} \tag{39}$$

Using the previous expression we can compute the coefficient of the $CP$ violating term:

$$\theta - \phi_1 - \phi_2 = \epsilon\frac{\sin\theta}{R} = \frac{\mu_1^2\mu_2^2\sin\theta}{a\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}} \tag{40}$$

It is vanishing if $\theta = 0$ or if $\mu_1^2$ and/or $\mu_2^2$ are equal to zero. If $\mu_1 \neq \mu_2$ it is also zero for $\theta = \pi$. But if $\mu_1 = \mu_2 \equiv \mu$ we get:

$$\theta - \phi_1 - \phi_2 = \frac{\mu^2}{a} \neq 0 \tag{41}$$

In conclusion if $\mu_1 = \mu_2$ then $CP$ is violated at $\theta = \pi$.

From the $CP$ violating term in Eq. (23) we can extract a cubic term in the fields of the pseudoscalar mesons that is given by:

$$-\frac{a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)}{3\sqrt{2}F_\pi}\,\mathrm{Tr}\left(\Phi^3\right) \Longrightarrow -\frac{a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)}{\sqrt{3}F_\pi}\pi^+\pi^-\eta_8 \tag{42}$$

from which we have extracted the decay amplitude $\eta_8 \to \Pi^+\Pi^-$given by:

$$T\left(\eta \to \pi^+\pi^-\right) = \frac{a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)}{\sqrt{3}F_\pi} = \frac{2m_\pi^2(\theta)}{\sqrt{3}F_\pi}\cdot\frac{\mu_1^2\mu_2^2\sin\theta}{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta} \tag{43}$$

where

$$m_\pi^2(\theta) = \frac{\mu_1^2\cos\phi_1 + \mu_2^2\cos\phi_2}{2} = \frac{1}{2}\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta} \tag{44}$$

For small values of $\theta$ we get

$$T\left(\eta \rightarrow \pi^{+}\pi^{-}\right) \sim \frac{2m_{\pi}^{2}}{\sqrt{3}F_{\pi}} \frac{\theta}{\left(\sqrt{\frac{m_1}{m_2}} + \sqrt{\frac{m_2}{m_1}}\right)^{2}} \tag{45}$$

where $m_i$ is the quark mass related to the meson mass through Eq. (6). This implies that

$$\Gamma\left(\eta \rightarrow \pi^{+}\pi^{-}\right)) = \theta^{2} \cdot (135\text{KeV}) \quad : \quad \frac{\Gamma\left(\eta \rightarrow \pi^{+}\pi^{-}\right)}{\Gamma_{\text{tot}}} = 159\theta^{2} \tag{46}$$

From experiments we get:

$$\frac{\Gamma\left(\eta \rightarrow \pi^{+}\pi^{-}\right)}{\Gamma_{tot}} < 3 \cdot 10^{-4} \tag{47}$$

that gives an upper limit to the value of $\theta < 10^{-3}$. We will get a much better limit from the electric dipole moment of the neutron.

Notice that the decay amplitude of $\eta \rightarrow \pi^{+}\pi^{-}$is zero for $\theta = 0, \pi$ if $\mu_1^2 \neq \mu_2^2$, while if $\mu_1^2 = \mu_2^2$ it is not vanishing anymore at $\theta = \pi$. In the previous analysis we have assumed that there are only two quark flavours. A more realistic case is the one with three flavours. In this case one finds that

1. If $|\mu_2^2 - \mu_1^2|\, \mu_3^2 > \mu_1^2\mu_2^2$ then $CP$ is conserved at $\theta = \pi$
2. If $|\mu_2^2 - \mu_1^2|\, \mu_3^2 > \mu_1^2\mu_2^2$ then $CP$ is violated at $\theta = \pi$.

From the meson mass matrix one can easily get the mass of the pseudoscalar mesons as a function of the angle $\theta$. One gets:

$$m_{\pi^0,\pi^{\pm}}^{2} = \frac{\mu_1^2 \cos\phi_1 + \mu_2^2 \cos\phi_2}{2}; m_{k^{\pm}}^{2} = \frac{\mu_1^2 \cos\phi_1 + \mu_3^2 \cos\phi_3}{2} \tag{48}$$

and

$$m_{k^0;\bar{k}^0}^{2} = \frac{\mu_2^2 \cos\phi_1 + \mu_3^2 \cos\phi_3}{2} \tag{49}$$

They imply:

$$R(\theta) \equiv \frac{m_{k^0}^2 - m_{k^+}^2 - m_{\pi^0}^2 + m_{\pi^+}^2}{m_{\pi}^2} = \frac{\mu_2^2 \cos\phi_2 - \mu_1^2 \cos\phi_1}{\mu_2^2 \cos\phi_2 + \mu_1^2 \cos\phi_1} = \frac{\left(\mu_2^2 - \mu_1^2\right)\left(\mu_2^2 + \mu_1^2\right)}{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta} \tag{50}$$

where we have used Eqs. (32) and (33). In particular we get:

$$R(\theta=0)=\frac{\mu_2^2-\mu_1^2}{\mu_2^2+\mu_1^2} \quad ; \quad R(\theta=\pi)=\frac{\mu_2^2+\mu_1^2}{\mu_2^2-\mu_1^2} \tag{51}$$

Experimentally $R=0.3$ that is consistent with $\theta=0$. The ratio of masses for the two lighest quarks is determined from the following relation:

$$\frac{m_1}{m_2}=\frac{\mu_1^2}{\mu_2^2}=\frac{2m_{\pi^0}^2-m_{\pi^+}^2+m_{k^+}^2-m_{k^0}^2}{m_{k^0}^2-m_{k^+}^2+m_{\pi^+}^2}=0.56 \tag{52}$$

For the sake of completeness we give also the ratio between the mass of the strange and that of the down quarks:

$$\frac{m_3}{m_2}=\frac{\mu_3^2}{\mu_2^2}=\frac{m_{k^0}^2-m_{\pi^+}^2+m_{k^+}^2}{m_{k^0}^2-m_{k^+}^2+m_{\pi^+}^2}=20.1 \tag{53}$$

# 4 Strong $CP$ violating amplitudes with baryons

In order to compute the $CP$ violating terms involving baryons it is convenient to add to the effective Lagrangian terms involving baryons. The baryons belong to an octet of $SU(3)$ and are described by the following matrix:

$$\begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & 2\frac{\Lambda}{\sqrt{6}} \end{pmatrix} \tag{54}$$

Remember that $B$ is also a Dirac spinor. Under the chiral $U(3)\times U(3)$ the baryons transform as follows:

$$R\equiv\frac{1+\gamma_5}{2}B\rightarrow ARB^\dagger \quad ; \quad L\equiv\frac{1-\gamma_5}{2}B\rightarrow BLA^\dagger \tag{55}$$

Remember that the meson fields transform as in Eq. (8). The Lagrangian involving baryons can be written as follows:

$$\begin{aligned} L_{bar}=\mathrm{Tr}\left[\bar{B}i\gamma^\mu\partial_\mu B\right]-\tfrac{\sqrt{2}\alpha}{F_\pi}\mathrm{Tr}\left[\bar{L}URU+\bar{R}U^\dagger LU^\dagger\right]+\\ +\delta\,\mathrm{Tr}\left[\bar{L}URM+\bar{R}U^\dagger LM^\dagger\right]+\gamma\,\mathrm{Tr}\left[\bar{L}MRU+\bar{R}M^\dagger LU^\dagger\right] \end{aligned} \tag{56}$$

As before we introduce

$$U_{ij}=V_{ij}e^{-i\phi_j} \quad ; \quad R_{ij}=e^{i\phi_i}R'_{ij} \quad ; \quad \bar{L}_{ij}=e^{i\phi_i}\bar{L}'_{ij} \tag{57}$$

and the previous Lagrangian becomes:

$$\begin{aligned}
L_{bar} &= \mathrm{Tr}\left[\bar{B}' i\gamma^\mu \partial_\mu B'\right] - i\frac{\sqrt{2}\alpha}{F_\pi}\,\mathrm{Tr}\left[\bar{L}' V R' V + \bar{R}' V^\dagger L' V^\dagger\right] + \\
&+\delta\,\mathrm{Tr}\left[\left(\bar{L}' V R' + \bar{R}' V^\dagger L'\right) M(\theta)\right] + \gamma\,\mathrm{Tr}\left[\bar{L}' M(\theta) R' V + \bar{R}' M(\theta) L' V^\dagger\right] + \\
&+i\left(\theta - \sum_i \phi_i\right)\left[\delta\,\mathrm{Tr}\left(\bar{L}' V R' - \bar{R}' V^\dagger L'\right) + \gamma\,\mathrm{Tr}\left(\bar{L}' R' V + \bar{R}' L' V^\dagger\right)\right]
\end{aligned} \tag{58}$$

The Lagrangian has the same structure as the one before with in addition a $CP$ violating term. One can determine $\alpha, \gamma$ and $\delta$ in terms of the baryon masses:

$$\begin{aligned}
\alpha &= \frac{\sqrt{2}}{F_\pi}\left[m_\Sigma + \frac{3\mu^2}{2\left(\mu_3^2 - \mu^2\right)}\left(m_\Sigma - m_\Lambda\right)\right] \\
\gamma &= \frac{\sqrt{2}}{2F_\pi\left(\mu_3^2 - \mu^2\right)}\left[\frac{3}{2}\left(m_\Sigma - m_\Lambda\right) - \left(m_\Xi - m_N\right)\right] \\
\delta &= \frac{\sqrt{2}}{2F_\pi\left(\mu_3^2 - \mu^2\right)}\left[\frac{3}{2}\left(m_\Sigma - m_\Lambda\right) + \left(m_\Xi - m_N\right)\right]
\end{aligned} \tag{59}$$

It is easy to check that the baryon masses satisfy the Gell-Mann-Okubo mass formula:

$$3m_\Lambda + m_\Sigma = 2\left(m_\Xi + m_N\right) \tag{60}$$

From the previous Lagrangian one can extract the $\pi N$ coupling constants:

$$\sqrt{2}\bar{N}\left[i\gamma_5 g_{\pi NN} + \bar{g}_{\pi NN}\right]\pi^i \tau^i N \tag{61}$$

that are given by:

$$F_\pi g_{\pi NN} = m_N + \frac{\mu^2}{2\left(\mu_3^2 - \mu^2\right)}\left[\frac{3}{2}\left(m_\Sigma - m_\Lambda\right) - \left(m_\Xi - m_N\right)\right] \tag{62}$$

that is the Goldeberger-Treiman relation apart from terms that vanish in the chiral limit and

$$\begin{aligned}
\bar{g}_{\pi NN} &= \frac{m_1 m_2 \theta}{2F_\pi\left(m_1 + m_2\right)\left(m_3 - m\right)}\left[\frac{3}{2}\left(m_\Sigma - m_\Lambda\right) - \left(m_\Xi - m_N\right)\right] \\
&\times\left[1 + \frac{3m\left(m_\Sigma - m_\Lambda\right)}{2\left(m_3 - m\right) m_N}\right]
\end{aligned} \tag{63}$$

Having computed $\bar{g}_{\pi NN}$ we can use it to estimate the electric dipole moment of the neutron that, if different from zero, implies a violation of $CP$. The dominant contribution comes from the two diagrams discussed and computed in Ref. [5] and one gets:

$$D_n = \frac{1}{4\pi^2 m_N} \cdot g_{\pi NN} \bar{g}_{\pi NN} \log \frac{m_N}{m_\pi} = 3.6 \cdot 10^{-16} \theta \mathrm{cm} \tag{64}$$

in units where the electric charge $e = 1$. The experimental limit is:

$$D_n < 6 \cdot 10^{-26} \Longrightarrow \theta < 10^{-9} \tag{65}$$

# 5 Including the axion

From the analysis of the previous sections, we have seen that, if none of the quark masses is exactly zero, the $\theta$ angle must be very small and is actually consistent with zero. If instead one of the quark masses were zero, CP violation would be absent thanks to an exact classical symmetry (the chiral rotation of the massless quark) which allows to rotate $\theta$ away.

The Peccei-Quinn (PQ) solution of the strong-CP problem uses a similar mechanism, but is based on extending QCD to include, in the matter sector, some new degrees of freedom. The essential property of the PQ model is that such an extension should provide a new classically exact, but anomalous and spontaneously broken, $U(1)_{PQ}$ symmetry.

The low-energy effective action of such a theory will have to contain, besides the usual QCD degrees of freedom, an extra would-be Goldstone boson related to the spontaneously broken $U(1)_{PQ}$ symmetry. If we denote by $a_{PQ}$ the coefficient of the $U(1)_{PQ}$ anomaly and by $F_\alpha$ the scale of its spontaneous breaking (the analog of $F_\pi$ ), we can easily write down an effective action that incorporates all the relevant (anomalous and non-anomalous) Ward identities. It consists of adding a couple of terms to the effective Lagrangian of Eq. (15) to give (Ref. [9]):

$$\begin{aligned} L = \frac{1}{2} \mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + \frac{1}{2} \mathrm{Tr}\left(\partial_\mu N \partial_\mu N^\dagger\right) + \frac{F_\pi}{2\sqrt{2}} \mathrm{Tr}\left(M\left(U + U^\dagger\right)\right) \frac{q^2}{aF_\pi^2} - \theta q + \\ + \frac{i}{2} q(x) \left(\mathrm{Tr}\left(\log U - \log U^\dagger\right) + a_{PQ}\left(\log N - \log N^\dagger\right)\right) \end{aligned} \tag{66}$$

where $U$

$$U(x) = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \quad ; \quad N(x) = \frac{F_\alpha}{\sqrt{2}} e^{i\sqrt{2}\alpha(x)/F_\alpha} \tag{67}$$

Notice that, following our assumptions, the only term that breaks $U(1)_{PQ}$ is the one related to the anomaly. Under the axial $U(1)$ and the additional $U(1)_{PQ}$ defined by:

$$U \to e^{i\beta} U \quad ; \quad N \to e^{i\gamma} N \tag{68}$$

the effective Lagrangian transforms as follows:

$$\delta L = -\left(N_f \beta + a_{PQ}\gamma\right) q(x) \tag{69}$$

It is invariant if we choose $N_f\beta + a_{PQ}\gamma = 0$. This is an anomaly-free $U(1)$ subgroup, whose spontaneous and explicit breaking (by quark masses) implies a new, pseudo-Goldstone boson, the (Peccei-Quinn-Weinberg-Wilczek) axion. Proceeding as in the previous sections $\left(< U_{ij} >= e^{-i\phi_i}\delta_{ij}F_\pi/\sqrt{2}\right.$ and $\left.\langle N\rangle = e^{-i\phi}F_\alpha/\sqrt{2}\ \right)$, we have to minimize the energy given by:

$$E = \frac{F_\pi^2}{2}\left[\frac{a}{2}\left(\theta - \sum_{i=1}^{N_f}\phi_i - \phi\right)^2 - \sum_{i=1}^{N_f}\mu_i^2\cos\phi_i\right] \tag{70}$$

obtaining

$$a\left(\theta - \sum_{i=1}^{N_f}\phi_i - \phi\right) = \mu_i^2 \sin\phi_i \quad ; \quad \theta - \phi - \sum_{i=1}^{N_f}\phi_i = 0 \tag{71}$$

that imply $\phi_i = 0$ and $\theta - \phi = 0$. In this case there is no dependence on the $\theta$ angle and no $CP$ violation because $\theta - \phi - \sum_{i=1}^{N_f}\phi_i = 0$ (in analogy, again, with the case of a single massless quark). The mass matrix involving the axion and the components of $\Phi$ belonging to the Cartan subalgebra of $U\left(N_f\right)\left(\Phi_{ij} = v_i\delta_{ij}\right)$ is given by:

$$-\frac{1}{2}\left[\sum_{i=1}^{N_f}\mu_i^2 v_i^2 - \frac{a}{2}\left(\sum_{i=1}^{N_f} v_i + b\alpha\right)^2\right] \tag{72}$$

where $b \equiv a_{PQ}\frac{F_\pi}{F_\alpha}$. The masses of the neutral mesons and of the axion are given by setting to zero the determinant of the following matrix:

$$\begin{pmatrix} b^2a-\lambda & ba & ba & ba & \dots & ba \\ ba & \mu_1^2+a-\lambda & a & a & \dots & a \\ ba & a & \mu_2^2+a-\lambda & a & \dots & a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ ba & a & a & a & \dots & \mu_{N_f}^2+a-\lambda \end{pmatrix} \tag{73}$$

that is by solving the equation:

$$\lambda\left[\frac{1}{a}+\sum_{i=1}^{N_f}\frac{1}{\mu_i^2-\lambda}\right]=b^2 \tag{74}$$

Since $b << 1$ the lowest eigenvalue, corresponding to the mass of the axion, can be easily written down:

$$m_\alpha=\frac{b^2}{\frac{1}{a}+\sum_{i=1}^{N_f}\frac{1}{\mu_i^2}}\sim\frac{b^2}{\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}}=2m_\pi^2b^2\cdot\frac{m_1m_2}{(m_1+m_2)^2} \tag{75}$$

In order to be consistent with experiments we have to require that $F_\alpha \geq 10^9$GeV corresponding to an axion mass $m_\alpha < 0.01$eV.

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