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Neutrino physics – exercises

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Exercise 1: The oscillation phase

Departing from the amplitude for vacuum oscillations

$$\mathcal{A}_{\nu_\alpha \to \nu_\beta} = \sum_i U_{\beta i} U^*_{\alpha i} e^{-i(E_i t - p_i x)} \tag{1}$$

derive the oscillation probability

$$P_{\nu_\alpha \to \nu_\beta} = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta} \right|^2 = \sum_{jk} U_{\alpha j} U^*_{\beta j} U^*_{\alpha k} U_{\beta k} \exp\left[-i \frac{\Delta m^2_{kj} x}{2\, E_\nu} \right] . \tag{2}$$

Derive the oscillation phase $\phi_{kj} = \Delta m^2_{kj} x / (2\, E_\nu)$ in the case of two neutrinos only. Avoid the assumption of equal energy or equal momentum for the neutrino mass states, but use that neutrinos are ultra-relativistic.

Hint: use the definitions

$$\Delta X = X_2 - X_1 \,, \quad \Delta X^2 = X_2^2 - X_1^2 \,, \quad \bar{X} = (X_1 + X_2)/2 \,, \tag{3}$$

which imply $\Delta X^2 = 2\bar{X}\Delta X$, for $X = E, p, m$. Furthermore, use the average velocity $v = \bar{p}/\bar{E}$ and $x \approx vt$.

Think about conceptual problems of this derivation. An overview over a consistent calculation and references can be found in Ref. [1]

Exercise 2: Mass and mixing angle in constant matter

Consider two neutrino flavours and start from the effective Hamiltonian in matter

$$H_{\rm mat} = \frac{1}{2E} U(\theta)\, \mathrm{diag}(m_1^2, m_2^2)\, U^\dagger(\theta) + \mathrm{diag}(V, 0) \,, \qquad U(\theta) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \tag{4}$$

with $c = \cos\theta, s = \sin\theta$ and $V = \sqrt{2} G_F N_e$ is the effective matter potential, where N_e is the electron density along the neutrino path, which is assumed to be constant.

Show that

$$H_{\rm mat} = \frac{1}{2E} U(\theta_{\rm mat})\, \mathrm{diag}(m^2_{1\rm mat}, m^2_{2\rm mat})\, U^\dagger(\theta_{\rm mat}) \tag{5}$$

with

$$\sin^2 2\theta_{\rm mat} = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - A)^2} \tag{6}$$

$$\Delta m^2_{\rm mat} = \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - A)^2} \tag{7}$$

where $A \equiv 2EV/\Delta m^2$, $\Delta m^2 \equiv m_2^2 - m_1^2$ and similar for $\Delta m^2_{\rm mat}$.

Discuss the behaviour of $m^2_{1\rm mat}$, $m^2_{2\rm mat}$, and $\sin^2 2\theta_{\rm mat}$ as a function of A (including its sign).

Long-baseline appearance experiments

The appearance probability in vaccum to second order in θ_{13} and $\Delta m^2_{21}/|\Delta m^2_{31}|$ is given by

$$P_{\mu\to e} \approx s_{23}^2\, S^2\, \sin^2\Delta + \sin 2\theta_{23}\,\tilde\alpha\, S\, \sin\Delta\cos(\Delta \pm \delta_{\rm CP}) + c_{23}^2\,\tilde\alpha^2 \tag{8}$$

with

$$S \equiv \sin 2\theta_{13}\,, \quad \Delta \equiv \frac{\Delta m^2_{31} L}{4E}\,, \quad \tilde\alpha \equiv \sin 2\theta_{12}\,\frac{\Delta m^2_{21} L}{4E}\,, \tag{9}$$

and $s_{23} \equiv \sin\theta_{23}$, $c_{23} \equiv \cos\theta_{23}$. For neutrinos (anti-neutrinos) holds the upper (lower) sign, for $e \to \mu$ transitions exchange $\delta_{\rm CP} \to -\delta_{\rm CP}$.[1] The neutrino mass ordering is determined by the sign of Δ. A discussion of LBL oscillation probabilities can be found for example in Ref. [2], see also Ref. [3].

Exercise 3: The $\tilde\alpha^2$-term

Consider an experiment at the first oscillation maximum and estimate the size of the $\tilde\alpha^2$ term in the oscillation probability (third term in Eq. 8). Give the range for $\sin^2 2\theta_{13}$, where this term can be neglected. What does this imply, given the value of $\sin^2 2\theta_{13}$ as determined by reactor experiments?

Exercise 4: $\sin^2 2\theta_{13}$-determination in appearance experiments

Consider an experiment at the first oscillation maximum which measures some value for $P_{\mu\to e}$. Suppose this is just a counting experiment and ignore the energy dependence of the signal. To a good approximation this applies to current data from the T2K experiment.[2]

a) Using Eq. 8, estimate the shape of the allowed region for $\sin^2 2\theta_{13}$ as a function of $\delta_{\rm CP}$.

b) How does this shape depend on whether neutrinos or anti-neutrinos are used?

c) Discuss the dependence of the region on θ_{23}.

Hint: use the measured value of $\sin^2 2\theta_{13}$ from reactor experiments and the results of exercise 3 to motivate whether the $\tilde\alpha^2$-term in Eq. 8 has to be considered or not.

[1] Remember: in vacuum the CP-conjugation (exchaning ν with $\overline{\nu}$) is equivalent to the T-conjugation (exchaning initial and final neutrino flavours), as a consequence of CPT invariance.

[2] For the NOvA experiment, similar considerations apply, but in this case the matter effect is larger and leads to modifications.

Exercise 5: The $\mathrm{sign}(\Delta m_{31}^2)$-degeneracy

a) Show that in vacuum the relation

$$P_{\mu\to e}(\Delta m_{31}^2, S, \delta_{\rm CP}) = P_{\mu\to e}(-\Delta m_{31}^2, S, \delta'_{\rm CP}) \tag{10}$$

can be fulfilled simultaneously for neutrinos and anti-neutrinos, and independent of the neutrino energy. Determine $\delta'_{\rm CP}$.

b) Consider the case of small matter effect. Without performing any calculations, give an argument why the leading order matter effect correction to Eq. 8 cannot break the $\mathrm{sign}(\Delta m_{31}^2)$-degeneracy and similar to Eq. 10, a relation

$$P_{\mu\to e}(\Delta m_{31}^2, S, \delta_{\rm CP}) = P_{\mu\to e}(-\Delta m_{31}^2, S', \delta'_{\rm CP}) \tag{11}$$

still can be satisfied for neutrinos and anti-neutrinos simultaneously.

The degeneracy discussed in this exercise makes it hard to determine the neutrino mass ordering (normal versus inverted ordering) and is the reason why the ordering is not determined by present data. The neutrino mass ordering degeneracy was first noted in Ref. [4], a discussion with some analytical considerations can be found in Ref. [5]. The classical paper on the eight-fold degeneracy (including the intrinsic, $\mathrm{sign}(\Delta m_{31}^2)$, and octant degeneracies) is Ref. [6].

Exercise 6: Majorana mass term

The charge conjugated field is defined as

$$\psi^c \equiv C\overline{\psi}^T = C\gamma_0\psi^* \tag{12}$$

where the charge conjugation matrix C has the following properties:

$$C^\dagger = C^{-1}\,, \quad C^T = -C\,, \quad C\gamma_\mu^T C^{-1} = -\gamma_\mu\,. \tag{13}$$

a) Show the quivalence of the following notations for the Majorana mass term

$$\frac{m}{2}\psi_L^T C^{-1}\psi_L + \text{h.c.} = -\frac{m}{2}\overline{(\psi_L)^c}\psi_L + \text{h.c.} = -\frac{m}{2}\overline{\psi}\psi \quad \text{with} \quad \psi = \psi_L + (\psi_L)^c\,. \tag{14}$$

b) Show that a Majorana mass matrix has to be symmetric. (Hint: use the anti-commutation rule for fermion fields.)

c) Consider a Lagrangian with one left-handed and one right-handed fermion with mass terms of the following form:

$$\mathcal{L}_M = -m_D\overline{\psi_L}\psi_R + \frac{m_L}{2}\psi_L^T C^{-1}\psi_L + \frac{m_R}{2}\psi_R^T C^{-1}\psi_R\,. \tag{15}$$

Show that this can be cast into the form of a Majorana mass term in the following way:

$$\mathcal{L}_M = \frac{1}{2}\psi^T C^{-1}\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}\psi + \text{h.c.} \quad \text{with} \quad \psi = \begin{pmatrix} \psi_L \\ (\psi_R)^c \end{pmatrix}\,. \tag{16}$$

d) Assume that m_D, m_L, m_R are real (this corresponds to CP conservation). Diagonalize the mass matrix in eq. 16. What are the mass eigenvalues and the mass eigenfields?

e) Consider the two limiting cases (i) $m_L, m_R \ll m_D$ and (ii) $m_L \ll m_D \ll m_R$. In both cases discuss the mass eigenvalues and the mass eigenfields. Give an interpretation of your results.

A discussion along these lines can be found in [7].

References

[1] E. K. .Akhmedov and J. Kopp, "Neutrino oscillations: Quantum mechanics vs. quantum field theory," JHEP **1004** (2010) 008 [arXiv:1001.4815 [hep-ph]].

[2] E. K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson and T. Schwetz, "Series expansions for three-flavor neutrino oscillation probabilities in matter," JHEP **0404** (2004) 078 [arXiv:hep-ph/0402175].

[3] M. Freund, "Analytic approximations for three neutrino oscillation parameters and probabilities in matter," Phys. Rev. D **64** (2001) 053003, [arXiv:hep-ph/0103300].

[4] H. Minakata and H. Nunokawa, "Exploring neutrino mixing with low energy superbeams," JHEP **0110**, 001 (2001) [arXiv:hep-ph/0108085].

[5] T. Schwetz, "Determination of the neutrino mass hierarchy in the regime of small matter effect," JHEP **0705** (2007) 093 [arXiv:hep-ph/0703279].

[6] V. Barger, D. Marfatia and K. Whisnant, "Breaking eight-fold degeneracies in neutrino CP violation, mixing, and mass hierarchy," Phys. Rev. D **65**, 073023 (2002) [arXiv:hep-ph/0112119].

[7] S. M. Bilenky, C. Giunti and W. Grimus, "Phenomenology of neutrino oscillations," Prog. Part. Nucl. Phys. **43** (1999) 1 [hep-ph/9812360].

Solutions

Solution 1: The oscillation phase

From squaring the amplitude in the case of two neutrinos the oscillation phase is obtained as

$$\phi = (E_2 - E_1)t - (p_2 - p_1)x \quad \text{with} \quad E_i^2 = p_i^2 + m_i^2 \tag{17}$$

Then we write

$$\phi = \Delta E t - \frac{\Delta p^2}{2\bar{p}}x = \Delta E t - \frac{\Delta E^2 - \Delta m^2}{2\bar{p}}x \tag{18}$$

$$= \Delta E t - \frac{2\bar{E}}{2\bar{p}}\Delta E x + \frac{\Delta m^2}{2\bar{p}}x \tag{19}$$

With the "average velocity" of the neutrino $v = \bar{p}/\bar{E}$ and $x \approx vt$ one gets

$$\phi \approx \frac{\Delta m^2}{2\bar{p}}x \approx \frac{\Delta m^2}{2\bar{E}}x \tag{20}$$

In the last step $v \approx 1$ has been used.

The main problem with this derivation is setting $x \approx vt$. Note that we are using a plane wave for the neutrinos, which is delocalized in space and time. A consistent treatment either requires the introduction of wave packets or a QFT calculation, see e.g. [1].

Solution 2: Mass and mixing angle in constant matter

The Hamiltonian in matter eq. 4 takes the form

$$H_{\text{mat}} = \frac{1}{2E}\begin{pmatrix} c^2 m_1^2 + s^2 m_2^2 + V & sc(m_2^2 - m_1^2) \\ sc(m_2^2 - m_1^2) & s^2 m_1^2 + c^2 m_2^2 \end{pmatrix} \tag{21}$$

The eigenvalues of this matrix are

$$\lambda_{1,2} = \frac{m_1^2 + m_2^2}{4E} + \frac{V}{2} \pm \frac{\Delta m^2}{4E}\sqrt{\sin^2 2\theta + (\cos 2\theta - A)^2}\,, \tag{22}$$

which leads to eq. 7 by $\Delta m^2_{\text{mat}} = 2E(\lambda_2 - \lambda_1)$.

The expression for the mixing angle in matter, Eq. 6, is most easily obtained by calculating $U^\dagger(\theta_{\text{mat}})H_{\text{mat}}U(\theta_{\text{mat}})$ and demanding that the off-diagonal element of this matrix is zero. This gives

$$\tan 2\theta_{\text{mat}} = \frac{\sin 2\theta}{\cos 2\theta - A}\,, \tag{23}$$

which implies Eq. 6.

Solution 3: The $\tilde{\alpha}^2$-term

At the first oscillation maximum we have

$$\Delta \equiv \frac{\Delta m_{31}^2 L}{4E} = \frac{\pi}{2} \qquad \Rightarrow \qquad \frac{L}{4E} = \frac{\pi}{2}\frac{1}{\Delta m_{31}^2} \tag{24}$$

and

$$\tilde{\alpha} \equiv \sin 2\theta_{12}\frac{\Delta m_{21}^2 L}{4E} = \sin 2\theta_{12}\frac{\pi}{2}\frac{\Delta m_{21}^2}{\Delta m_{31}^2} \approx 0.046\,. \tag{25}$$

Hence, $\tilde{\alpha}^2 \approx 2.1 \times 10^{-3}$, and for $\sin^2 2\theta_{13} \gtrsim 0.01$ this term can be neglected. From reactor experiments we have $\sin^2 2\theta_{13} \simeq 0.1$, so it is save to neglect the $\tilde{\alpha}^2$ term.

Solution 4: $\sin^2 2\theta_{13}$-determination in appearance experiments

Neglecting the $\tilde{\alpha}^2$ term and using $\Delta = \pi/2$, Eq. 8 becomes

$$P_{\mu\to e} \equiv P \approx s_{23}^2\, S^2 \mp \sin 2\theta_{23}\, \tilde{\alpha}\, S\, \sin\delta_{\mathrm{CP}}\,. \tag{26}$$

Solving for S yields

$$S \approx \pm\cot\theta_{23}\,\tilde{\alpha}\sin\delta_{\mathrm{CP}} + \sqrt{\frac{P}{s_{23}^2}}\,, \tag{27}$$

where again $\tilde{\alpha}^2$ terms have been neglected wrt P and the sign in front of the square-root has been chosen "+", in order to keep S positive. Hence,

$$S^2 \approx \frac{1}{s_{23}^2}\left(P \pm 2\sqrt{P}\,c_{23}\,\tilde{\alpha}\sin\delta_{\mathrm{CP}}\right). \tag{28}$$

a) Eq. 28 shows that the region of S^2 follows the shape of a sin function in δ_{CP}.

b) For neutrinos (anti-neutrinos) the region goes as $\sin\delta_{\mathrm{CP}}$ ($-\sin\delta_{\mathrm{CP}}$).

c) The region in Eq. 28 is a linear function in $1/s_{23}^2$, the amplitude of the $\sin\delta_{\mathrm{CP}}$-term is proportional to c_{23}. Hence the region in S^2 shifts and the amplitude of the δ_{CP} term changes as θ_{23} varies.

Remark: Note that Eq. 28 is not always positive. In order to have a positive definite expression all higher order terms have to be kept.

Solution 5: The $\mathrm{sign}(\Delta m_{31}^2)$-degeneracy

a) Using Eq. 8, Eq. 10 leads to the condition

$$\begin{aligned} \cos(\Delta \pm \delta_{\mathrm{CP}}) &= -\cos(-\Delta \pm \delta'_{\mathrm{CP}}) \qquad (29)\\ &= -\cos(\Delta \mp \delta'_{\mathrm{CP}}) \qquad (30) \end{aligned}$$

Writing $\delta'_{\rm CP} = x - \delta_{\rm CP}$, one obtains

$$\begin{aligned} \cos(\Delta \pm \delta_{\rm CP}) &= -\cos(-\Delta \pm \delta_{\rm CP} \mp x) && (31) \\ &= -\cos(\Delta \pm \delta_{\rm CP})\cos x \mp \sin(\Delta \pm \delta_{\rm CP})\sin x && (32) \end{aligned}$$

This relation is fulfilled for $x = \pi$ and hence $\delta'_{\rm CP} = \pi - \delta_{\rm CP}$. Therefore, Eq. 10 holds for neutrinos and anti-neutrinos simultaneously, since Eq. 32 is fulfilled for both signs.

b) If the matter effect A is small, it will introduce a small perturbation to the solution found in case of vacuum:

$$S' = S + \epsilon_S\,, \quad \delta'_{\rm CP} = \pi - \delta_{\rm CP} + \epsilon_\delta\,. \tag{33}$$

For neutrinos and anti-neutrinos, Eq. 11 is a system of two equations, which can be linearized in the small quantities $\epsilon_S, \epsilon_\delta, A$. A linear system of two equations for two variables (ϵ_S and ϵ_δ) has in general a unique solution, and hence the first order matter effect cannot break the degeneracy.[3] In order to resolve it, one has to enter the regime of strong matter effect, i.e., close to the resonance.

Solution 6: Majorana mass term

Solutions to a), b), c) follow from eqs. 12 and 13.

d) A real symmetric matrix M can be diagonalized by

$$\hat{m} \equiv \mathrm{diag}(m_1, m_2) = OMO^T \quad \text{with} \quad O = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}. \tag{34}$$

The eigenvalues are given by

$$m_{1,2} = \frac{1}{2}\left[m_L + m_R \mp \sqrt{(m_L - m_R)^2 + 4m_D^2}\right], \tag{35}$$

which are real but may be negative. Therefore, the physical masses are $|m_1|$ and $|m_2|$, and we define ρ_i as the sign of m_i such that $\rho_i m_i = |m_i|$, and ρ_i is called the CP parity of the Majorana mass fiel ψ_i. Eq. 16 can now be written as

$$\mathcal{L}_M = \frac{1}{2}\psi^T C^{-1} M \psi = \frac{1}{2}\psi^T O^T C^{-1} \hat{m} O \psi = \frac{1}{2}\hat{\psi}^T C^{-1} |\hat{m}| \hat{\psi} \tag{36}$$

with the mass eigen fields defined by

$$\hat{\psi} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \mathrm{diag}(\sqrt{\rho_1}, \sqrt{\rho_2}) O \psi\,. \tag{37}$$

The mixing angle in O is defined by

$$\tan 2\theta = \frac{2m_D}{m_R - m_L}\,. \tag{38}$$

[3] An explicit calculation can be found in Ref. [5].

e)(i) The limit $m_L, m_R \ll m_D$. From the above results follows

$$m_{1,2} \approx \mp m_D + \frac{m_L + m_R}{2} \quad \text{and} \quad \theta \to \pi/4 \,. \tag{39}$$

In the limit of $m_L = m_R = 0$ we recover a Dirac particle. It follows that a Dirac field can be represented by two degenerate Majorana fields with mass m_D and oposite CP parity:

$$\psi_1 = \frac{i}{\sqrt{2}} \left(\psi_L + (\psi_R)^c\right) \,, \quad \psi_2 = \frac{1}{\sqrt{2}} \left(-\psi_L + (\psi_R)^c\right) \,. \tag{40}$$

The case of small but non-zero m_L, m_R is called pseudo-Dirac particle.

e)(ii) In the limit $m_L \ll m_D \ll m_R$ we obtain

$$m_1 \approx m_L - \frac{m_D^2}{m_R} \,, \quad m_2 \approx m_R \,, \quad \theta \approx \frac{m_D}{m_R} \,. \tag{41}$$

Setting $m_L \approx 0$ and assuming $m_R > 0$ we have

$$\psi_1 \approx i \left(\psi_L + \frac{m_D}{m_R} (\psi_R)^c \right) \,, \quad \psi_2 \approx (\psi_R)^c - \frac{m_D}{m_R} \psi_L \,. \tag{42}$$

Hence, there is a light (heavy) mass state with mass m_D^2/m_R (m_R), and the corresponding field coincides with ψ_L (ψ_R^c) up to order m_D/m_R. This is the Seesaw mechanism.