# Perturbative QCD 2

Giulia Zanderighi (Max Planck Institute für Physik)

2nd Lecture

Nordita Winter School 2024 in Particle Physics and Cosmology
January 2024

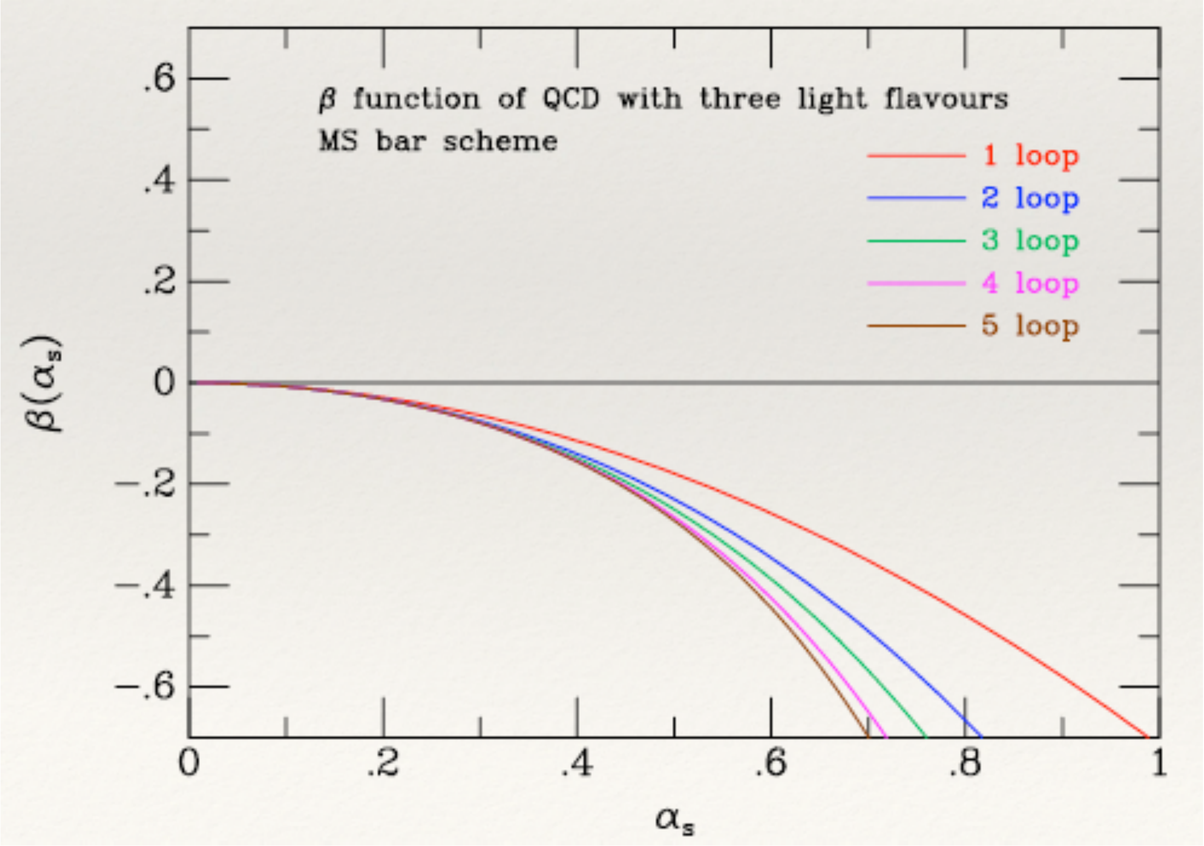
# Back to the QCD beta-function

Perturbative expansion of the beta-function:

$$\beta = -\alpha_s^2(\mu)\sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11N_c - 4n_f T_R}{12\pi}$$

$$b_1 = \frac{17N_c^2 - 5N_c n_f - 3C_F n_f}{24\pi^2}$$

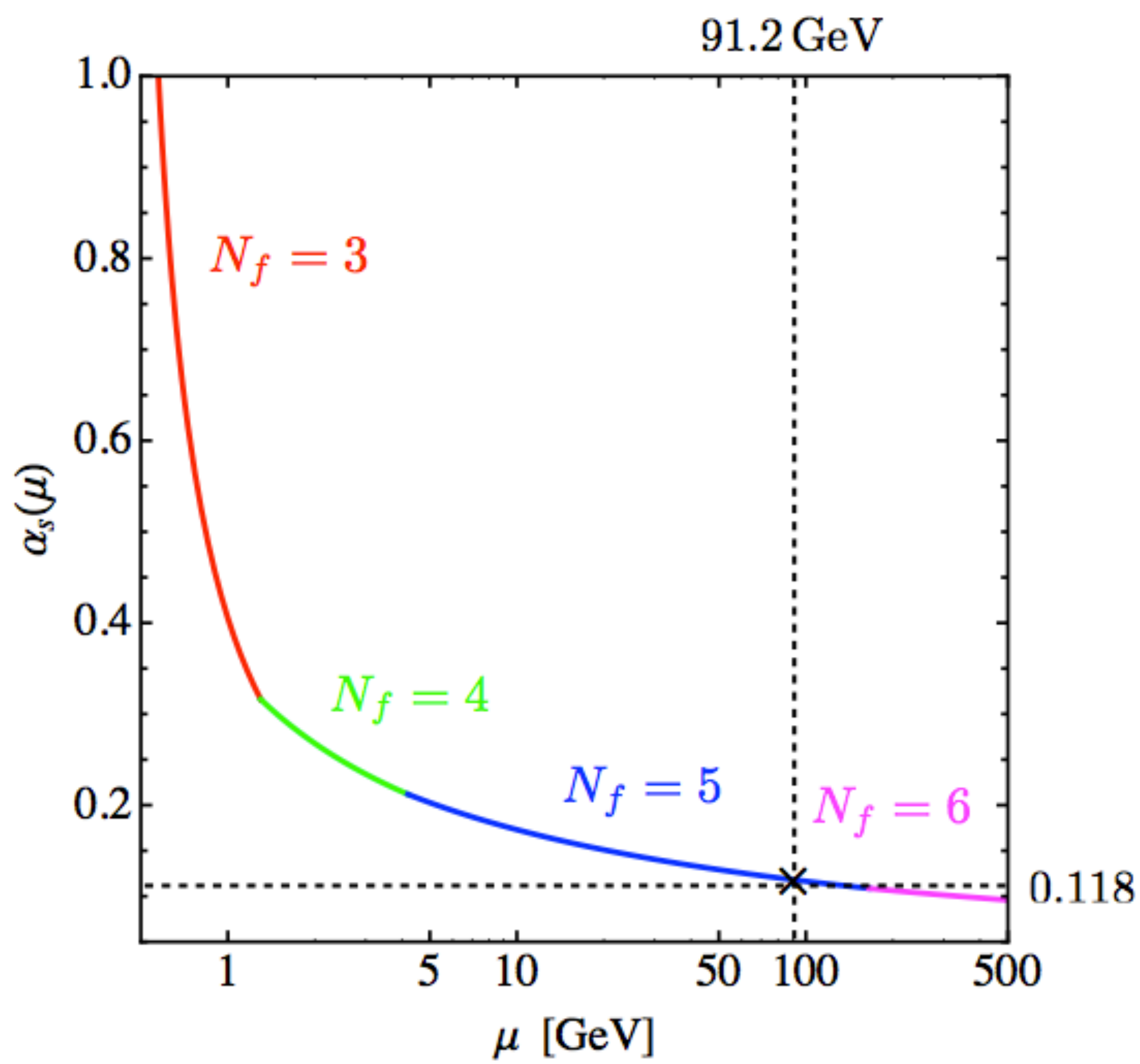



- $n_f$ is the number of active flavours (depends on the scale)
- today, the beta-function known up to five loops, but only first two coefficients are independent of the renormalization scheme

Exercise: proof the above statement [hint: use the fact that at $O(\alpha_s)$ the coupling in two different schemes is related by a finite change]

# Active flavours & running coupling

The active field content of a theory modifies the running of the couplings




Constrain New Physics by measuring the running at high scales?

# Renormalization Group Equation

Consider a dimensionless quantity A, function of a single scale Q. The dimensionless quantity should be independent of Q. However in quantum field theory this is not true, as renormalization introduces a second scale μ

But the renormalization scale is arbitrary. The dependence on it must cancel in physical observables up to the order to which one does the calculation.

So, for any observable A one can write a renormalization group equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] A\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 0$$

$$\alpha_s = \alpha_s(\mu^2) \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

Scale dependence of A enters through the running of the coupling: knowledge of $A(1, \alpha_s(Q^2))$ allows one to compute the variation of A with Q given the beta-function

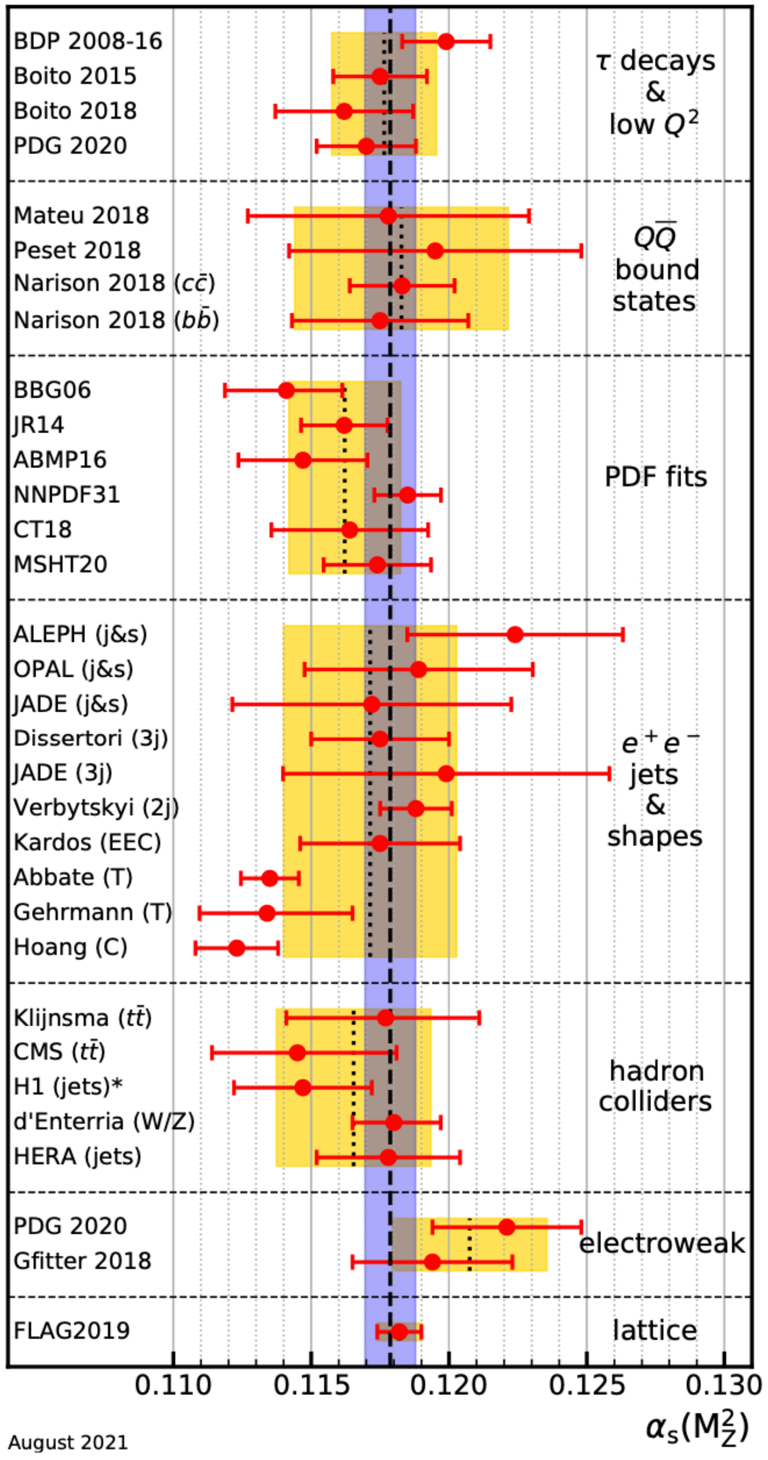
# Measurements of the running coupling

<u>Summarizing:</u>

- overall consistent picture: $\alpha_s$ from very different observables compatible
- $\alpha_s$ is not so small at current scales
- $\alpha_s$ decreases slowly at higher energies (logarithmic only)
- higher order corrections are and will remain important

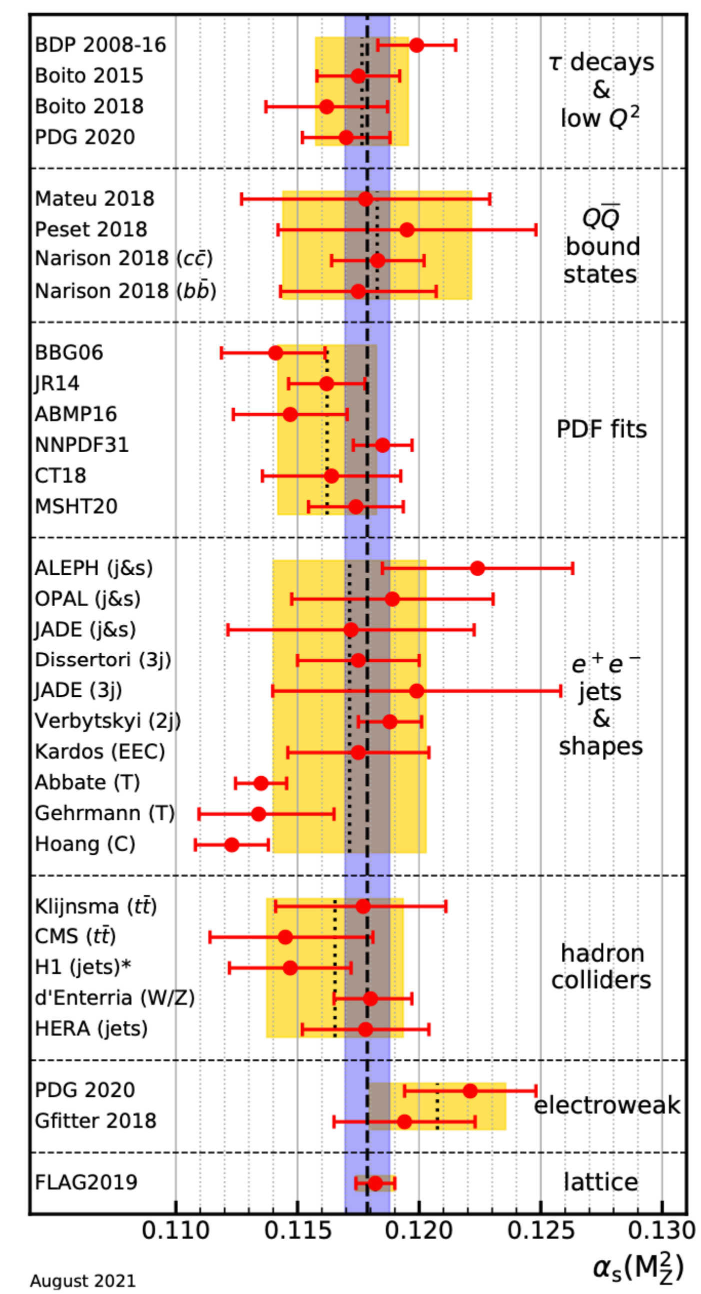
***World average***

$$\alpha_s(M_Z^2) = 0.1179 \pm 0.0009$$



# Measurements of the running coupling

Uncertainty on $\alpha_s$ (and PDF) is in several cases the dominant source of uncertainty to provide precise theory predictions




<u>Procedure to compute worlds average in PDG</u>:

- subdivide observables in categories
- provide an average for each category
- provide an average of all categories

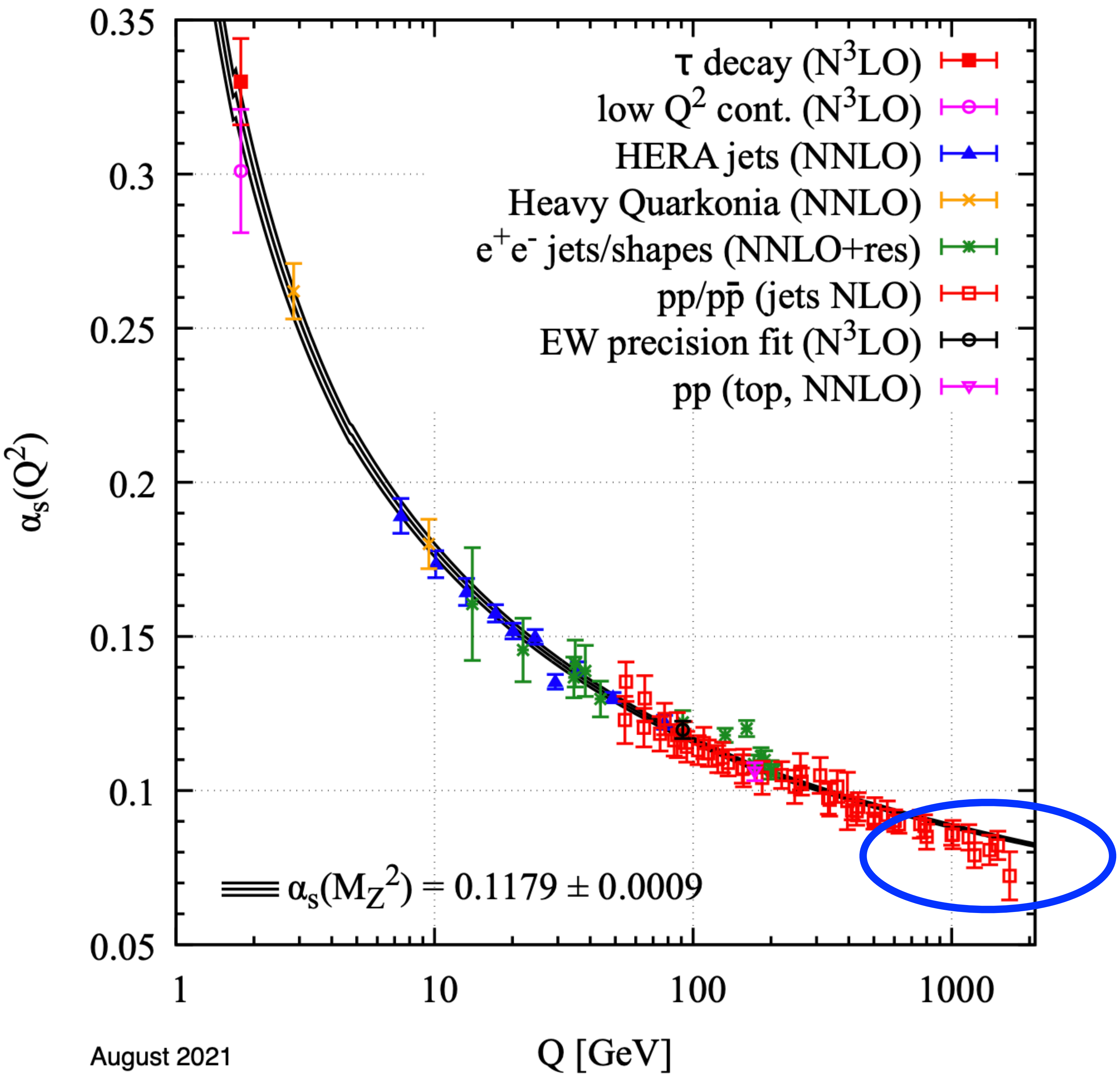
  ⇒ *the world average of $\alpha_s$*

$$\alpha_s(M_Z^2) = 0.1179 \pm 0.0009$$

$$\alpha_s(M_Z^2) = 0.1182 \pm 0.0008\,, \qquad \text{(lattice)}$$
$$\alpha_s(M_Z^2) = 0.1176 \pm 0.0010\,, \qquad \text{(without lattice)}$$

Many ambiguities, choices (e.g. treatment of correlations etc.), subtle aspects involved…

# Measurements of the running coupling



# Recap

The formulation of QCD as a non-abelian Quantum Field Theory allows to

- Describe the hadron spectrum
- Explain experimentally the observed symmetries in the strong iteration
- No mixing between strong and weak interactions
- Obtain a field-theoretical description of the strong force, opening the path to a unified formalism of all fundamental interactions

We have then discussed the UV behaviour of QCD

- discussed renormalisation of UV divergences
- introduced the running of the coupling constant and the beta-function
- discussed measurements of the coupling constant

As we will see, the perturbative description of QCD is very predictive (while we understand much less the regime governed by strong dynamics)

# Next

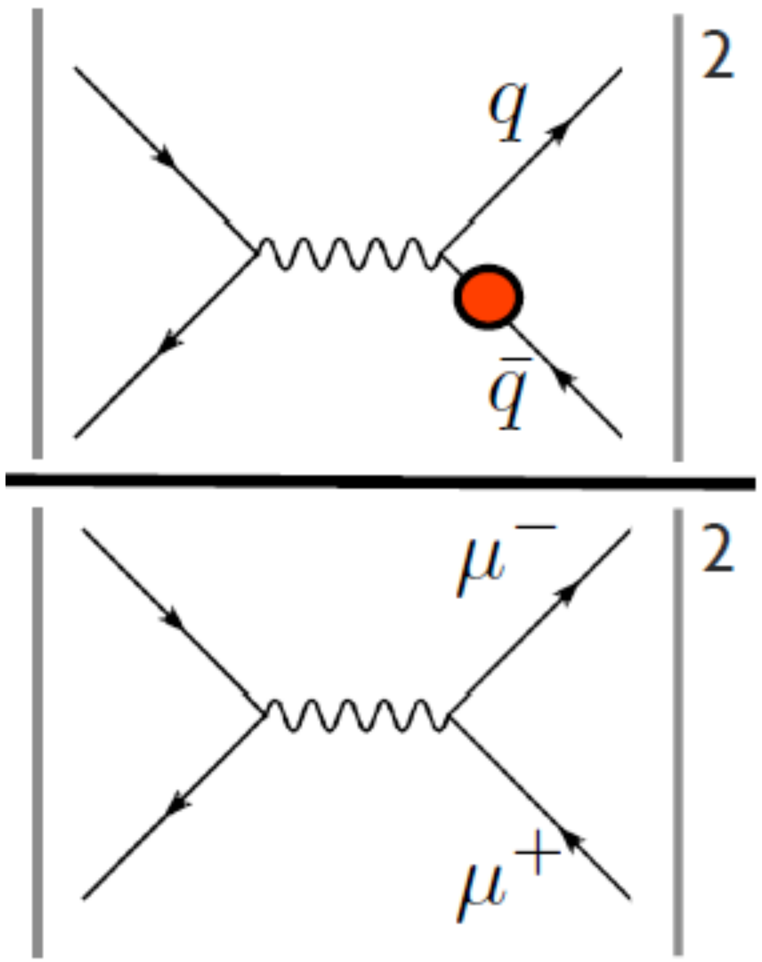
In the following we will concentrate on the perturbative regime of QCD.

In particular, we'll discuss generic properties of QCD amplitudes

- Soft-collinear divergences and how they are dealt with
- Kinoshita-Lee-Nauenberg theorem
- The concept of infrared finiteness
- Sterman Weinberg jets

# The soft approximation

Let's consider again the R-ratio

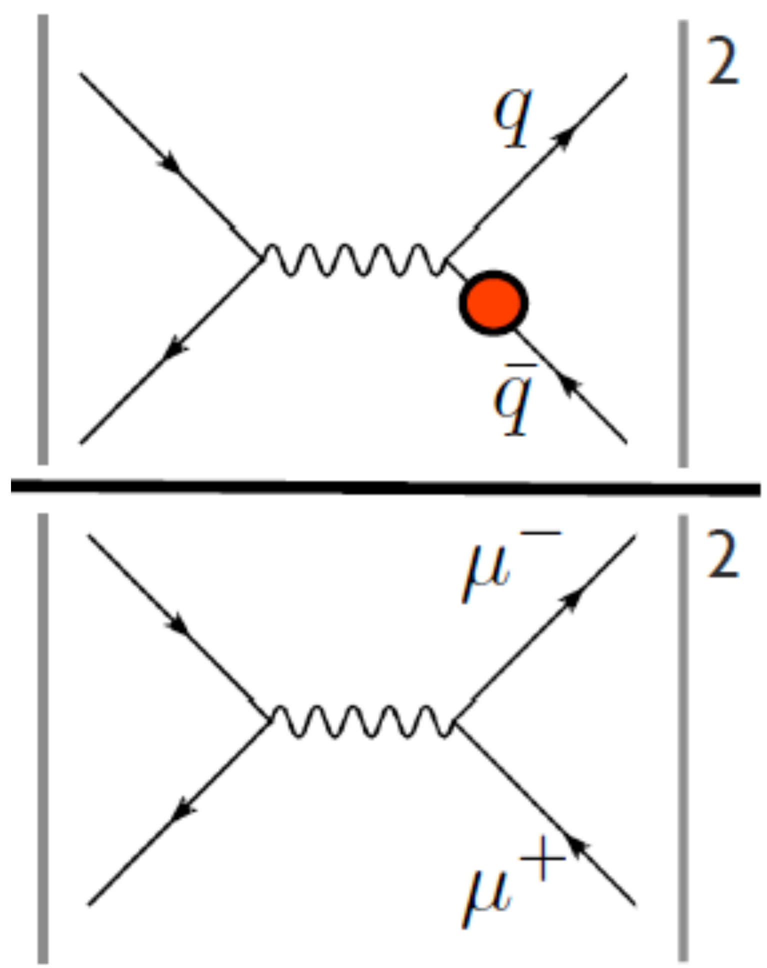



Leading order result

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q e_q^2$$

# The soft approximation

Let's consider again the R-ratio




We have seen a good agreement between the leading order result and data, but there are various unanswered questions

- Since free quarks do not exist, why is the leading order result so good?
- In particular, why can one identify the cross-sections for the production of quarks to that of hadrons?
- Can one probe QCD further by testing more exclusive observables?

# Quark-hadron duality

The reliability of parton-level calculations to describe hadron-level observables is known as quark-hadron duality.

This duality relies on the time separation between a hard scattering (partons are produced) and a soft process (quarks hadronize). Since the two processes happen at very different time-scales there is not quantum interference and the soft process does not alter the hard momentum flow "too much"

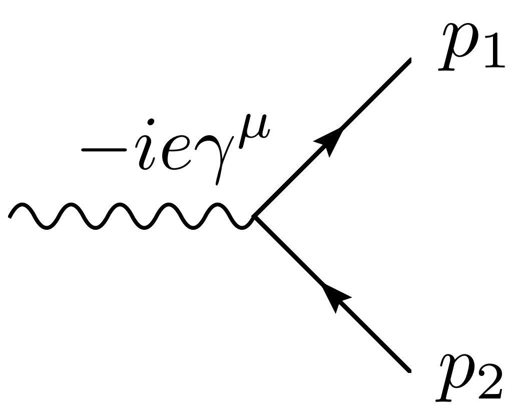
With this in mind, let's apply the parton description and look for a better approximation of R, i.e. let's compute QCD corrections, at least in some approximation

# The soft approximation

QCD corrections are only in the final state, i.e. corrections to $\gamma^* \to q\bar{q}$

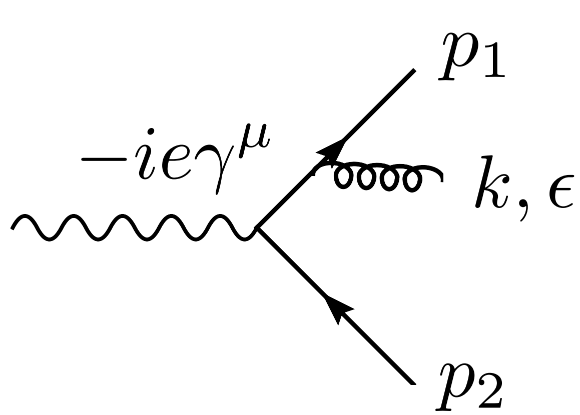
At leading order:

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$




Emit one gluon:

$$\begin{aligned} M^\mu_{q\bar{q}g} &= \bar{u}(p_1)(-ig_s t^a \not\epsilon)\frac{i(\not p_1 + \not k)}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2) \\ &+ \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not p_2 - \not k)}{(p_2-k)^2}(-ig_s t^a \not\epsilon)v(p_2) \end{aligned}$$




Consider the soft approximation: $k \ll p_1, p_2$

$$M^\mu_{q\bar{q}g} = \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)$$

$\Rightarrow$ factorization of soft part (crucial for resummed calculations)

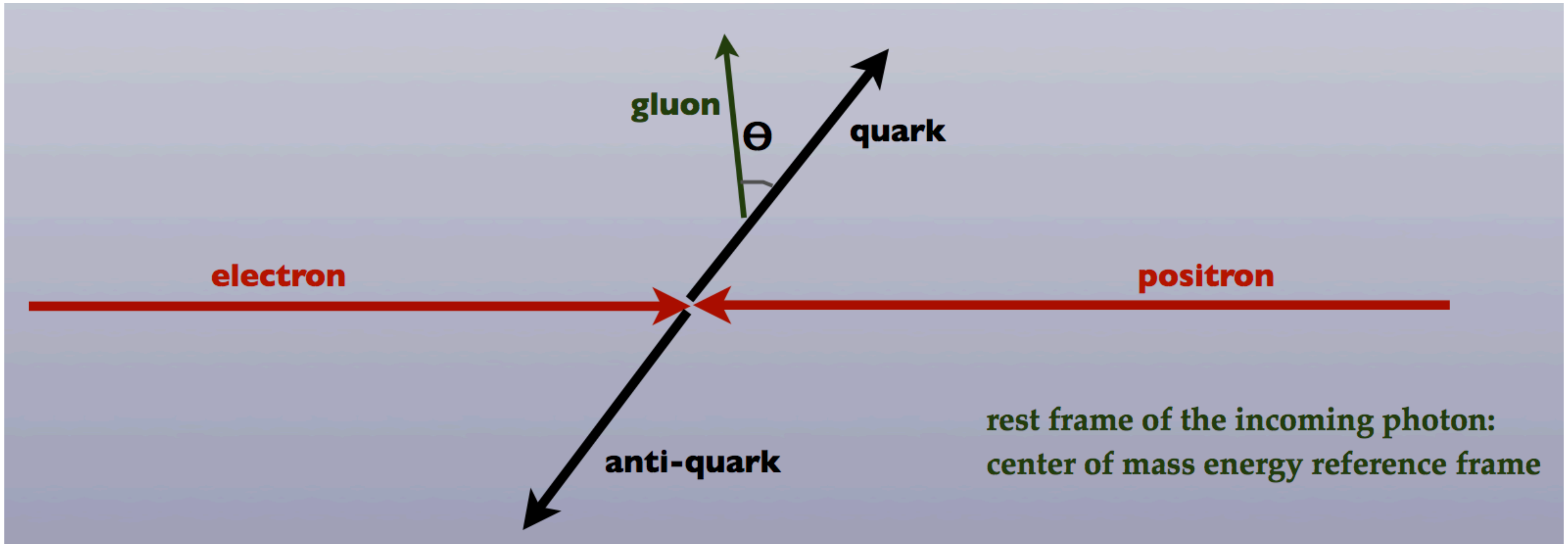
# Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

The above is a Lorentz-invariant amplitude. Go to the centre-of-mass frame:



# Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

Including phase space, in this frame, in terms of energy and angle of the gluon one contains

$$d\phi_{q\bar{q}g}|M_{q\bar{q}g}|^2 \quad = \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \frac{d^3k}{2\omega(2\pi)^3} C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

$$= \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \omega d\omega d\cos\theta \frac{d\phi}{2\pi}\frac{2\alpha_s C_F}{\pi}\frac{1}{\omega^2(1-\cos^2\theta)}$$

The differential cross section becomes

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi}\frac{d\omega}{\omega}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

# Soft & collinear divergences

Cross section for producing a q$\bar{q}$-pair and a gluon is infinite (IR divergent)

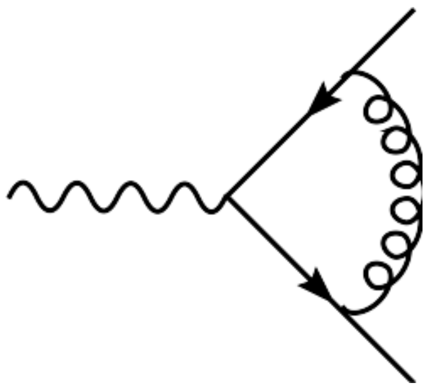
$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

<u>$\omega \to 0$:</u> soft divergence

<u>$\theta \to 0$:</u> collinear divergence

But the full O($\alpha_s$) correction to R is finite, because one must include a virtual correction which cancels the divergence of the real radiation

$$d\sigma_{q\bar{q},v} \sim -d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$



NB: here we kept only soft terms, if we do the full calculation one gets a finite correction of $\alpha_s/\pi$

# Soft & collinear divergences

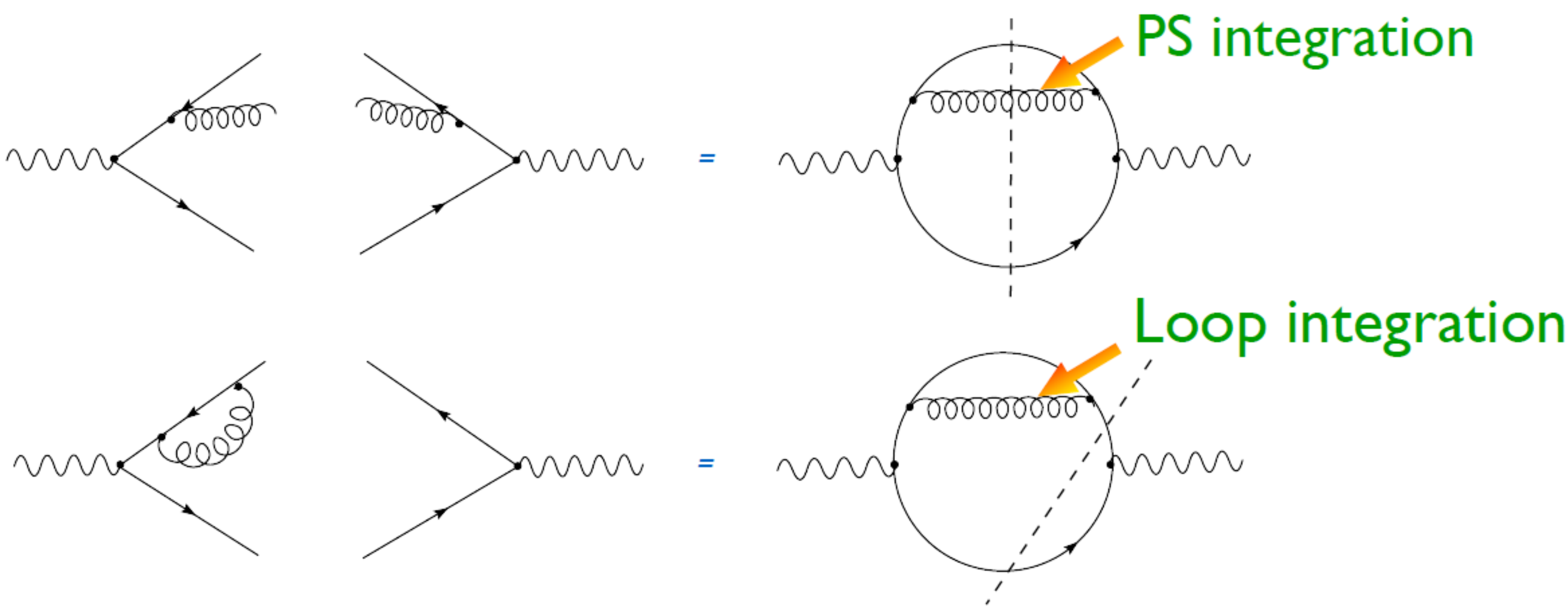
$\underline{\omega \to 0}$ soft divergence: the four-momentum of the emitted particle approaches zero, typical of gauge theories, even if matter (radiating particle) is massive

$\underline{\theta \to 0}$ collinear divergence: particle emitted collinear to emitter. Divergence present only if all particles involved are massless

NB: the appearance of soft and collinear divergences discussed in the specific contect of $e^+e^- \to qq$ are a general property of QCD

# Infrared finiteness

Cancellation of IR divergences in R is not a miracle. It follows directly from unitarity provided the measurement is inclusive enough



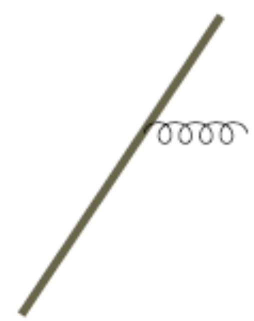

In the infrared region real and virtual are kinematically equivalent but for a (-1) from unitarity

Compute and regulate real and virtual separately, until a cancelation of divergences is achieved

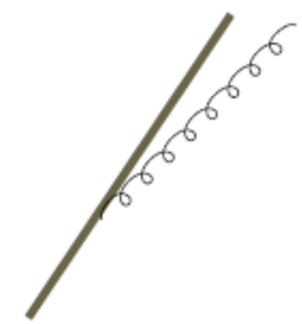
# KLN Theorem

Kinoshita-Lee-Nauenberg theorem: Infrared singularities in a massless theory cancel out after summing over degenerate (initial and final) states

hard



hard + soft gluon



2 collinear partons

Physically a hard parton can not be distinguished from a hard parton plus a soft gluon or from two collinear partons with the same energy. They are degenerate states.

Hence, one needs to add them to get a physically sound observable

# Infrared safety (= finiteness)

So, the R-ratio is an infrared safe quantity.

In perturbation theory one can compute only IR-safe quantities, otherwise get infinities, which can not be renormalized away (why not...?)

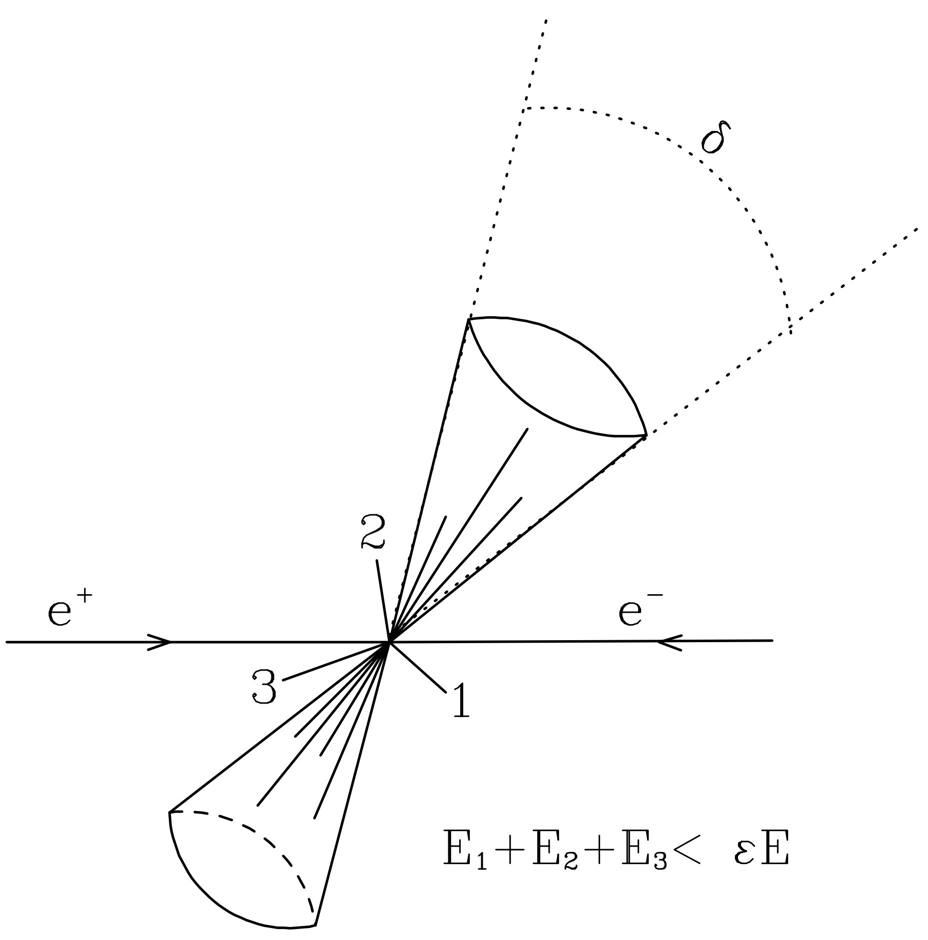
So, the natural questions are:

- are there other IR-safe quantities?
- what property of R guarantees its IR-safety?

# Sterman-Weinberg jets

First formulation of cross-sections which are finite in perturbation theory and describe the hadronic final state

Introduce two parameters $\varepsilon$ and $\delta$:
a pair of Sterman-Weinberg jets are two cones of opening angle $\delta$ that contain all the energy of the event excluding at most a fraction $\varepsilon$

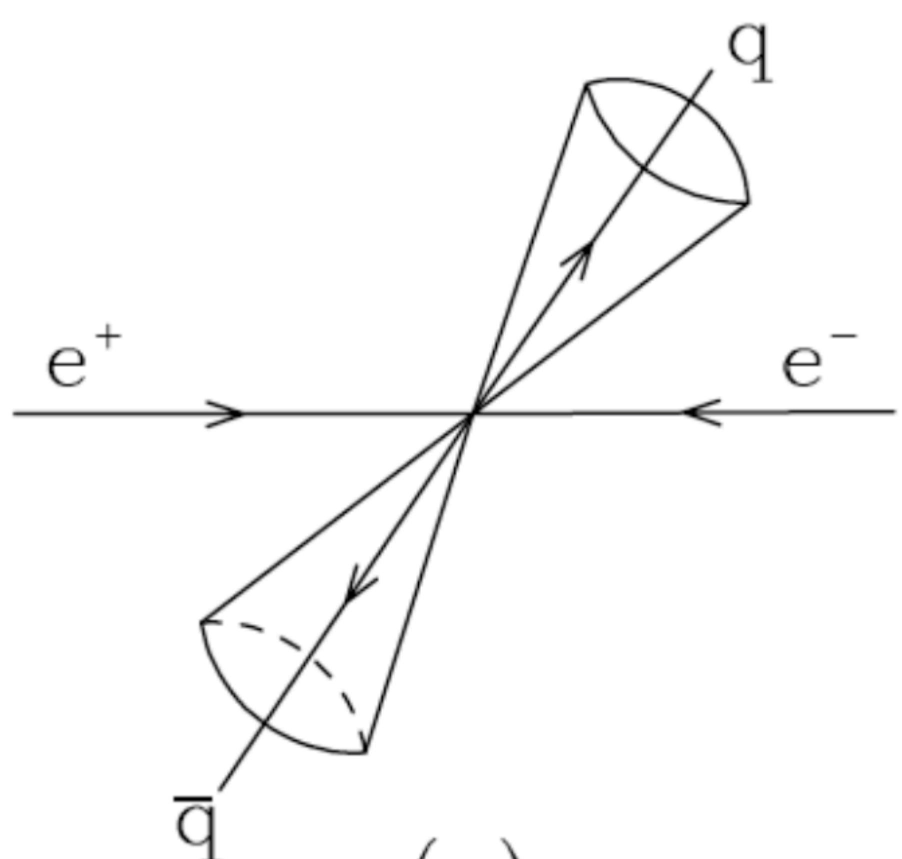



Why finite? the cancelation between real and virtual is not destroyed in the soft/collinear regions

# Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

a) We have a Born term $\sigma_B$ which is completely within the Sterman-Weinberg jet definition: since there are only two quarks they keep all the energy inside the cones

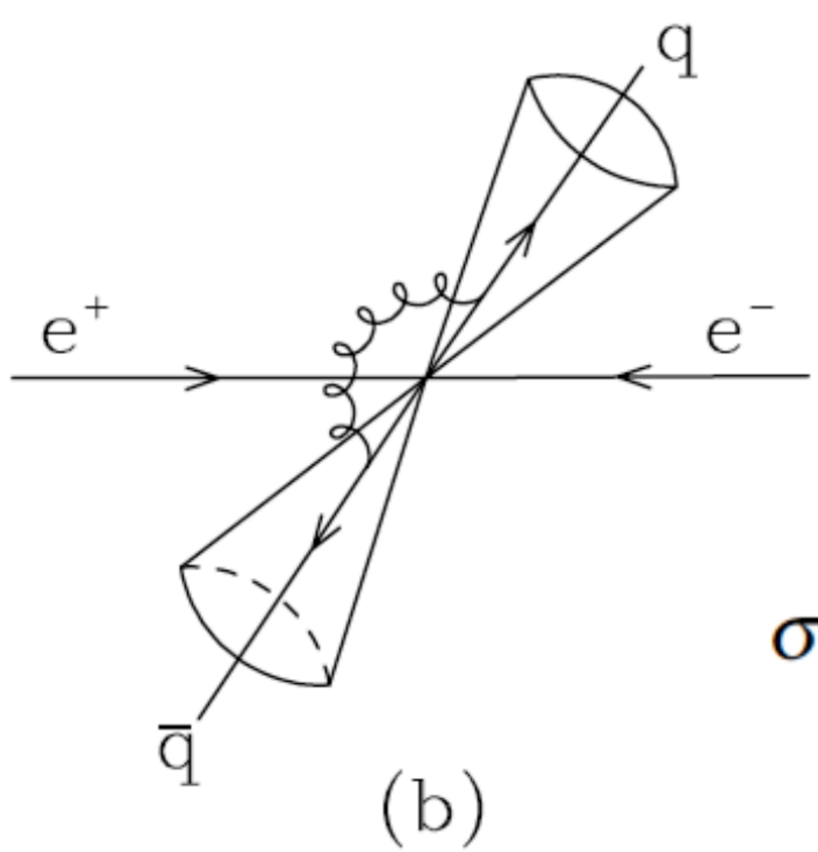


(a)

# Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

b) We have a virtual term which is also completely within the Sterman-Weinberg jet definition (only two quarks)

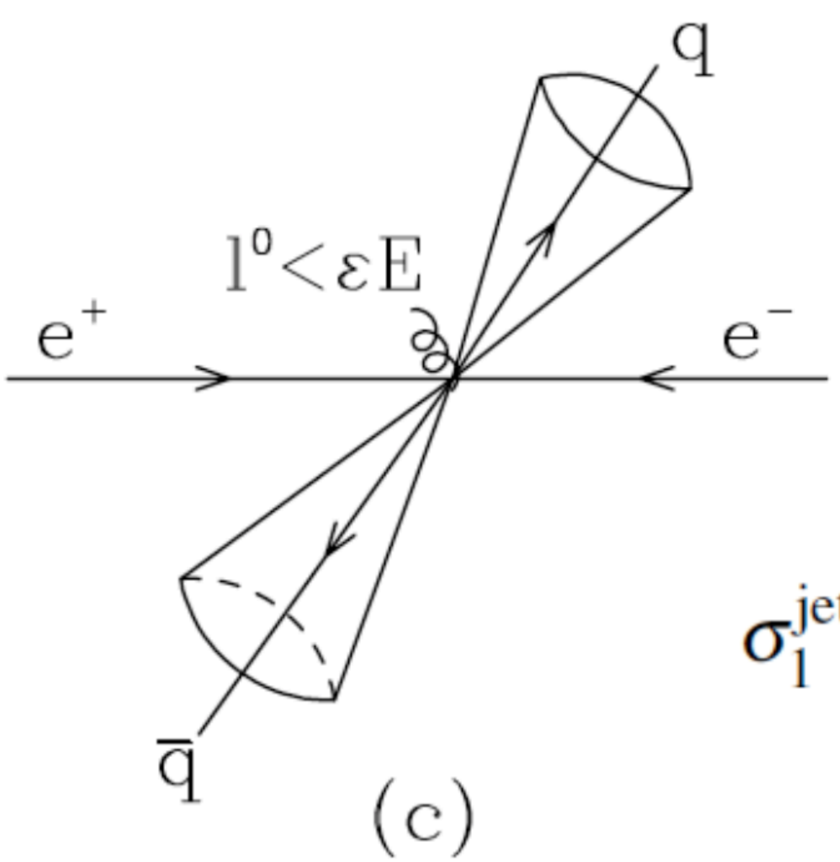



$$\sigma_1^{\text{jet,virt}} = -C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^E \frac{dl_0}{l_0} \int_0^\pi \frac{d\cos\theta}{1-\cos^2\theta}$$

# Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

c) We have a real term: the emitted gluon can be emitted also outside the jet provided it carries only little energy, or..

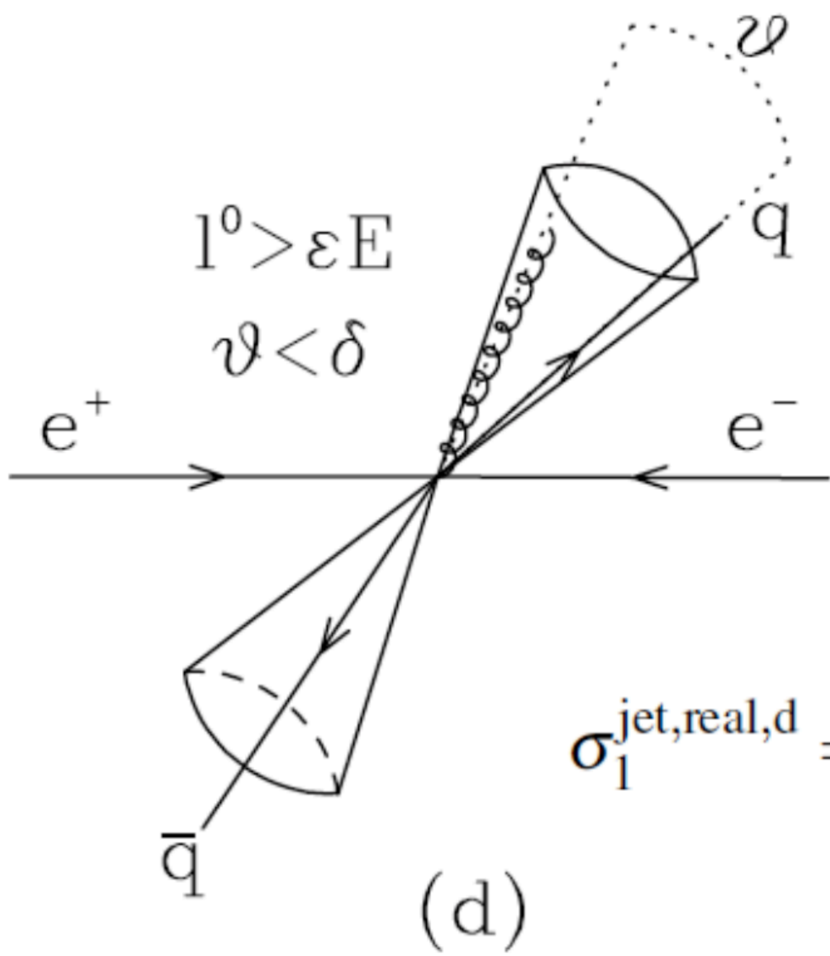



$$\sigma_1^{\text{jet,real,c}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^{\varepsilon E} \frac{dl_0}{l_0} \left[ \int_0^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

# Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

d) .. or it can carry a considerable fraction of energy provided it is emitted inside the cones




$$\sigma_1^{\text{jet,real,d}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_{\varepsilon E}^{E} \frac{dl_0}{l_0} \left[ \int_0^{\delta} \frac{d\cos\theta}{1-\cos^2\theta} + \int_{\pi-\delta}^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

# Sterman-Weinberg jets

Adding all the contributions, the Sterman-Weinberg jet cross-section up to $O(\alpha_s)$ in the soft-collinear approximation is given by

$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln\epsilon \ln\delta^2\right)$$

Effective expansion parameter in QCD is often $\alpha_s C_F/\pi$ not $\alpha_s$

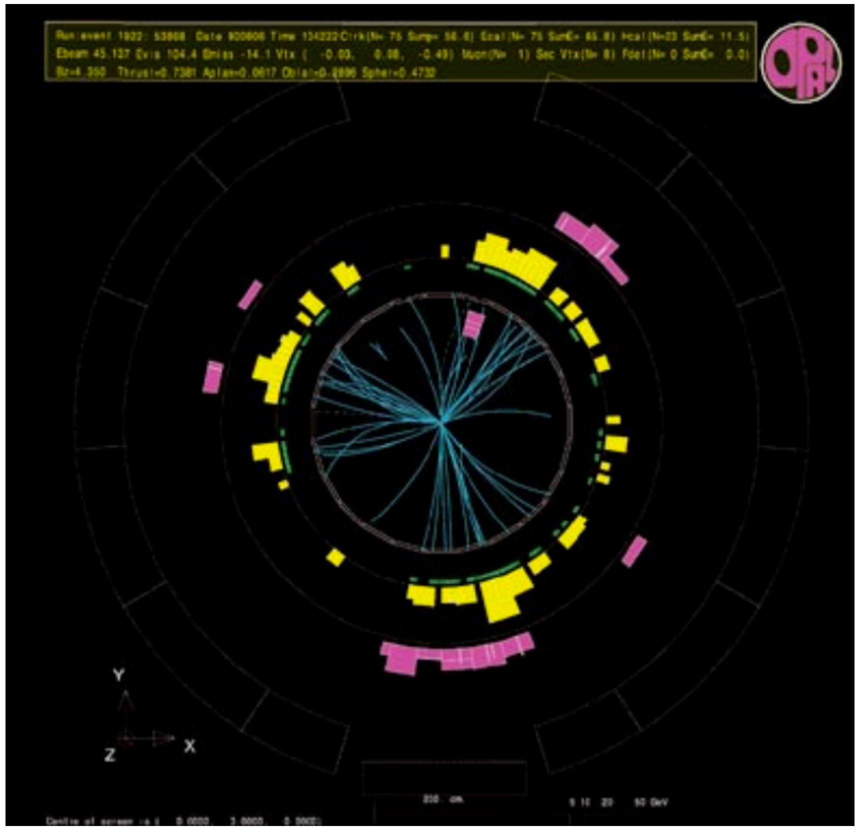
$\alpha_s$-expansion enhanced by a double log: left-over from real-virtual cancellation

- if more gluons are emitted, one gets for each gluon
  - a power of $\alpha_s C_F/\pi$
  - a soft logarithm $\ln\varepsilon$
  - a collinear logarithm $\ln\delta$
- if $\varepsilon$ and/or $\delta$ become too small the above result diverges
- if the logs are large, fixed order meaningless, one needs to resum large infrared and collinear logarithms to all orders in the coupling constant

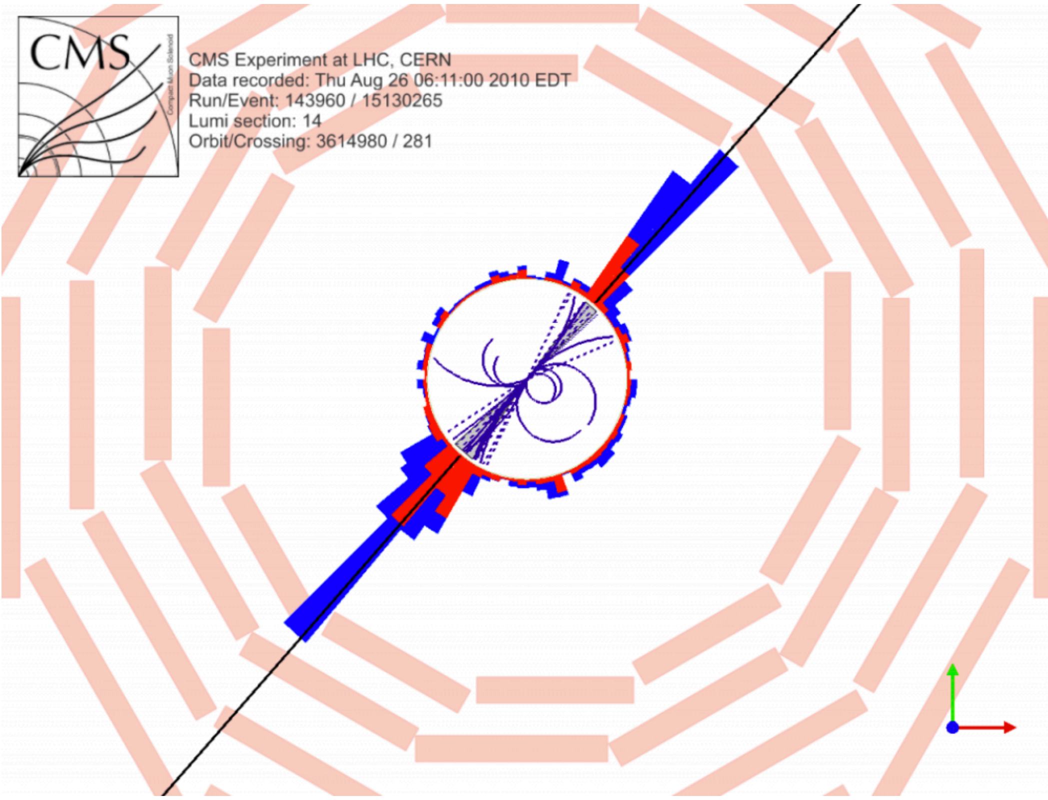
# Jets

- Jets were discovered in the late 70s in electron-position collision
- They provided the first direct evidence for the gluon (we'll discuss indirect evidence later)
- In the 80s and 90s jets provided many other stringent tests of QCD at LEP
- Today jets are one of the powerful tools to look for New Physics at the LHC

Gluon discovery: 3jet event in $e^+e^-$



High energy di-jet event at CMS



# Infrared safety: definition

An observable $\mathcal{O}$ is infrared and collinear safe if

$$\mathcal{O}_{n+1}(k_1, k_2, \ldots, k_i, k_j, \ldots k_n) \to \mathcal{O}_n(k_1, k_2, \ldots k_i + k_j, \ldots k_n)$$

whenever one of the $k_i/k_j$ becomes soft or $k_i$ and $k_j$ are collinear

i.e. the observable is insensitive to emission of soft particles or to collinear splittings

# Infrared safety: examples

## Infrared safe ?

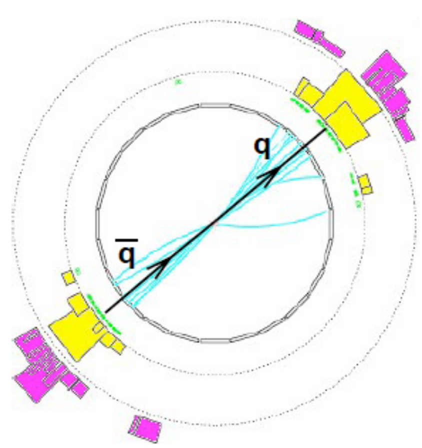
- energy of the hardest particle in the event — NO
- multiplicity of gluons — NO
- momentum flow into a cone in rapidity and angle — YES
- cross-section for producing one gluon with $E > E_{min}$ and $\theta > \theta_{min}$ — NO
- jet cross-sections — DEPENDS

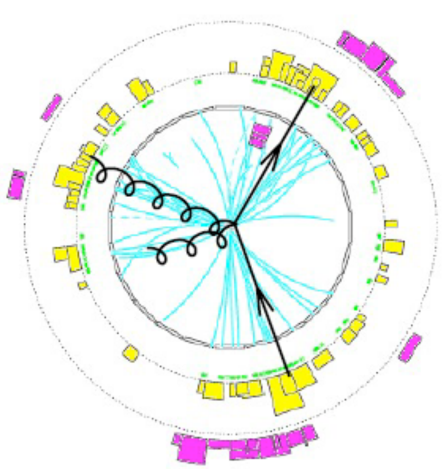
Only for infrared safe quantities is a comparison of data and theory well defined to all orders in perturbation theory

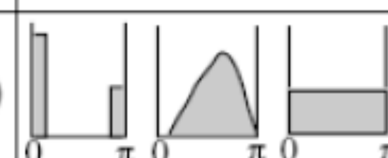
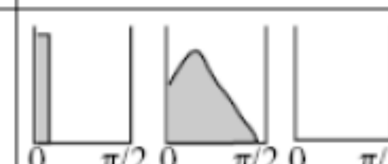
# Other IR safe quantities

Event shapes: describe the shape of the event, but are largely insensitive to soft and collinear branching

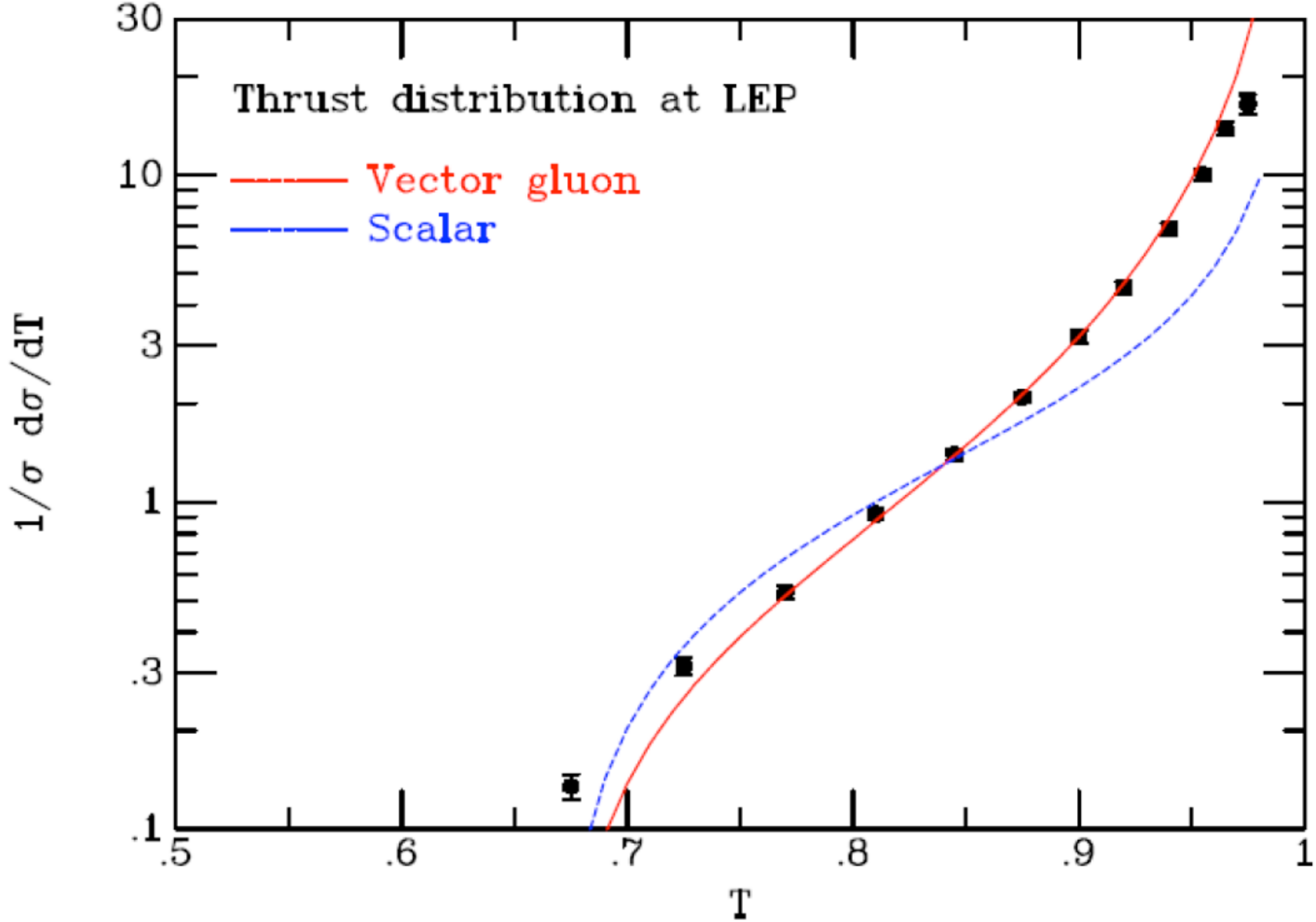
- widely used to measure $\alpha_s$
- measure color factors
- test QCD
- learn about non-perturbative physics



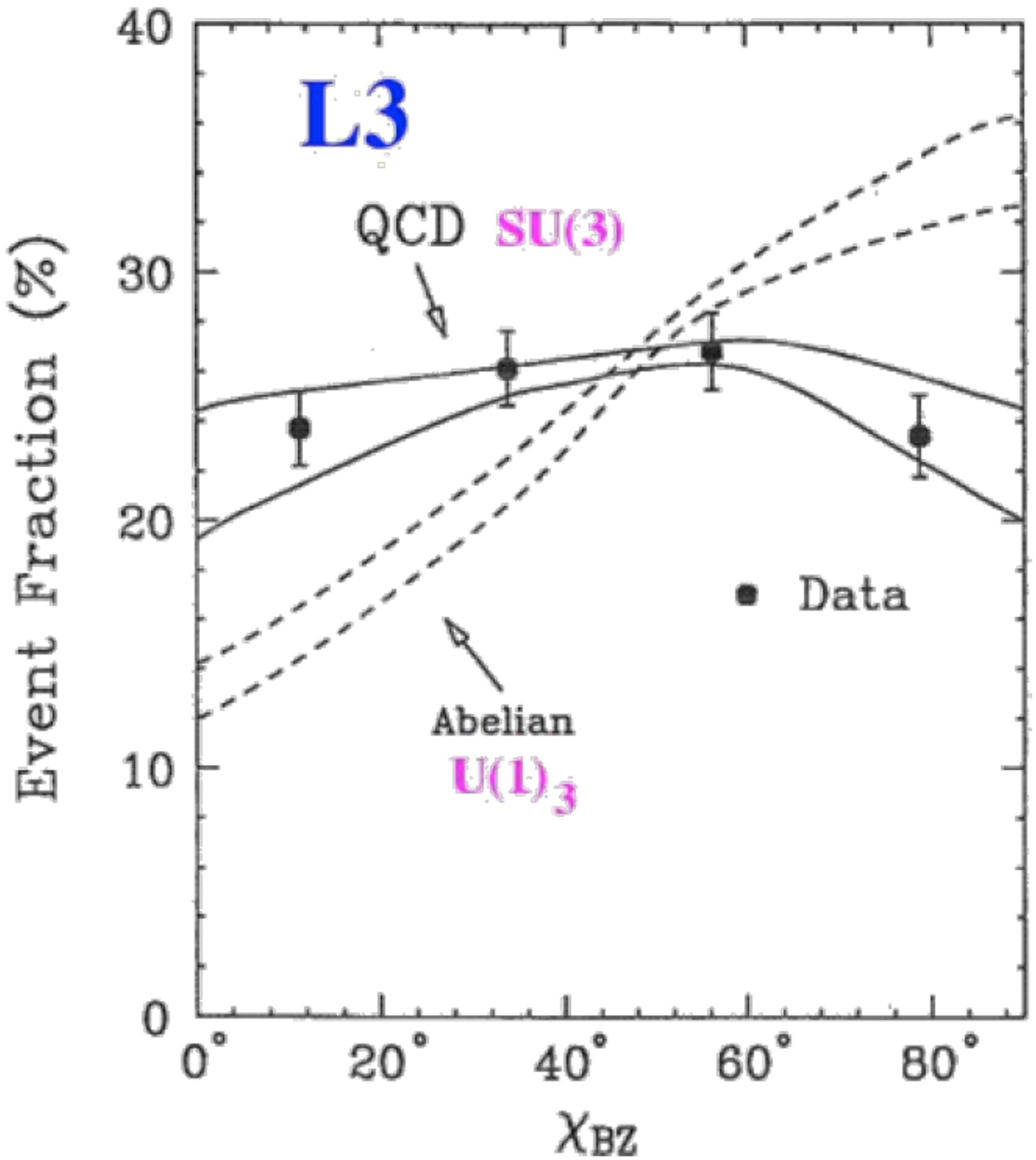



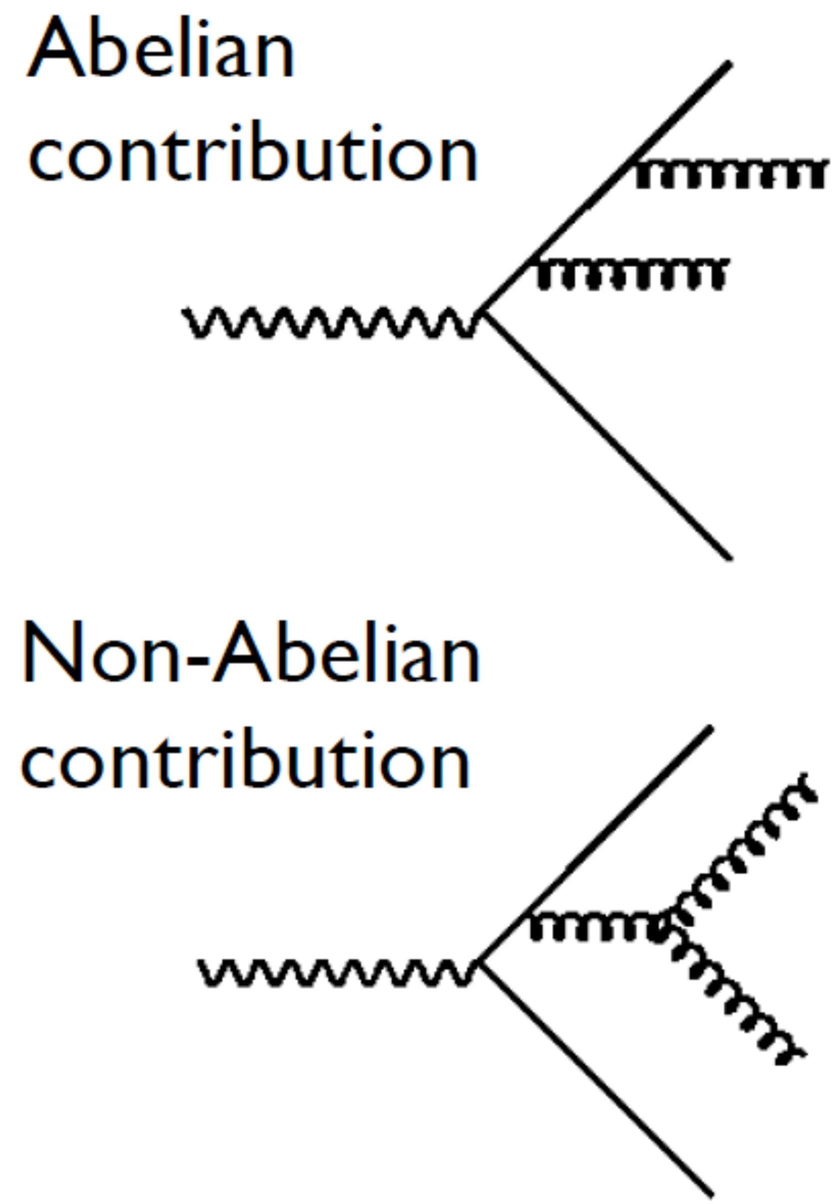
| Name of Observable | Definition | Typical Value for:  | | | QCD calculation |
|---|---|---|---|---|---|
| Thrust | $T = \max_{\vec{n}} \left( \frac{\sum_i \lvert \vec{p}_i \vec{n} \rvert}{\sum_i \lvert \vec{p}_i \rvert} \right)$ | 1 | $\geq 2/3$ | $\geq 1/2$ | (resummed) $O(\alpha_s^2)$ |
| Thrust major | Like T, however $T_{maj}$ and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$ | 0 | $\leq 1/3$ | $\leq 1/\sqrt{2}$ | $O(\alpha_s^2)$ |
| Thrust minor | Like T, however $T_{min}$ and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$ | 0 | 0 | $\leq 1/2$ | $O(\alpha_s^2)$ |
| Oblateness | $O = T_{maj} - T_{min}$ | 0 | $\leq 1/3$ | 0 | $O(\alpha_s^2)$ |
| Sphericity | $S = 1.5\,(Q_1 + Q_2)$; $Q_1 \leq \ldots \leq Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\sum_i p_i^\alpha p_i^\beta}{\sum_i p_i^2}$ | 0 | $\leq 3/4$ | $\leq 1$ | none (not infrared safe) |
| Aplanarity | $A = 1.5\,Q_1$ | 0 | 0 | $\leq 1/2$ | none (not infrared safe) |
| Jet (Hemisphere) masses | $M_\pm^2 = \left(\Sigma_i E_i^2 - \Sigma_i \vec{p}_i^{\,2}\right)_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$)<br>$M_H^2 = \max(M_+^2, M_-^2)$<br>$M_D^2 = \lvert M_+^2 - M_-^2 \rvert$ | <br>0<br>0 | <br>$\leq 1/3$<br>$\leq 1/3$ | <br>$\leq 1/2$<br>0 | (resummed) $O(\alpha_s^2)$ |
| Jet broadening | $B_\pm = \frac{\sum_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2 \sum_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$<br>$B_w = \max(B_+, B_-)$ | 0<br>0 | $\leq 1/(2\sqrt{3})$<br>$\leq 1/(2\sqrt{3})$ | $\leq 1/(2\sqrt{2})$<br>$\leq 1/(2\sqrt{3})$ | (resummed) $O(\alpha_s^2)$ |
| Energy-Energy Correlations | $EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi - \frac{\Delta\chi}{2}}^{\chi + \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$ |  | | | (resummed) $O(\alpha_s^2)$ |
| Asymmetry of EEC | $AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$ |  | | | $O(\alpha_s^2)$ |
| Differential 2-jet rate | $D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$ | | | | (resummed) $O(\alpha_s^2)$ |

# Example: spin of the gluon

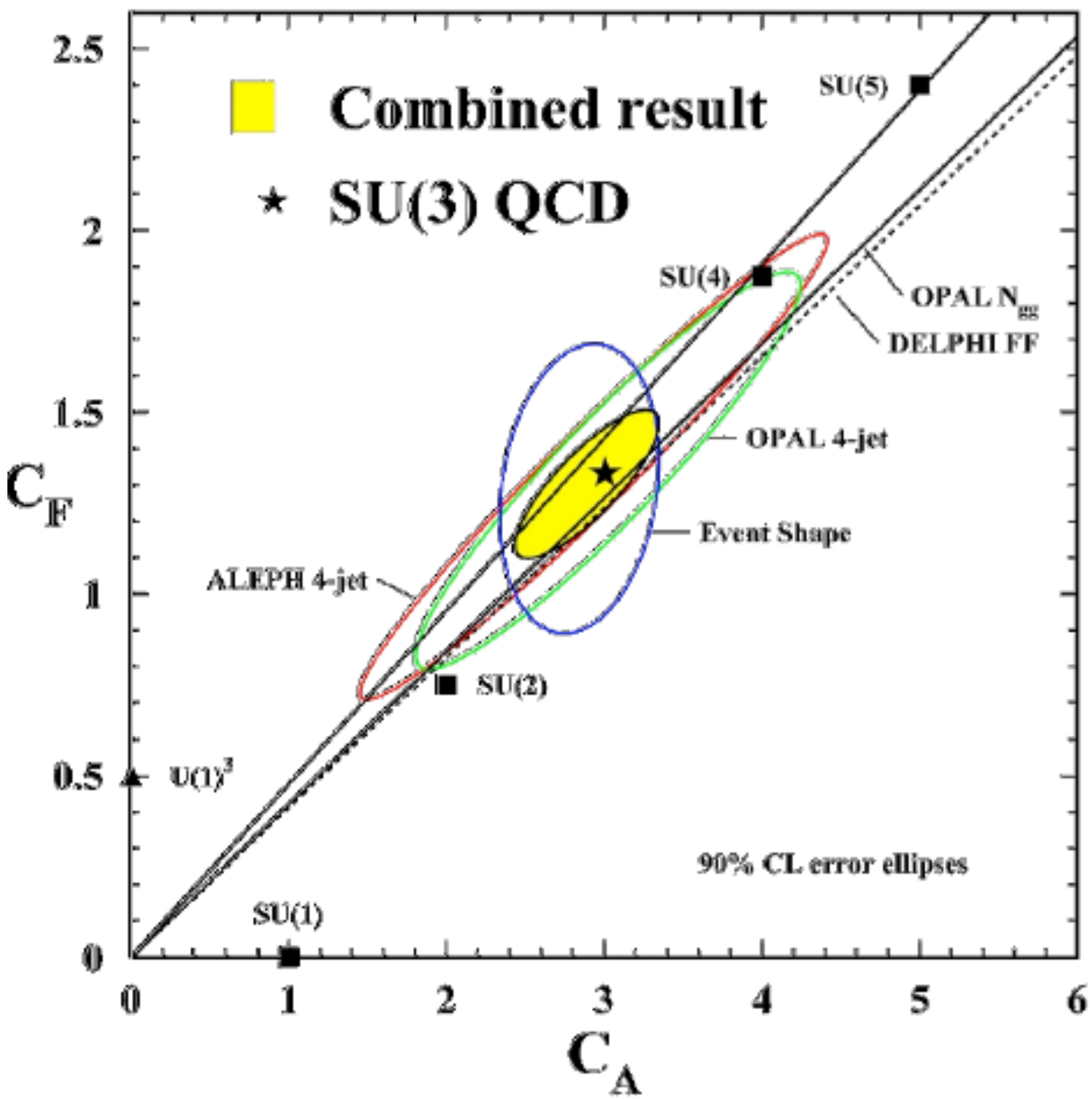


# Example: non-abelian nature of QCD



# Example: fits of colour fators




Fits of colour factors from 4-jet rates and event shapes

$$C_A = 2.89 \pm 0.21$$
$$C_F = 1.30 \pm 0.09$$

Well compatible with QCD:

$$C_A = 3$$
$$C_F = \frac{4}{3}$$

# Recap

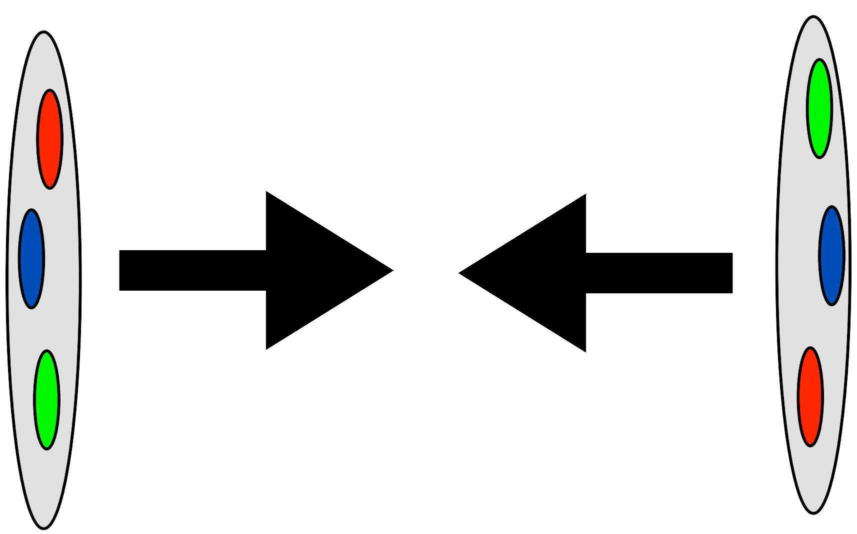
Brief recap on the infrared behaviour of QCD

- we have seen that soft and collinear divergences arise universally in QCD calculations
- these divergences cancel in $e^+e^-$ observables in inclusive observables (KLN theorem)
- we have performed a first genuine QCD calculation: the cross-section for Sterman Weinberg jets in $e^+e^-$ collisions
- perturbative QCD can be used to compute jet-cross section and other infrared-safe event shape variables
- comparison of theory and calculations provide stringent tests of QCD

# Partons in the initial state

- We talked a lot about final state QCD effects
- This is the only thing to worry about at $e^+e^-$ colliders (LEP)
- Hera/Tevatron/LHC involve protons in the initial state
- Proton are made of QCD constituents

Next we will focus mainly on aspects related to initial state effects

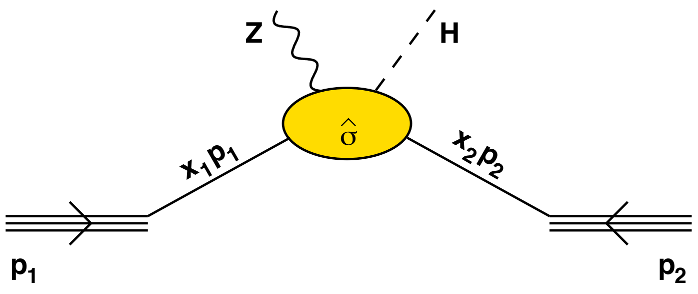


# The parton model

Basic idea of the parton model: intuitive picture where in a high transverse momentum scattering partons behave as quasi free in the collision

$\Longrightarrow$ cross section is the incoherent sum of all partonic cross-sections

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$




*NB: This formula is wrong/incomplete (see later)*

$f_i^{(P_j)}(x_i)$: parton distribution function (PDF) is the probability to find parton i in hadron j with a fraction $x_i$ of the longitudinal momentum (transverse momentum neglected), extracted from data

$\hat{\sigma}(x_1 x_2 s)$ : partonic cross-section for a given scattering process, computed in perturbative QCD

# Sum rules

Momentum sum rule: conservation of incoming total momentum

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

Conservation of flavour: e.g. for a proton

$$\int_0^1 dx \left( f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left( f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

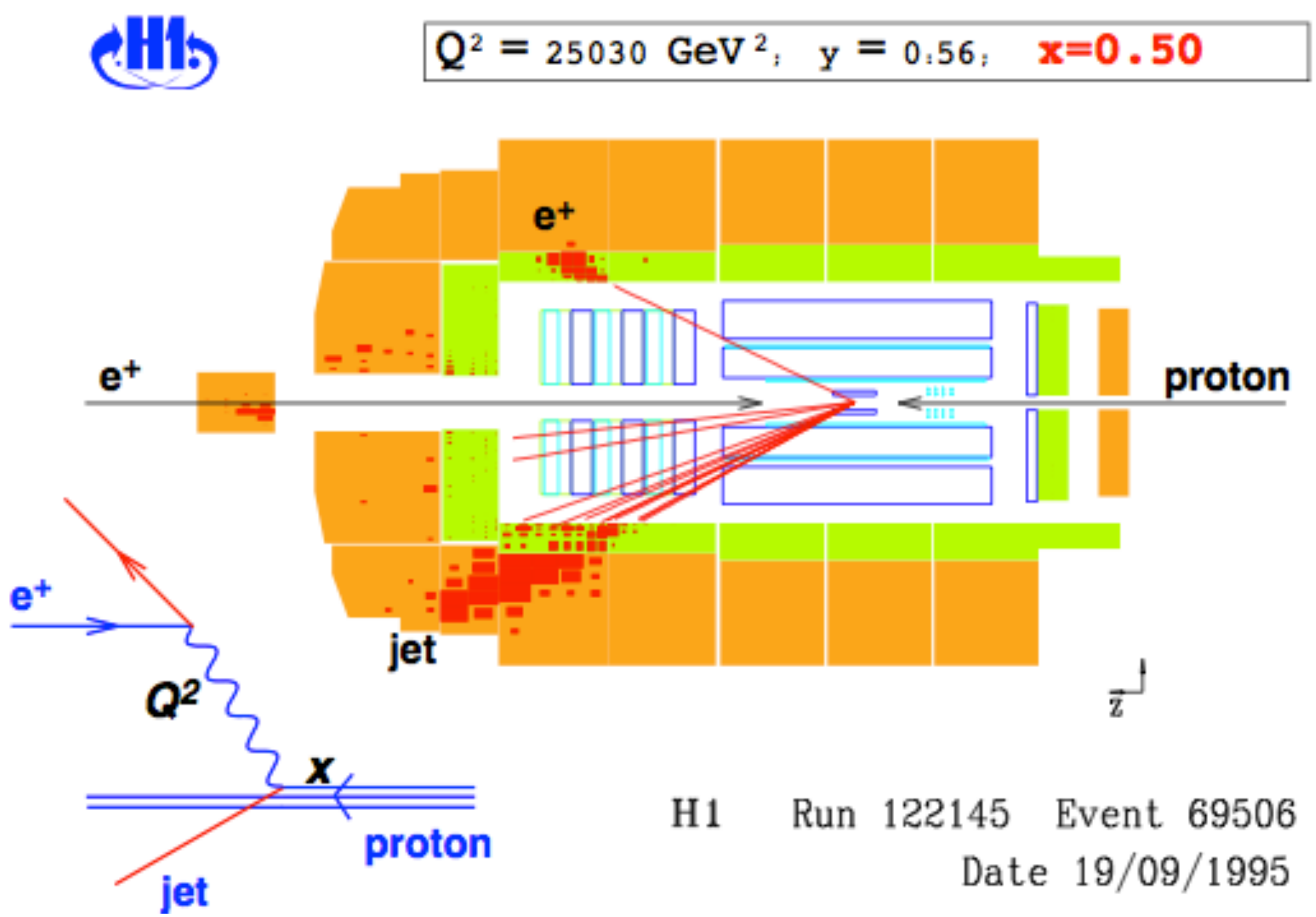
$$\int_0^1 dx \left( f_s^{(p)}(x) - f_{\bar{s}}^{(p)}(x) \right) = 0$$

In the proton: u, d valence quarks, all other quarks are called sea-quarks

How can parton densities be extracted from data?

# Deep inelastic scattering

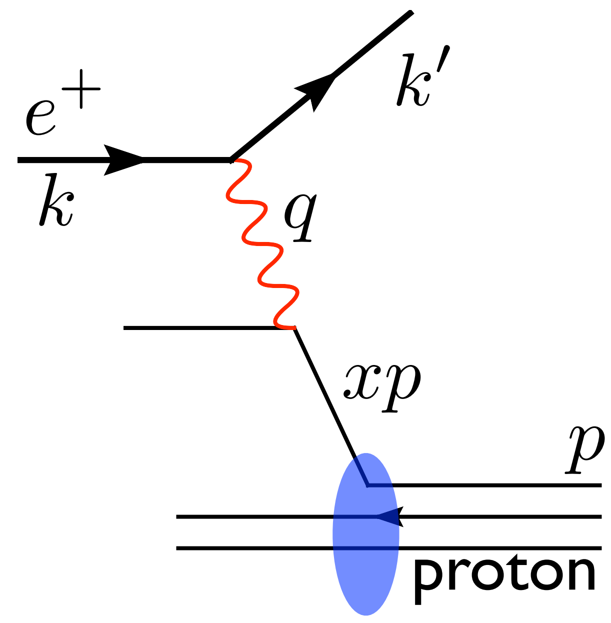
Easier than processes with two incoming hadrons is the scattering of a lepton on a (anti)-proton



# Deep inelastic scattering

Protons made up of point-like quarks.
Different momentum scales involved:

- hard photon virtuality (sets the resolution scale) Q
- hard photon-quark interaction Q
- soft interaction between partons in the proton $m_p \ll Q$



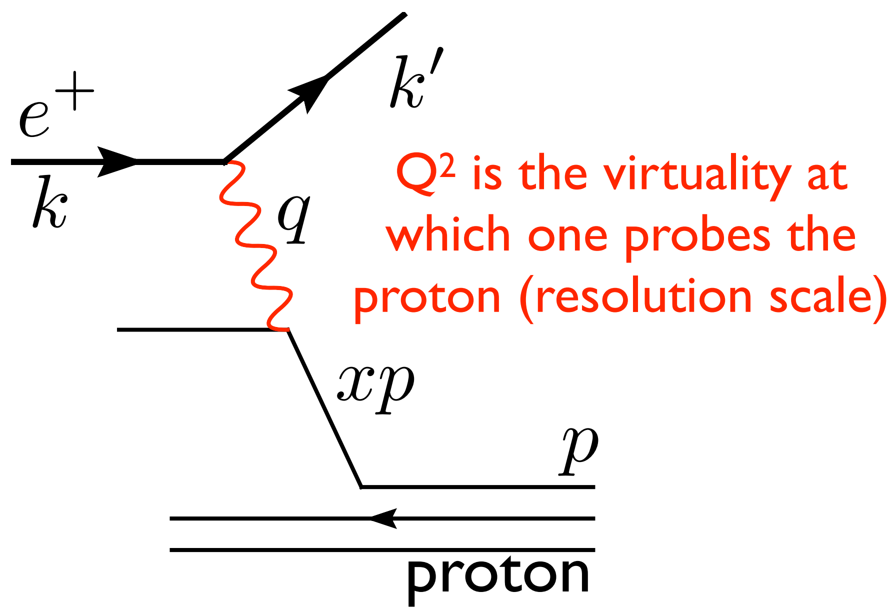

During the hard interaction, partons do not have time to interact among them, they behave as if they were free

$\Rightarrow$ approximate as incoherent scattering on single partons

# Deep inelastic scattering

Kinematics:

$$Q^2 = -q^2 \quad s = (k+p)^2 \quad x_{Bj} = \frac{Q^2}{2p\cdot q} \quad y = \frac{p\cdot q}{k\cdot p}$$




Partonic variables:

$$\hat{p} = xp \quad \hat{s} = (k+\hat{p})^2 = 2k\cdot\hat{p} \quad \hat{y} = \frac{\hat{p}\cdot q}{k\cdot\hat{p}} = y \quad (\hat{p}+q)^2 = 2\hat{p}\cdot q - Q^2 = 0$$

$$\Rightarrow x = x_{Bj}$$

Hence at leading order, the experimentally accessible $x_{Bj}$ coincides with the momentum fraction carried by the quark in the proton

Partonic cross section:

(apply QED Feynman rules and add phase space)

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2 \frac{\hat{s}}{Q^4} 2\,\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)$$

Exercise: show that in the CM frame of the electron-quark system y is given by $(1-\cos\theta_{\rm el})/2$, with $\theta_{\rm el}$ the scattering angle of the electron in this frame

Exercise:

- show that the two particle phase space is $\frac{d\phi}{16\pi}$
- show that the squared matrix element is $\frac{16\pi\alpha q_l^2}{Q^4}\hat{s}xpk\left(1+(1-y)^2\right)$
- show that the flux factor is $\frac{1}{4xpk}$

Hence derive that

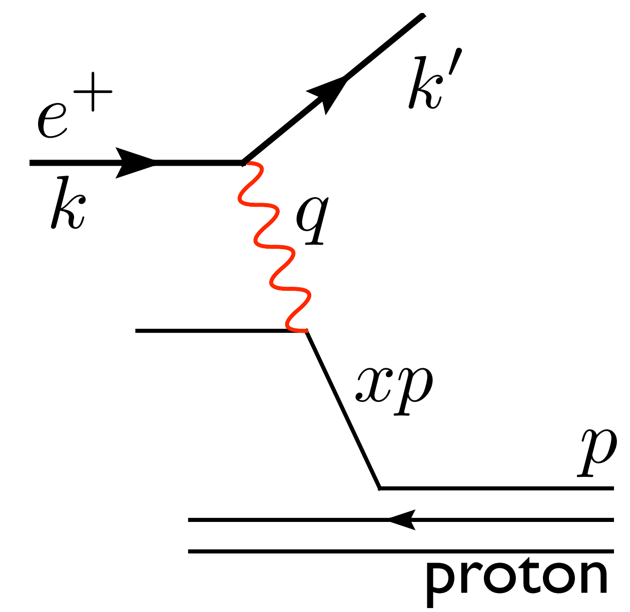
$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2\frac{\hat{s}}{Q^4}2\,\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)$$

# Deep inelastic scattering

Hadronic cross section (factorization):

$$\frac{d\sigma}{dy} = \int dx \sum_l f_l^{(p)}(x) \frac{d\hat{\sigma}}{d\hat{y}}$$

Using x = $x_{BJ}$




$$\frac{d\sigma}{dy\,dx_{Bj}} = \sum_l f_l^{(p)}(x) \frac{d\hat{\sigma}}{d\hat{y}}$$
$$= \frac{2\pi\,\alpha_{em}^2 s x_{Bj}}{Q^4} \left(1 + (1-y)^2\right) \sum_l q_l^2 f_l^{(p)}(x_{Bj})$$

1. at fixed $x_{Bj}$ and y the cross-section scales with s
2. the y-dependence of the cross-section is fully predicted and is typical of vector interaction with fermions ⇒ Callan-Gross relation
3. can access (sums of) parton distribution functions
4. Bjorken scaling: pdfs depend on x and not on $Q^2$ (violated by logarithmic radiative corrections, see later)

# The structure function $F_2$

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1+(1-y^2\right)F_2(x) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

$F_2$ is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Bjorken scaling: the fact the structure functions are independent of Q is a direct evidence for the existence of point-like quarks in the proton (violated by logarithmic corrections)

# The structure function $F_2$

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1 + (1-y^2)\right)F_2(x) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

$F_2$ is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Question: $F_2$ gives only a linear combination of u and d. How can they be extracted separately?

# Isospin

**Neutron is like a proton with u & d exchanged**

For electron scattering on a proton

$$F_2^p(x) = x\left(\frac{4}{9}u_p(x) + \frac{1}{9}d_p(x)\right)$$

For electron scattering on a neutron

$$F_2^n(x) = x\left(\frac{1}{9}d_n(x) + \frac{4}{9}u_n(x)\right) = x\left(\frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)\right)$$

$F_2^n$ and $F_2^p$ allow determination of $u_p$ and $d_p$ separately

NB: experimentally get $F_2^n$ from deuteron: $F_2^d(x) = F_2^p(x) + F_2^n(x)$

# Sea quark distributions

Inside the proton there are fluctuations, and pairs of $u\bar{u}$,$d\bar{d}$,$c\bar{c}$,$s\bar{s}$ ... can be created

An infinite number of pairs can be created as long as they have very low momentum, because of the momentum sum rules.

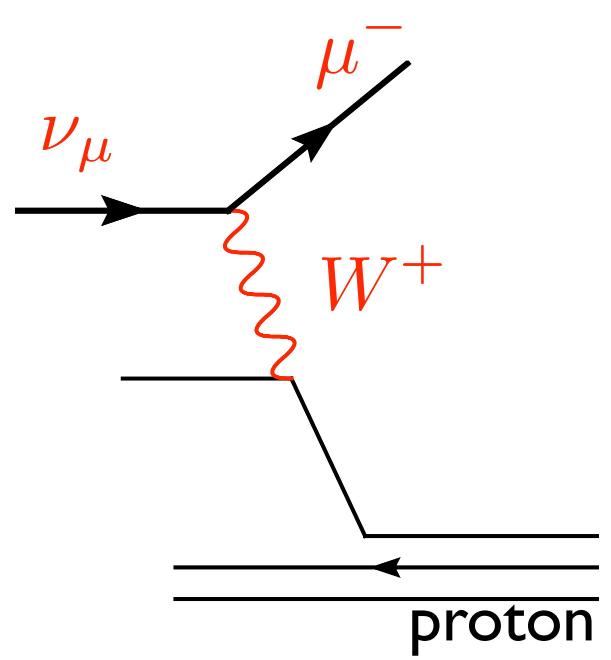
We saw before that when we say that the proton is made of uud what we mean is

$$\int_0^1 dx\,(u_p(x) - \bar{u}_p(x)) = 2 \qquad \int_0^1 dx\,\left(d_p(x) - \bar{d}_p(x)\right) = 1$$

Photons interact in the same way with u(d) and $\bar{u}$($\bar{d}$)

How can one measure the difference?

Question: What interacts differently with particle and antiparticle? $W^+/W^-$ from neutrino scattering



# Check of the momentum sum rule

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

| | |
|---|---|
| $u_v$ | 0,267 |
| $d_v$ | 0,111 |
| $u_s$ | 0,066 |
| $d_s$ | 0,053 |
| $s_s$ | 0,033 |
| $c_c$ | 0,016 |
| total | 0,546 |

➠ *half of the longitudinal momentum carried by gluons*

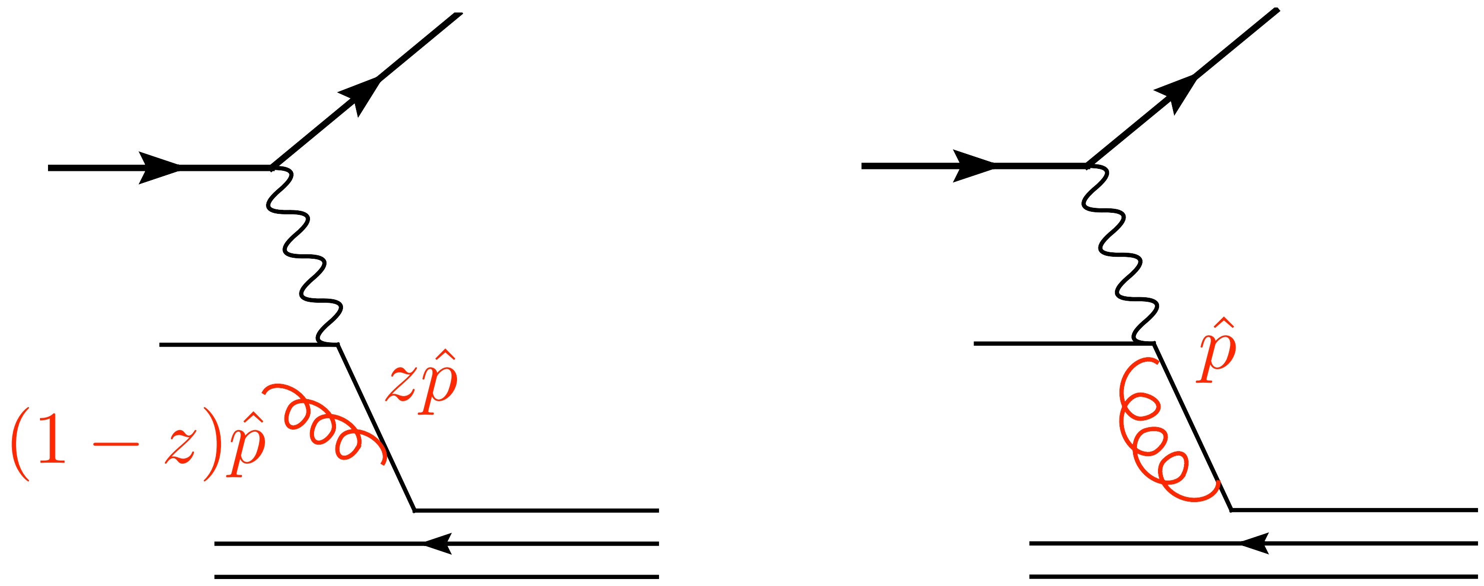
$\gamma/W^{+/-}$ don't interact with gluons

How can one measure gluon parton densities?

We need to discuss radiative effects first

# Radiative corrections

To first order in the coupling:
need to consider the emission of one real gluon and a virtual one




Adding real and virtual contributions, the partonic cross-section reads

$$\sigma^{(1)} = \frac{C_F \alpha_s}{2\pi} \int dz \frac{dk_\perp^2}{k_\perp^2} \frac{1+z^2}{1-z} \left( \sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p}) \right)$$

Partial cancellation between real (positive), virtual (negative), but real gluon changes the energy entering the scattering, the virtual does not

# Radiative corrections

Partonic cross-section:

$$\sigma^{(1)} = \frac{\alpha_s}{2\pi}\int dz \int_{\lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} P(z)\left(\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p})\right), \qquad P(z) = C_F\frac{1+z^2}{1-z}$$

Soft limit: singularity at z=1 cancels between real and virtual terms

Collinear singularity: $k_\perp \to 0$ with finite z. Collinear singularity does not cancel because partonic scatterings occur at different energies

$\Longrightarrow$ naive parton model does not survive radiative corrections

Similarly to what is done when renormalizing UV divergences, collinear divergences from initial state emissions are absorbed into parton distribution functions

# The plus prescription

Partonic cross-section:

$$\sigma^{(1)} = \frac{\alpha_s}{2\pi} \int_{\lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz\, P(z) \left( \sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p}) \right)$$

Plus prescription makes the universal cancelation of singularities explicit

$$\int_0^1 dz f_+(z) g(z) \equiv \int_0^1 f(z) \left( g(z) - g(1) \right)$$

The partonic cross section becomes

$$\sigma^{(1)} = \frac{\alpha_s}{2\pi} \int dz \int_{\lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} P_+(z) \sigma^{(0)}(z\hat{p}) \,, \quad P(z) = C_F \left( \frac{1+z^2}{1-z} \right)$$

Collinear singularities still there, but they factorize.

# Factorization scale

Schematically use
$$\ln\frac{Q^2}{\lambda^2} = \ln\frac{Q^2}{\mu_F^2} + \ln\frac{\mu_F^2}{\lambda^2}$$

$$\sigma = \sigma^{(0)} + \sigma^{(1)} = \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\lambda^2}P_+\right) \times \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu_F^2}P_+\right)\sigma^{(0)}$$

So we define

$$f_q(x,\mu_F) = f_q(x) \times \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\lambda^2}P_{qq}^{(0)}\right) \qquad \hat{\sigma}(p,\mu_F) = \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu_F^2}P_{qq}^{(0)}\right)\sigma^{(0)}(p)$$

NB:

- universality, i.e. the PDF redefinition does not depend on the process
- choice of $\mu_F \sim Q$ avoids large logarithms in partonic cross-sections
- PDFs and hard cross-sections don't evolve independently
- the factorization scale acts as a cut-off, it allows to move the divergent contribution into non-perturbative parton distribution functions

# Improved parton model

Naive parton model:

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$

After radiative corrections:

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1, \mu^2) f_2^{(P_2)}(x_2, \mu^2) \hat{\sigma}(x_1 x_2 s, \mu^2)$$

# Intermediate recap

- With initial state parton collinear singularities don't cancel
- Initial state emissions with $k_\perp$ below a given scale are included in PDFs
- This procedure introduces a scale $\mu_F$, the so-called factorization scale which factorizes the low energy (non-perturbative) dynamics from the perturbative hard cross-section
- As for the renormalization scale, the dependence of cross-sections on $\mu_F$ is due to the fact that the perturbative expansion has been truncated
- The dependence on $\mu_F$ becomes milder when including higher orders
- The redefinition of PDFs is universal and process-independent

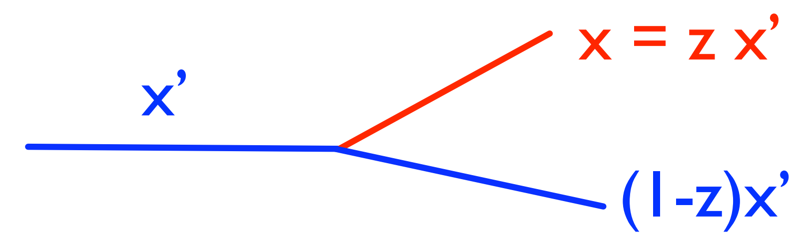
One incoming hard parton: $\sigma = \int dx f^{(P)}(x,\mu^2)\hat{\sigma}(xs,\mu^2)$

Two incoming hard partons: $\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1,\mu^2) f_2^{(P_2)}(x_2,\mu^2)\hat{\sigma}(x_1 x_2 s,\mu^2)$

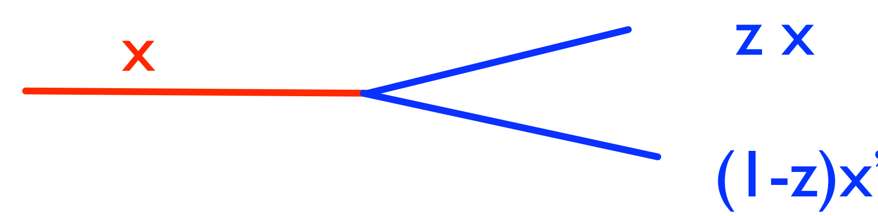
# Evolution of PDFs

A parton distribution changes when

- a different parton splits and produces it




- the parton itself splits




$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \int_0^1 dx' \int_x^1 dz \frac{\alpha_s}{2\pi} P(z) f(x',\mu^2)\delta(zx'-x) - \int_0^1 dz \frac{\alpha_s}{2\pi} P(z) f(x,\mu^2)$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z},\mu^2\right) - \int_0^1 dz \frac{\alpha_s}{2\pi} P(z) f\left(x,\mu^2\right)$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_+(z) f\left(\frac{x}{z},\mu^2\right)$$

<u>The plus prescription</u> $\int_0^1 dz f_+(z) g(z) \equiv \int_0^1 dz f(z) \left(g(z) - g(1)\right)$

# DGLAP equation

$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z},\mu^2\right)$$

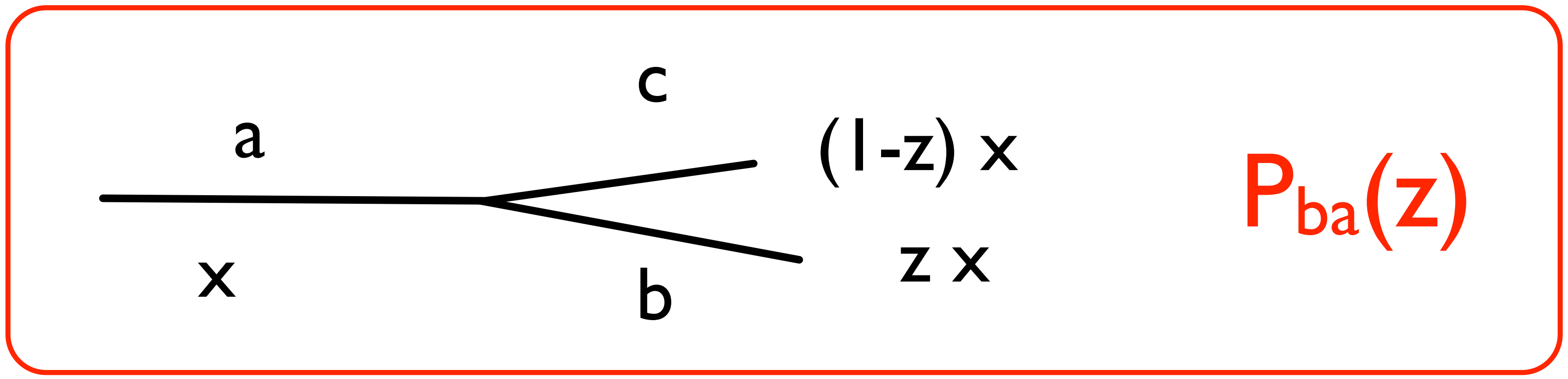
*Altarelli, Parisi; Gribov-Lipatov; Dokshitzer '77*

Master equation of QCD: we can not compute parton densities, but we can predict how they evolve from one scale to another

Universality of splitting functions: we can measure pdfs in one process and use them as an input for another process

# Conventions for splitting functions

There are various partons types. Standard notation:




Accounting for the different species of partons the DGLAP equations become:

$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \sum_j \int_x^1 \frac{dz}{z} P_{ij}(z) f_j\left(\frac{x}{z},\mu^2\right)$$

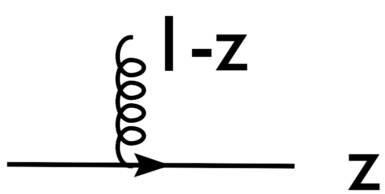
This is a system of coupled integro/differential equations

The above convolution in compact notation:

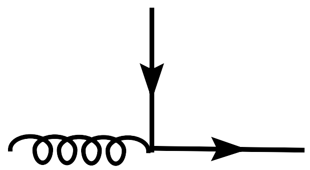
$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \sum_j P_{ij} \otimes f_j(\mu^2)$$

# Properties of splitting functions

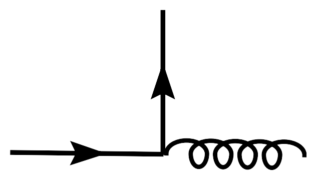
$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F \left[ \left( \frac{1+z^2}{1-z} \right)_+ \right]$$



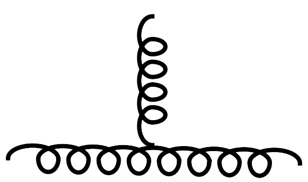

$$P_{qg}^{(0)} = P_{\bar{q}g}^{(0)} = T_R \left( z^2 + (1-z) \right)$$



$$P_{gq}^{(0)} = P_{g\bar{q}}^{(0)} = C_F \frac{1+(1-z)^2}{z}$$



$$P_{gg}^{(0)} = 2C_A \left[ z \frac{1}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + b_0 \delta(1-z) \right]$$



- $P_{qg}$ anf $P_{gg}$ symmetric under z (1-z)
- $P_{qq}$ and $P_{gg}$ divergence for z=1 (soft gluon)
- $P_{gq}$ and $P_{gg}$ divergenge for z=0 (soft gluon)
- $P_{qg}$ no soft divergence for gluon splitting to quarks

➠ gluon PDF grows at small x

# History of splitting functions

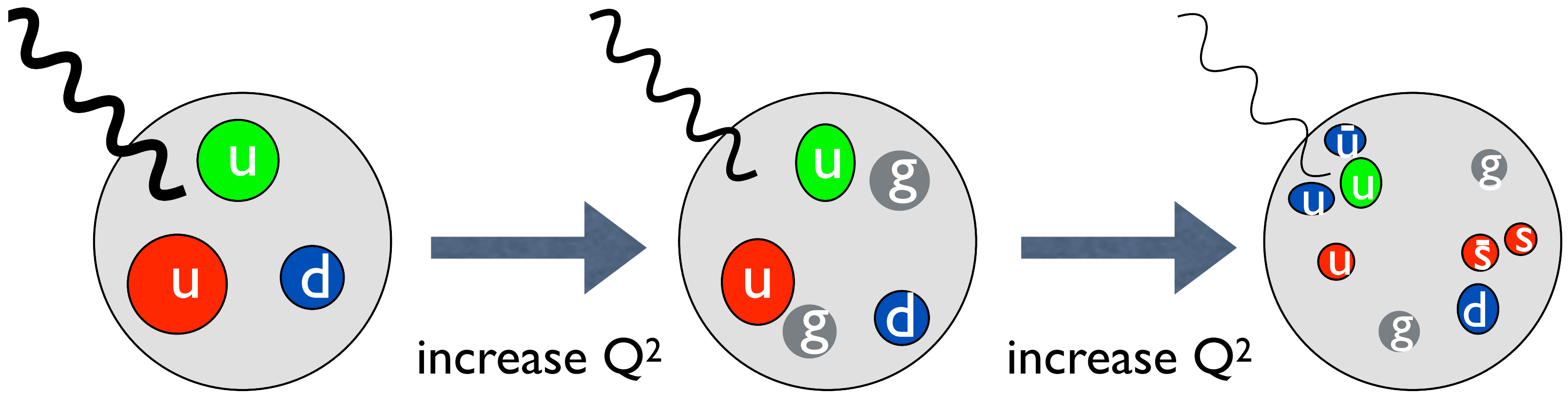
- ☑ $P^{(0)}_{ab}$ : Altarelly, Parisi; Gribov-Lipatov; Dokshitzer (1977)
- ☑ $P^{(1)}_{ab}$ : Curci, Furmanski, Petronzio (1980)
- ☑ $P^{(2)}_{ab}$ : Moch, Vermaseren, Vogt (2004)

☛ Essential input for NNLO pdfs determination (state of the art today)

# Evolution

So, in perturbative QCD we can not predict values for

- the coupling
- the masses
- the parton densities
- ...




What we can predict is the evolution with the $Q^2$ of those quantities. These quantities must be extracted at some scale from data.

- not only is the coupling scale-dependent, but partons have a scale dependent sub-structure
- we started with the question of how one can access the gluon pdf: Because of the DGLAP evolution, we can access the gluon pdf indirectly, through the way it changes the evolution of quark pdfs. Today also direct measurements using Tevatron jet data and LHC tt and jet data

# Recap.

- Parton model: incoherent sum of all partonic cross-sections
- Sum rules (momentum, charge, flavor conservation)
- Determination of parton densities (electron & neutrino scattering)
- Radiative corrections: failure of parton model
- Factorization of initial state divergences into scale dependent parton densities
- DGLAP evolution of parton densities $\Rightarrow$ measure gluon PDF
- While PDFs loose the naive probabilistic interpretation basic conservation principle still hold (momentum sum rules, energy, flavour conservation)