(GR) (QFT)

Gravitational scattering amplitudes & Black Holes

(classical sol.)

Lecture 1

GR : QFT for spin-2 particles $h_{\mu\nu}$

- massless $\Leftrightarrow$ long range force
- gauge symmetry = diffeomorphisms = general covariance
- No local operators $\mathcal{O}(x)$
- Asymptotic states & observables $\Rightarrow$ Scattering amplitudes
- Perturbation theory $\Rightarrow$ Feynman diagrams
  - $\hookrightarrow$ UV div. at 2 loops
  - 1 loop (matter)
  - = non-renormalizable

BHs : classical sol. to GR (m, J, Q)

- Schwarzschild,
- Kerr metric

$Q = 0$ for astrophy. BHs

- Static/stationary $\Rightarrow$ not dynamical $\Rightarrow$ asymptotic states
- Astrophysical BHs are dynamical $\Rightarrow$ move, "vibrate" interact with matter & energy
- Does GR predict BH dynamics?
  - $\hookrightarrow$ Yes, but with great difficulty
    - Numerical GR
    - BH perturbation theory $\nearrow$ Regge-Wheeler $J=0$, $\searrow$ Teukolsky $J\neq0$
    - . . . .
- Test-body limit : dynamics = geodesic motion $(J=0)$

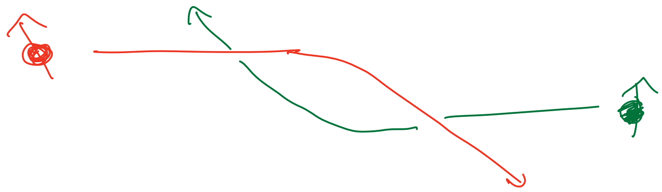
- For rotating BHs, test-body limit does not exist (according to GR)

  since $J < m \approx 0$

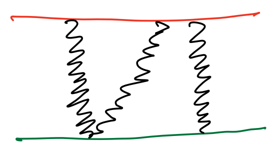
- Note: electron is a spinning test-body $J > m$

  so is a photon!

- Use scattering ampl's to infer dynamics!



Classical GR



QFT

Amplitudes in pure GR vs. YM

GR $\quad (\kappa^2 = 32\pi G_N)$	YM
$\frac{2}{\kappa^2}\sqrt{-g}\, R$	$-\frac{1}{4 g_{YM}} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$
$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$
EoM: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$	$D_\mu F^{\mu\nu} = 0$

Weak coupling: plane waves $\longleftrightarrow$ asymptotic states

$$h_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu e^{ip\cdot x} \qquad \longleftarrow \qquad A_\mu^a = c^a \varepsilon_\mu e^{ip\cdot x}$$

$\varepsilon_\mu \leftarrow c^a$

$(p^2 = 0)$

polarizations: $\varepsilon\cdot p = 0$, $\varepsilon\cdot\varepsilon = 0$ $\Rightarrow$ $h_{\mu\nu}$ $\begin{cases}\text{transverse}\\ \text{traceless}\\ \text{symmetric}\end{cases}$

$\varepsilon^\mu \sim \varepsilon^\mu + p^\mu$ (gauge invariance)

$\varepsilon^\mu\varepsilon^\nu \sim \varepsilon^\mu\varepsilon^\nu + \varepsilon^\mu p^\nu + p^\mu\varepsilon^\nu$ diffeomorphism

Two states both theories:

- $\varepsilon^+_\mu, \varepsilon^-_\mu$ gluon helicity
- $\varepsilon^+_\mu\varepsilon^+_\nu$, $\varepsilon^-_\mu\varepsilon^-_\nu$ graviton helicity

3pt amplitudes

$$A(123) = g_{YM} f^{a_1a_2a_3}\left(\eta_{\mu\nu}(p_1-p_2)_\rho + \text{cyclic}\right)\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho = V_{123}$$

$$M(123) = \frac{\kappa}{2}\left[\left(\eta_{\mu\nu}(p_1-p_2)_\rho + \text{cyclic}\right)\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\right]^2$$

$$A(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \quad M(1^{--}2^{--}3^{++}) = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2$$

Spinor-helicity notation [Elvang & Huang textbook]

For massless momentum p^μ, define bi-spinor

$$p_{\alpha\dot\alpha} \equiv \sigma^\mu_{\alpha\dot\alpha}\,p_\mu\,, \qquad \det p_{\alpha\dot\alpha} = p^2 = 0$$

$SL(2,\mathbb{C})$ indices

$$\Rightarrow \quad p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$$

More convenient notation: $\lambda_\alpha = |p\rangle$ Left-handed spinor

$\tilde\lambda^{\dot\alpha} = |p]$ Right-handed spinor

raise & lower indices with

$\epsilon^{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$

Automatically satisfy Weyl eqns

$$p|p\rangle = 0 \qquad p|p] = 0$$

polarizations:

$$\varepsilon_-^{\mu}(p) = \frac{\langle p|\sigma^\mu|q]}{\sqrt{2}\,[pq]} \qquad q^2 = 0$$

reference vector

$$\varepsilon_+^{\mu}(p) = \frac{\langle q|\sigma^\mu|p]}{\sqrt{2}\,\langle qp\rangle}$$

"light-cone gauge"

$$\langle qp\rangle \equiv \langle q|^{\alpha}\,|p\rangle_\alpha = \varepsilon^{\alpha\beta}|q\rangle_\beta\,|p\rangle_\alpha$$

$$[pq] \equiv [p|_{\dot\alpha}\,|q]^{\dot\alpha} \quad \ldots$$

For many particles $|p_i\rangle \longrightarrow |i\rangle \qquad i = 1, \ldots, n$

3pt & 4pt amplitudes:

$$A_3^{YM}(1^-2^-3^+) = \frac{\langle 12\rangle^2}{\langle 23\rangle\langle 31\rangle^2}\,, \qquad M_3^{GR} = \left(A_3^{YM}\right)^2$$

$$A_4^{YM}(1^-2^-3^+4^+) = g_{YM}^2\left(\underbrace{\frac{\langle 12\rangle^2[34]^2}{st}}\, f^{a_1a_4b}f^{ba_2a_3} + \underbrace{\frac{\langle 12\rangle^2[34]^2}{su}}\, f^{a_1a_3b}f^{ba_2a_4}\right).$$

$$M_4^{GR} = \frac{\langle 12\rangle^4[23]^4}{stu} = s\,A(1234)\,A(1243)$$

color-ordered amplitudes

$\rightarrow$ KLT formula "double copy"

Double copy: • Scattering ampl's in GR can be obtained from Yang-Mills ampl's

• $GR \sim (YM)^2$

Refs. • 1909.01358 "BCJ review"

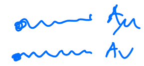
2203.13013 "SAGEX review"

2204.06547 "Snowmass white paper"

Diagrammatic double copy

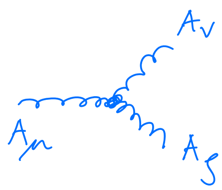
$\Longleftrightarrow$

$\sim$



$\epsilon_\mu e^{ip\cdot x}$

$\epsilon_\mu \epsilon_\nu e^{ip\cdot x}$



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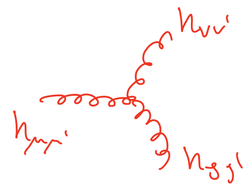

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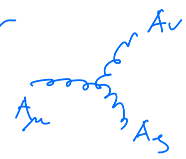


$\Big)^2$

Lecture 2

Classical double copy

[Monteiro, O'Connell, White]

Kerr-Schild metric ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu \phi$$

cf. plane wave
$[g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon_\mu \varepsilon_\nu e^{ix\cdot p}]$

where k^μ null vector w.r.t $g^{\mu\nu}$ & $\eta^{\mu\nu}$

$$\Rightarrow \quad g^{\mu\nu} = \eta^{\mu\nu} - \phi k^\mu k^\nu$$

where $k^\mu = \eta^{\mu\nu} k_\nu$

$\Rightarrow$ Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$
become linear in $h_{\mu\nu} = k_\mu k_\nu \phi$

$\Rightarrow$ $A_\mu = k_\mu \phi$ solves $\partial_\mu F^{\mu\nu} = 0$
Maxwell's eqns

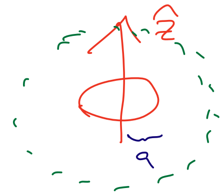
Schwarzchild solution

$$k_\mu = (1, \hat{r}), \quad \phi = \frac{2GM}{r}$$

Coulomb solution

$$k_\mu = (1, \hat{r}), \quad \phi = \frac{Q}{4\pi r}, \quad A_\mu \to A_\mu - \partial_\mu\left(\frac{Q}{4\pi} \log r\right)$$

Kerr solution



$$\phi(r) = \frac{2MG}{r + a^2 z^2/r^3}\,, \quad a \text{ ring radius}$$

$$k_\mu = \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right)$$

Spheroidal coordinates: $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$

Root-Kerr

$A_\mu = k^\mu \phi$

Massive, spinning, charged disk?

$R^{\pm\,\text{Kerr}}_{\mu\nu\rho\sigma} = R^{\pm\,\text{Sch.}}_{\mu\nu\rho\sigma}(z \to z \pm ia)$ Newman-Janis shift

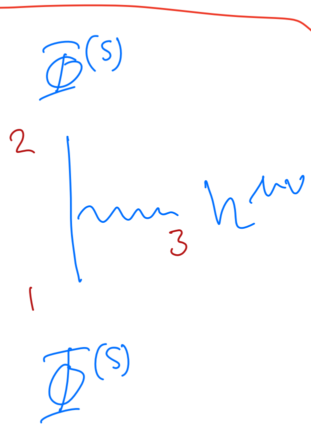
$F^{\pm\,\sqrt{\text{Kerr}}}_{\mu\nu} = F^{\pm\,\text{Coulomb}}_{\mu\nu}(z \to z \pm ia)$ $\quad F^{\pm} = \frac{1}{2}(F \pm i*F)$

Kerr BH 3pt amplitudes & Spin

Arkani-Hamed, Huang-Huang 17

$$M(1^s 2^s 3^+) = \underbrace{M(1 2 3^+)}_{[(p_1-p_2)\cdot\varepsilon_3^+]^2} \frac{\langle 12\rangle^{2s}}{m^{2s}}$$

Double Copy



$\boxed{\hbar \to 0}$

$$= M(123^+)\, e^{p_{3\mu} S^\mu/m}$$

[Vines 17]

where $S^\mu = m a^\mu = m a \hat{z}^\mu$

Massive momentum & Weyl spinors

$$p^\mu = k^\mu + m^2 q^\mu \quad , \text{with } k^2 = q^2 = 0$$
$$2k\cdot q = 1$$

then $p^2 = m^2$

$$p_\mu \sigma^\mu = |k\rangle[k| + m^2 |q\rangle[q|$$
$$\left(\lambda_k \tilde\lambda_k + m \lambda_q \tilde\lambda_q\right)$$

$$|p^I\rangle = \begin{pmatrix} |k\rangle \\ m|q\rangle \end{pmatrix}, \quad |p^I] = \begin{pmatrix} |k] \\ m|q] \end{pmatrix}$$

$\langle kq\rangle = [kq] = 1$

$I = 1,2$ is the little group $SU(2)$ index

spin

consider a QM wave function $|\psi\rangle = \underline{z_1} |\uparrow\rangle + \underline{z_2} |\downarrow\rangle$

basis

coefficients

$$|p\rangle = |p^1\rangle z_1 + |p^2\rangle z_2 = |p^I\rangle z_I$$

$$\langle 12\rangle^{2s} = \left(z_I \langle p_2^I | p_1^J\rangle z_J\right)^{2s}$$

$$\langle 12\rangle^{2s} = f\left(\langle \hat S^\mu\rangle\right)$$

where $\hat{S}^\mu = \frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} p_{1\nu} M_{\rho\sigma}$

Pauli-Lubanski pseudo-vector (operator)

What is : $\langle \hat{S}^\mu \rangle = \langle \bar{1}|^{2s} \hat{S}^\mu |1\rangle^{2s}$

Boost

$$|2\rangle = |\bar{1}\rangle + \frac{p_3 \cdot \sigma}{2m} |\bar{1}]$$

2

$\cancel{h}$ m_3

1

$$\langle 12\rangle = \underbrace{\langle 1\bar{1}\rangle}_{= 1} + \underbrace{\frac{1}{2m}\langle 1|\sigma^\mu|\bar{1}] p_{3\mu}}_{\langle \hat{S}^\mu_{(1/2)}\rangle \frac{p_{3\mu}}{m}}$$

$$\langle 12\rangle^{2s} = \left(1 + \langle \hat{S}^\mu_{(1/2)}\rangle \frac{p_{3\mu}}{m}\right)^{2s}$$

$$= \sum_{k=0}^{2s} \left(\frac{\langle \hat{S}_{(1/2)} \cdot p_3\rangle}{m}\right)^k \frac{(2s)!}{(2s-k)!\, k!}$$

$$\left[\left(S^\mu_{(s)}\right)^k = \frac{(2s)!}{(2s-k)!} \left(S^\mu_{(1/2)}\right)^k + \underbrace{\cdots}_{p_1^\mu} \right]$$

$$= \left\langle \sum_{k=0}^{2s} \left(\frac{\hat{S}_{(s)} \cdot p_3}{m}\right)^k \frac{1}{k!} \right\rangle = \left\langle e^{\frac{\hat{S}_{(s)} \cdot p_3}{m}} \right\rangle$$

$$\begin{bmatrix} \hbar \to 0 \\ s \to \infty \end{bmatrix} \Rightarrow \langle \hat{S}\rangle^2 = \langle \hat{S}^2\rangle \quad \text{variance is zero}$$

$$\langle 12\rangle^{2s} \longrightarrow e^{\frac{\langle \hat{S}\rangle \cdot p_3}{m}} = e^{\frac{S\cdot p_3}{m}} = e^{a\cdot p_3}$$

<u>Newman – Janis shift again</u>

Schwartzchild $\longrightarrow$ Kerr

$$x^\mu \to x^\mu + i a^\mu$$

Amplitude with planewave factor:

$$\int d^4x\, M_3\, e^{-ix\cdot(p_1+p_2+p_3)} \longrightarrow \underbrace{\int d^4x\, M_3\, e^{-i(x\pm ia)\cdot(p_1+p_2+p_3)}}_{\delta^4(p_1+p_2+p_3)\, M_3\, \underbrace{e^{\pm a\cdot p_3}}_{\text{Kerr exponential!}}}$$

(since $p_1 \cdot a = 0$ by transversality)