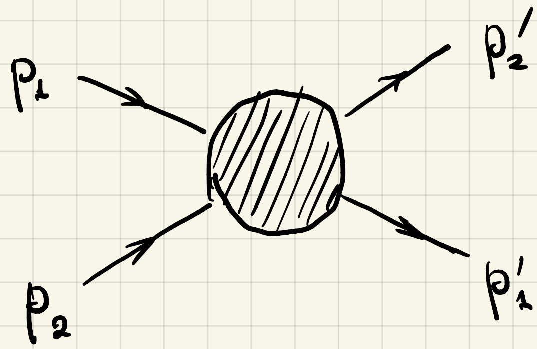
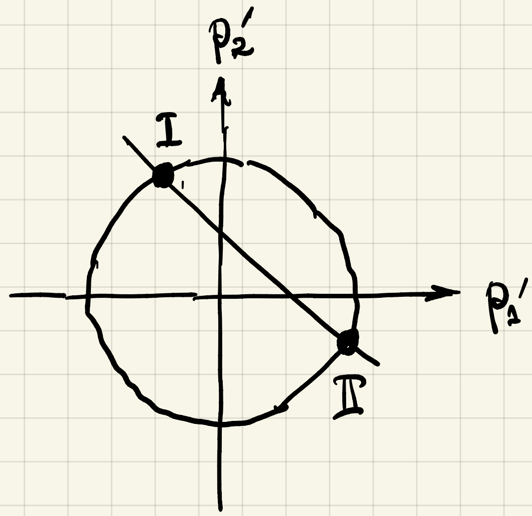
Exact S-matrices

2d kinematics





$$\mathrm{I}: \quad p_1' = p_1 \qquad p_2' = p_2$$

$$\mathrm{II}: \quad p_1' = p_2 \qquad p_2' = p_1$$

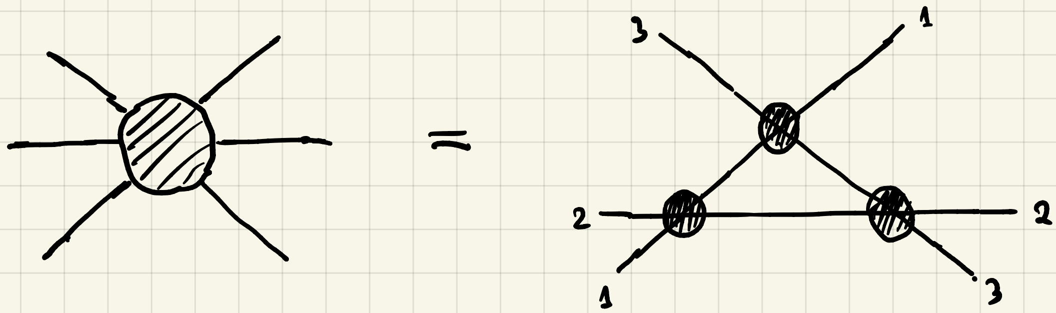
Integrability $=$ ∞ many conservation laws:

$$\sum_{j \in \text{out}} q_n(p_j') = \sum_{j \in \text{in}} q_n(p_j)$$

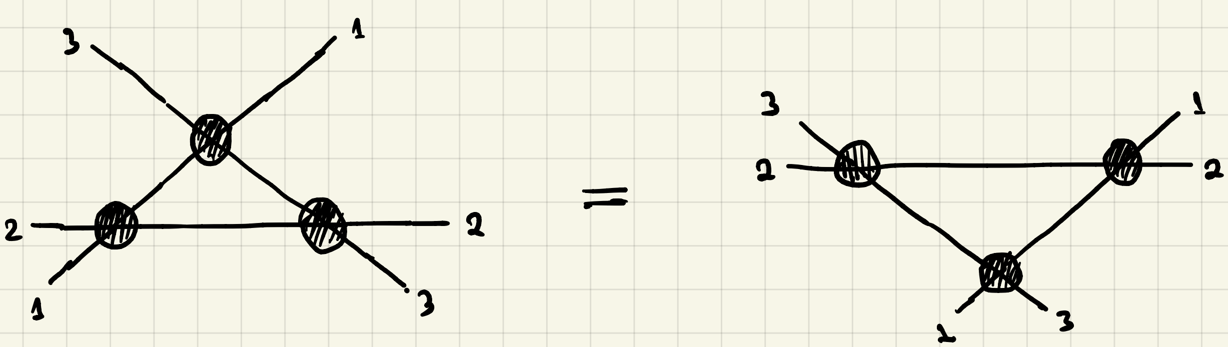
$$N' = N : \quad p_j' = p_{\sigma(j)}$$

$$N' \neq N : \quad \text{no solutions}$$

- no particle production
- factorized scattering:



$$S_{123} = S_{12}\, S_{13}\, S_{23}$$



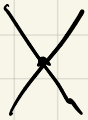
Yang - Baxter equation:

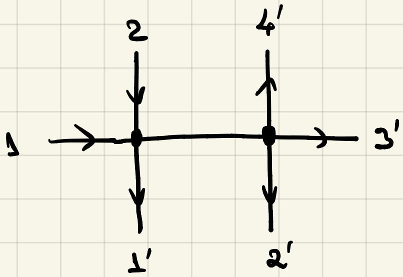
$$S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}$$

<u>Integrable QFTs</u>

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!} \lambda \varphi^4$$

- Is this model integrable?

$\mathcal{M}_{2\to 2} =$  $= -\lambda$

$\mathcal{M}_{2\to 4} =$  $+$ perm.

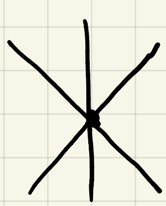
$$= \frac{\lambda^2}{(P_1 + P_2 - P_{1'})^2 - m^2} + \text{cross-channels}$$

Ex (1)* Show that $\mathcal{M}_{2\to 4} = \dfrac{\lambda^2}{m^2}$

Ex (1') Same as (1) but in simplified kinematics: $p_2 = -p_1 \quad p_3' = -p_1 \quad p_4' = -p_2$

$p_1' \sim p_1 \gg m \gg p_2'$

- φ^4 theory is not integrable
- easy fix: add a new vertex $\quad -\dfrac{\lambda_6}{6!}\varphi^6$

 $-\lambda_6$

and set $\lambda_6 = \dfrac{\lambda^2}{m^2}$

then $\quad \mathcal{M}_{2\to 4} = 0.$

Dimensionless coupling:

$$\lambda = b^2 m^2$$

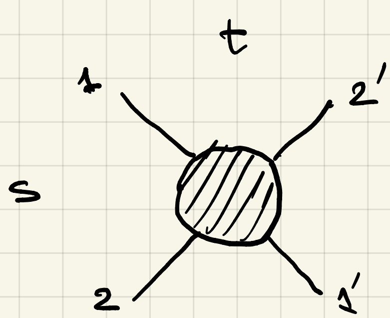
$$\lambda_6 = \frac{\lambda^2}{m^2} = b^4 m^2$$

$$V(\varphi) = m^2 \left(\frac{1}{2!}\varphi^2 + \frac{b^2}{4!}\varphi^4 + \frac{b^4}{6!}\varphi^6 + \dots \right)$$

$$= \frac{m^2}{b^2}\left(\cosh b\varphi - 1 \right)$$

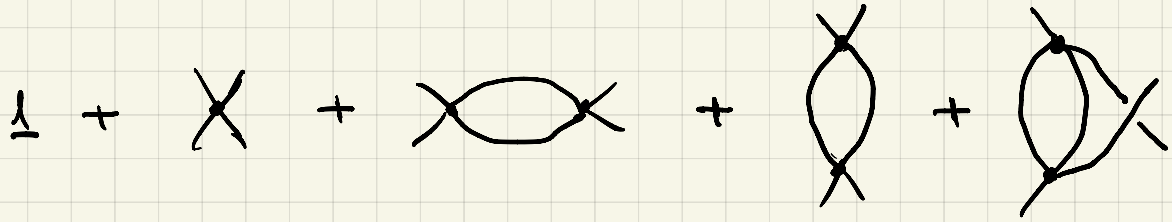
<u>Sinh-Gordon theory</u>

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{b^2}\cosh b\varphi}$$

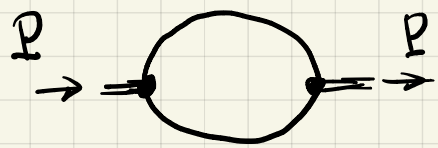


$u = 0$

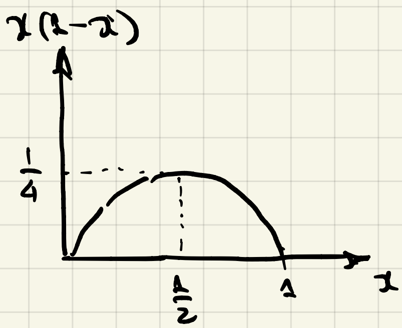
$s + t = 4m^2$



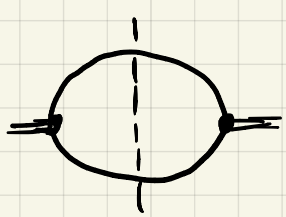
$B(s)$ $\quad B(t)$ $\quad B(u)$

$B(s) =$  $= \frac{1}{4\pi}\int_0^1 \frac{dx}{m^2 - x(1-x)s}$

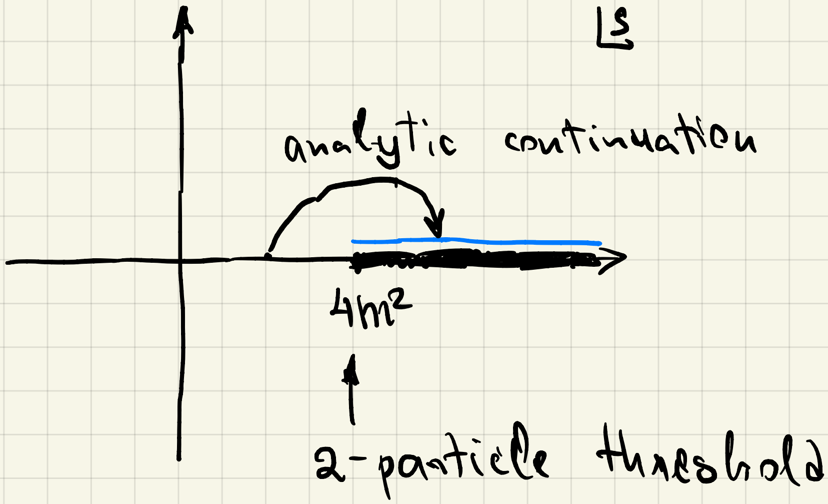
$s = p^2$



- If $s > 4m^2$ there is a pole on integration domain: manifestation of particles in the loop simultaneously going on-shell.



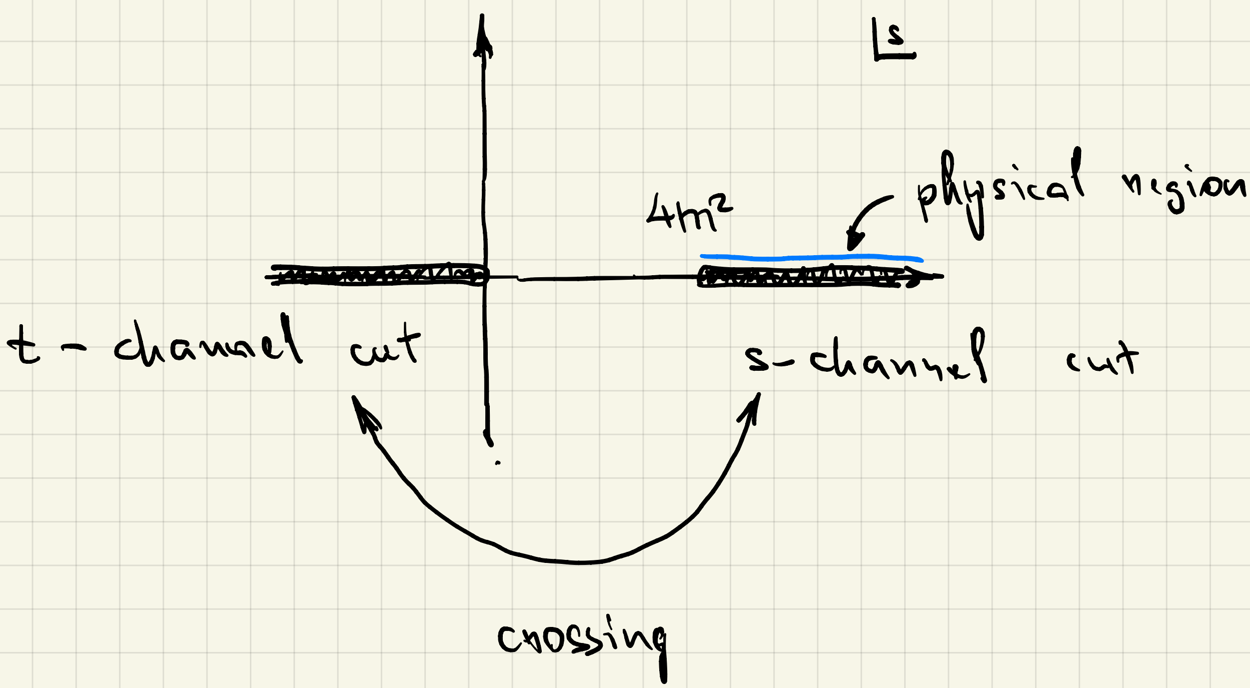
ε - prescription: $\quad s \to s + i\varepsilon$



$$B(s) = \frac{\arctan\sqrt{\frac{s}{t}}}{\pi\sqrt{st}}$$

$$\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$$

$$B(s) + B(t) + B(u) = \frac{1}{2\sqrt{st}} + \frac{1}{4\pi m^2} \qquad t = 4m^2 - s$$



Exact S-matrix:

- has the same analyticity properties

  / integrability $\Rightarrow$ only 2-particle cuts /

- unitary: $S^\dagger(s)\, S(s) = 1$

  / in physical region /

- crossing symmetric: $S(4m^2 - s) = S(s)$

Uniformization

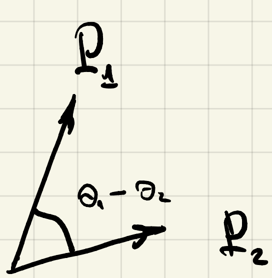
$$\varepsilon^2 - p^2 = m^2$$

Solved by

$$\varepsilon = m \cosh\theta \approx m + \frac{m\theta^2}{2}$$

$$p = m \sinh\theta \approx m\theta$$

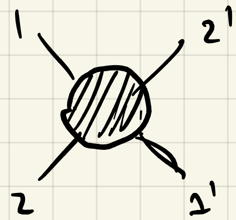
θ – rapidity



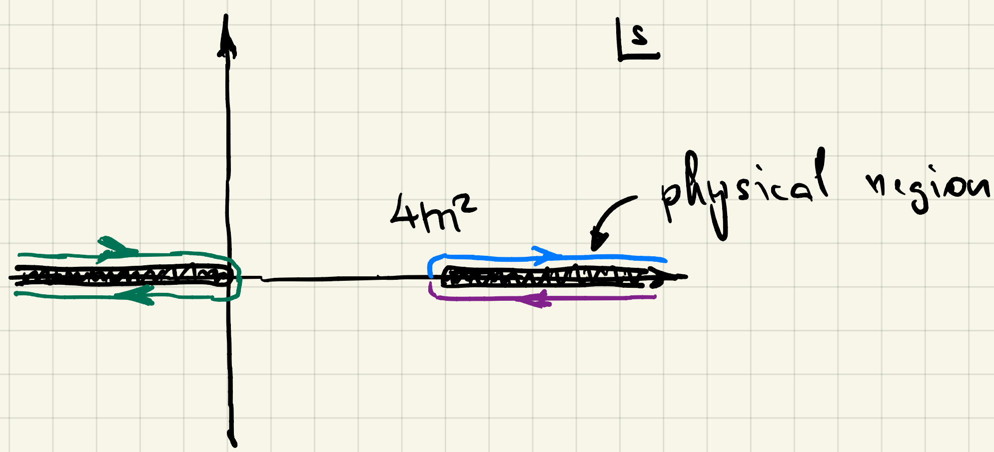
$$p_1 \cdot p_2 = m^2 \cosh(\theta_1 - \theta_2)$$

$$s = (p_1 + p_2)^2 = 2m^2 + 2m^2 \cosh(\theta_1 - \theta_2)$$

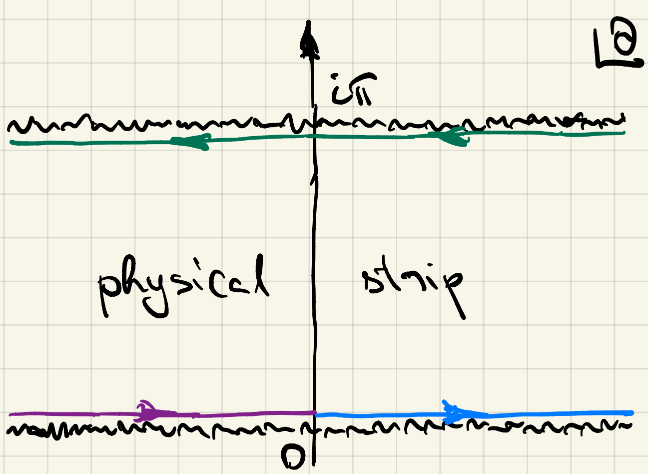
$$= 4m^2 \cosh^2 \frac{\theta_1 - \theta_2}{2}$$



$$S \equiv S(s) \quad \text{or} \quad S(\theta_1 - \theta_2)$$



$$s = 4m^2 \sinh^2 \frac{\theta}{2}$$



Crossing : $\theta \longrightarrow i\pi - \theta \qquad s \leftrightarrow t$

Crossing symmetry:

$$S(i\pi - \theta) = S(\theta)$$

Unitarity (+ real analyticity):

$$S(-\theta)\, S(\theta) = 1$$

Solving ShG

$$\boxed{S\left(\theta + \frac{i\pi}{2}\right) S\left(\theta - \frac{i\pi}{2}\right) = 1}$$

$$S(\theta) = e^{i\Phi(\theta)}$$

$$\Phi\left(\theta + \frac{i\pi}{2}\right) + \Phi\left(\theta - \frac{i\pi}{2}\right) = 2\pi\Big(\vartheta(\theta - ia) + \vartheta(\theta - ia)\Big)$$

step function

$$\Phi \sim \Phi + 2\pi$$

$$K = \frac{d\Phi}{d\theta}$$

$$K\left(\theta \pm \frac{i\pi}{2}\right) = e^{\pm \frac{i\pi}{2} \frac{d}{d\theta}}$$

$$2\cos\left(\frac{\pi}{2} \frac{d}{d\theta}\right) K(\theta) = 2\pi\Big(\delta(\theta + ia) + \delta(\theta - ia)\Big)$$

Fourier transform:

$$\frac{d}{d\theta} = -i\omega$$

$$K(\omega) = \frac{\pi \cosh a\omega}{\cosh \frac{\pi\omega}{2}}$$

$$K(\theta) = \frac{2\cos a \cosh\theta}{\sinh^2\theta + \cos^2 a}$$

$$a = \frac{\pi}{2} - \alpha \qquad \cos a = \sin\alpha$$

$$\boxed{S(\theta) = \frac{\sinh\theta - i\sin\alpha}{\sinh\theta + i\sin\alpha}} \qquad \alpha = \frac{\pi b^2}{8\pi + b^2}$$

Exact S-matrix of ShG theory

Ex (2) Show that the exact result matches with the first two orders of perturbation theory.

Tip:

$$\langle p_1' c; p_2' d | \hat{S} | p_1 a; p_2 b \rangle = S_{ab}^{cd}(p_1, p_2) (2\pi)^2 2E_1 2E_2$$

$$\times \left[\delta(p_1' - p_1)\, \delta(p_2' - p_2) \pm \delta(p_1' - p_2)\, \delta(p_2' - p_1) \right] \qquad (1)$$

$$\langle p | p' \rangle = 2\varepsilon(p)\, 2\pi\, \delta(p - p')$$

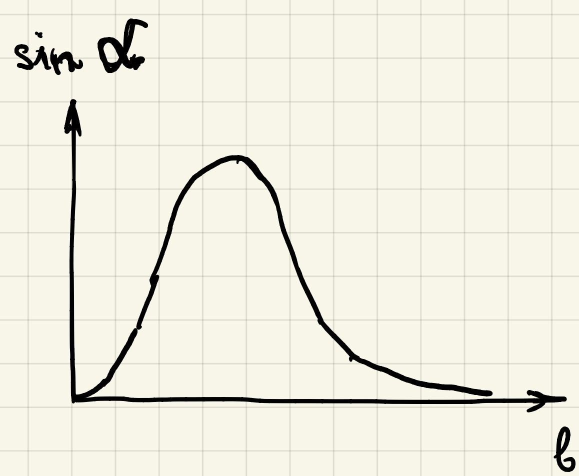
$\pm$: bosons / fermions

$$\hat{S} - 1 = i\, \mathcal{M}\, (2\pi)^2\, \delta^{(2)}\left(\underline{p}_1' + \underline{p}_2' - \underline{p}_1 - \underline{p}_2 \right) \quad (2)$$

- perturbation theory computes matrix elements of $\mathcal{M}$

(1) $\longleftrightarrow$ (2) has a non-trivial Jacobian

<u>Rem</u> α is the true measure of interaction strength



Duality:

$$b \longrightarrow \frac{8\pi}{b}$$

$$\alpha \longrightarrow \frac{8\pi^2}{8\pi + b^2} = \pi - \alpha$$

$$\sin\alpha \longrightarrow \sin\alpha$$

Sine - Gordon model

- Analytic continuation of ShG:

$$b = i\beta$$

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{\beta^2}\cos\beta\varphi}$$

$$\varphi \sim \varphi + \frac{2\pi}{\beta}$$

Exact S-matrix:

$$\boxed{S(\theta) = \frac{\sinh\theta + i\sin\gamma}{\sinh\theta - i\sin\gamma}} \qquad \gamma = \frac{\pi\beta^2}{8\pi - \beta^2}$$

- Effective coupling becomes infinite at $\beta^2 = 8\pi$.

Conformal perturbation theory

$$\mathcal{L} = \mathcal{L}_{CFT} + \frac{\mu^2}{2}(\mathcal{O} + \mathcal{O}^\dagger)$$

$\Delta[\mathcal{O}] < 2$ — relevant (μ^2 has positive dim)

$\Delta[\mathcal{O}] > 2$ — irrelevant (μ^2 has negative dim)

th. is not renormalizable

$\mathcal{L}_{CFT}$ — free compact boson

$$\mathcal{O} = e^{i\beta\varphi}$$

$$\Delta[\mathcal{O}] = ?$$

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \langle e^{i\beta\varphi(x)} e^{-i\beta\varphi(y)}\rangle = e^{\beta^2 D(x-y)}$$

In 2D:
$$D(x) = -\frac{1}{2\pi}\ln|x|$$

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = e^{-\frac{\beta^2}{2\pi}\ln|x-y|} = \frac{1}{|x-y|^{\frac{\beta^2}{2\pi}}}$$

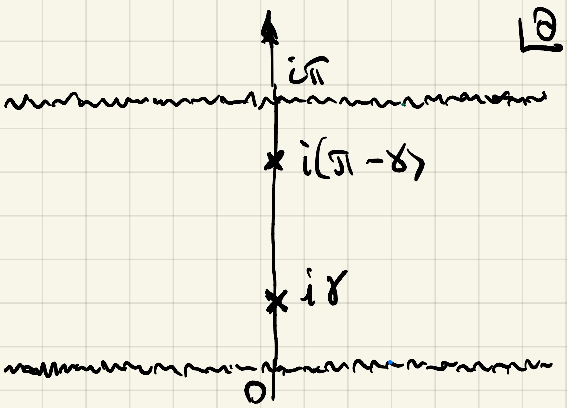
$$\boxed{\Delta[\mathcal{O}] = \frac{\beta^2}{4\pi}}$$

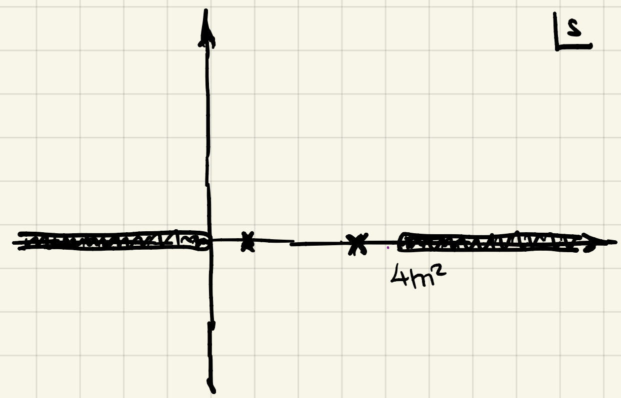
$$\cos\beta\varphi - \begin{cases} \text{relevant for } \beta^2 < 8\pi \\ \text{irrelevant for } \beta^2 > 8\pi \end{cases}$$

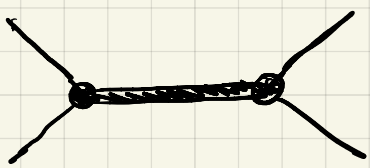
$\beta^2 = 8\pi$: Kosterlitz–Thouless phase transition

Bound states

- S-matrix has a pole at $\theta = i\gamma$







$$\approx \frac{1}{p^2 - M^2} = \frac{1}{s - M^2}$$

$$s = 4m^2 \cosh^2 \frac{\theta}{2}$$

Bound state mass:

$$\boxed{m_2 = 2m \cos \frac{\gamma}{2}}$$

- At $\gamma = \frac{\pi}{2}$ the s-channel and t-channel poles collide and for larger γ no longer correspond to a bound state

- Exists for $\gamma < \frac{\pi}{2}$ or $\beta^2 < \frac{8\pi}{3}$

<u>Ex</u> (3) Consider a 2-particle QM system

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{\beta^2}{8}\,\delta(x)$$

- Show that at weak coupling ($\beta \ll 1$) and in the non-relativistic approximation ($\theta \ll 1$) the SG S-matrix agrees with the scattering amplitude computed from this Hamiltonian.

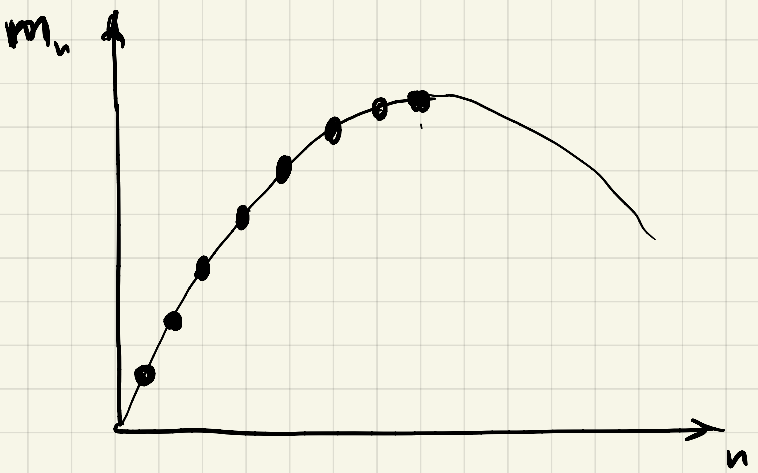
Tip: Use Born approximation

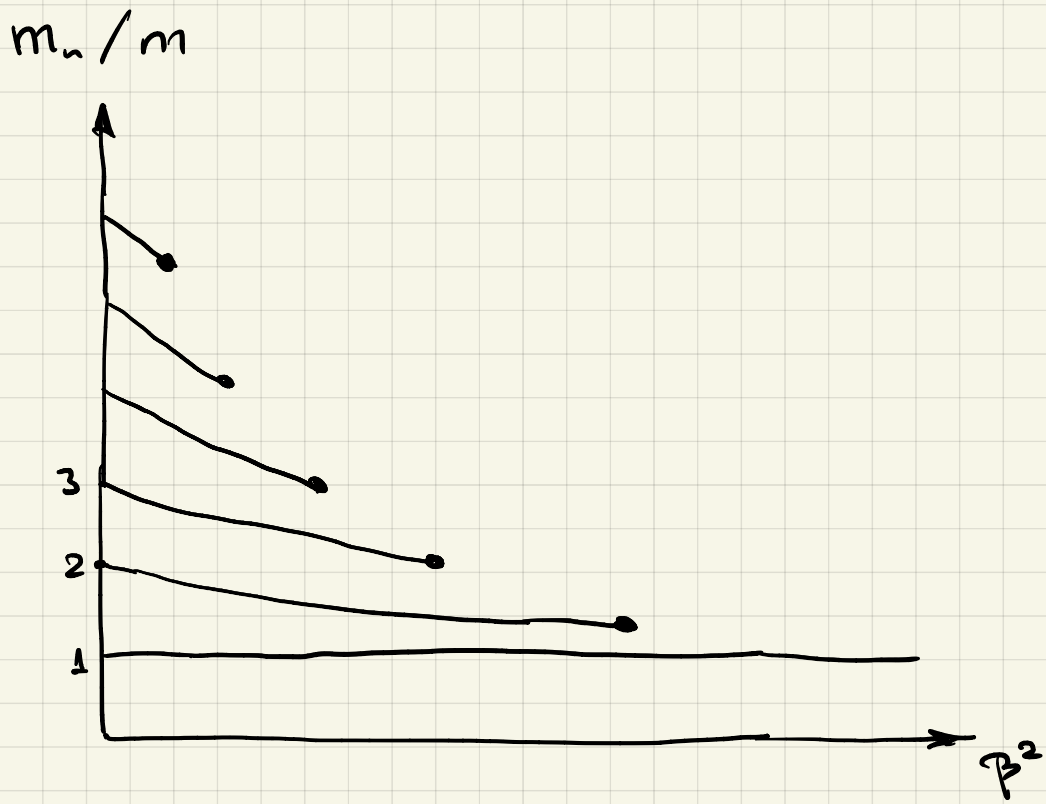
- Show that the QM Hamiltonian has exactly 1 normalizable state. Compare its energy with the binding energy of SG particles: $E_b = 2m - m_2$

• n-particle bound states:

$$\boxed{m_n = m\,\frac{\sin\frac{n\gamma}{2}}{\sin\frac{\gamma}{2}}} \qquad n = 1\ldots\left[\frac{\pi}{\gamma}\right]$$

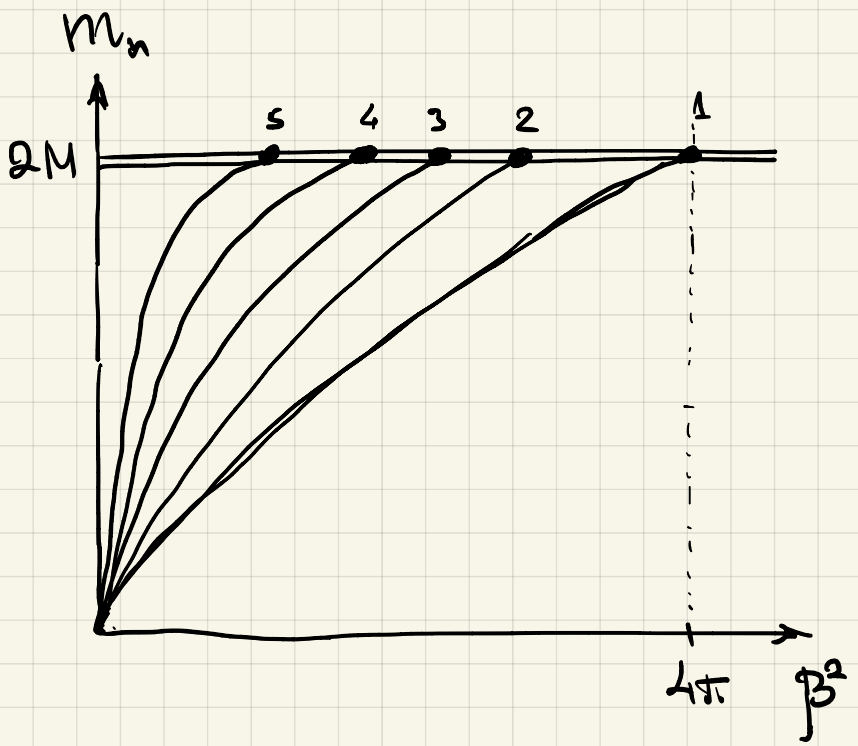
At weak coupling $(\gamma \ll 1)$: $m_n \approx nm$





$$M \equiv \frac{m}{\sin \frac{\gamma}{2}}$$

$$m_n = 2M \sin \frac{n\gamma}{2}$$



$$m_n = 2M - E_n^{\text{binding}}$$

- 2-particle bound states
- Elementary boson ($n=1$) is also a bound state, at $\beta^2 = 4\pi$ it disappears!

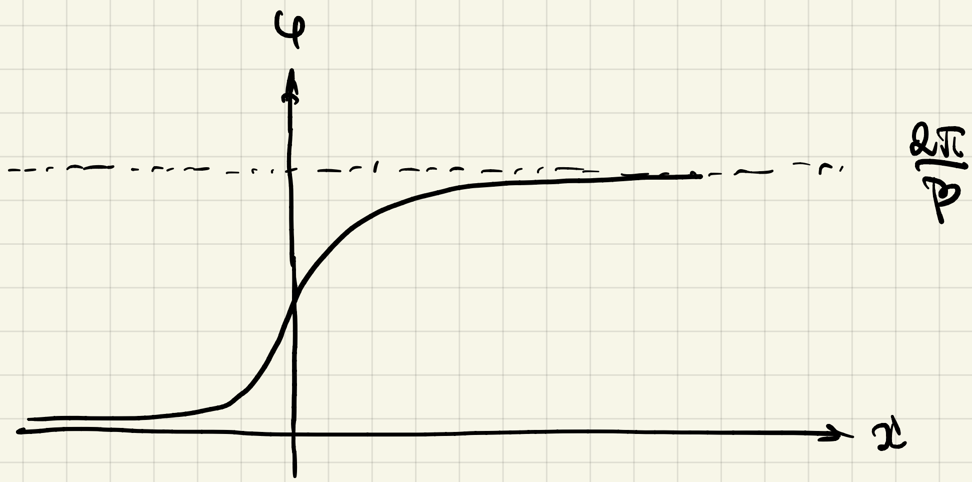
Solitons

Topological current:

$$j^\mu = -\frac{\beta}{2\pi}\,\varepsilon^{\mu\nu}\partial_\nu\varphi$$

(Anti-) solitons carry $Q = \pm 1$

Semiclassical description:



Ex (4) Find soliton solution, compute its energy and compare with the exact formula for the soliton mass:

$$M = \frac{m}{\sin\frac{\gamma}{2}} \qquad \gamma = \frac{\pi\beta^2}{8\pi-\beta^2}$$

Coleman–Mandelstam duality

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{\beta^2}\cos\beta\varphi \qquad \text{SG}$$

$$\Updownarrow$$

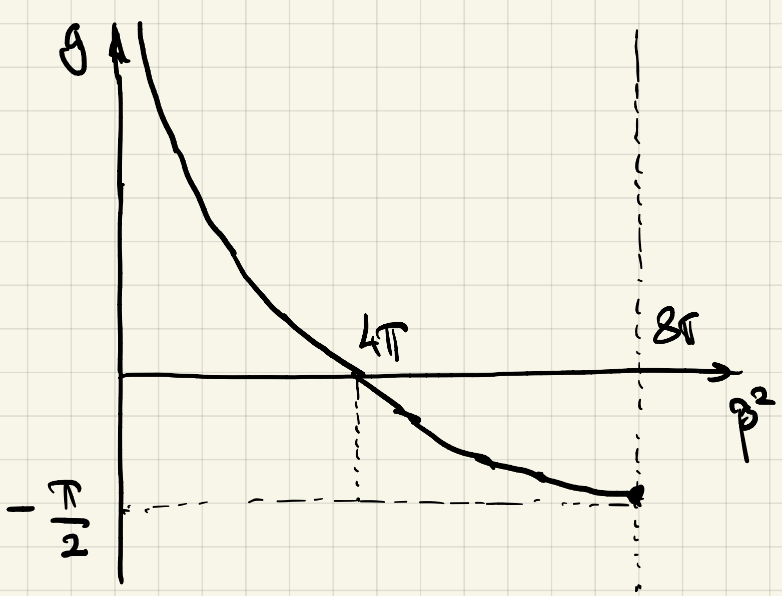
$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - M\bar{\Psi}\Psi - \frac{g}{2}(\bar{\Psi}\gamma^\mu\Psi)^2$$

Thirring model

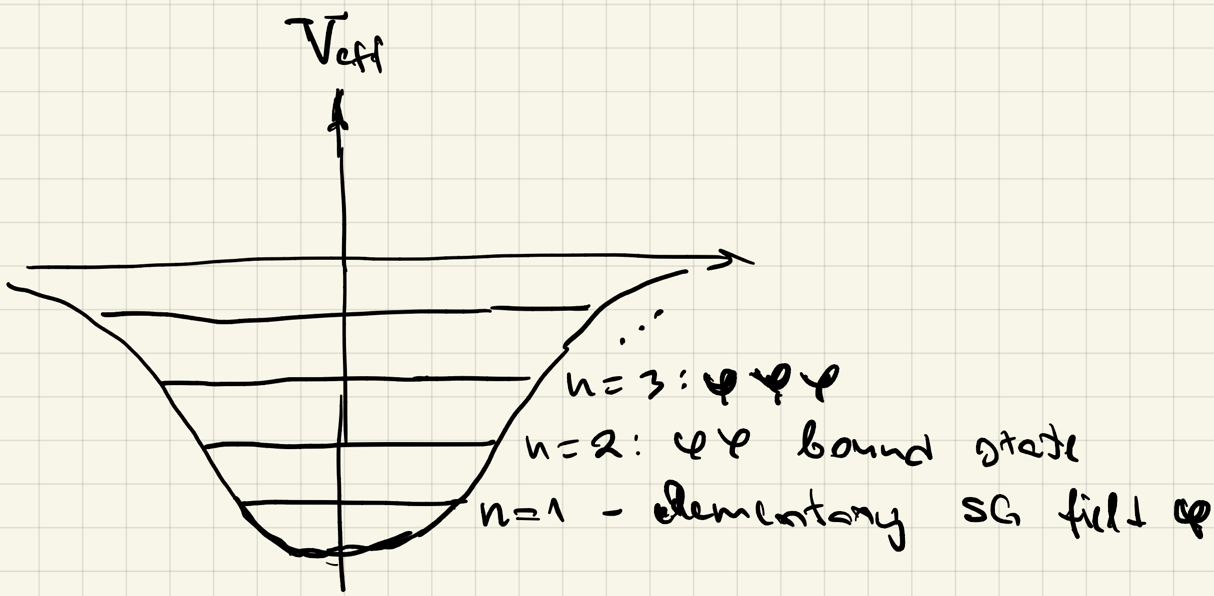
$$\frac{g}{\pi} = \frac{4\pi - \beta^2}{\beta^2}$$

- Soliton is the elementary field of TM

$$j^\mu = -\frac{\beta}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\varphi \qquad j^\mu = \bar{\Psi}\gamma^\mu\Psi$$



$\beta^2 < 4\pi$: $g > 0$ ($f\bar{f}$ interaction is attractive)



$\beta^2 = 4\pi$: $g = 0$ (free fermions)

$8\pi > \beta^2 > 4\pi$: $g < 0$ (repulsion)

$\beta^2 > 8\pi$: inconsistent