INTRODUCTION COSMOLOGY

Stockholm, Jan '24

* aim: initial conditions universe <-> CMB

* online ressources: link on indico
(recommendation: Baumann's notes & book)

A. HOMOGENEOUS UNIVERSE

A.1 GEOMETRY

* on large scales ($\gtrsim 100\,\mathrm{Mpc} = 3.26 \times 10^6\,\mathrm{ly}$):

(i) isotropic (same in all directions)

$\Downarrow$ (our position not special)

(ii) homogeneous (same everywhere)

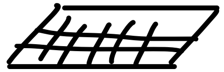
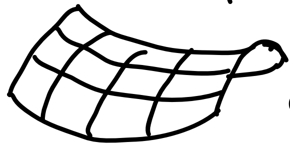
* FLRW: ($\hbar = c = 1$)

spatial slice

$$ds^2 = \underbrace{g_{\mu\nu}}_{\text{metric}} dx^\mu dx^\nu = -dt^2 + \underbrace{a^2(t)}_{\text{scale factor}}\, d\ell_3 \,,$$

where $$d\ell_3 = \frac{dr^2}{(1 - kr^2/R_0^2)} + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$K =$

- 0 $\quad E^3$  flat; $\alpha+\beta+\gamma = 180°$
- 1 $\quad S^3$  closed; $\alpha+\beta+\gamma > 180°$
- -1 $\quad H^3$  open; $\alpha+\beta+\gamma < 180°$

* flat case: $d\ell_3 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \delta_{ij}\, dx^i dx^j$; $\delta_{ij} = \begin{cases} 1; & i=j \\ 0; & i \neq j \end{cases}$

$x^\mu = (t, x^i)$

* r is not physical (co-moving)

-> phys. coord: $r_{phys} = a(t)\, r$

-> $r_{phys}(t)$ changes in proportion to scale factor

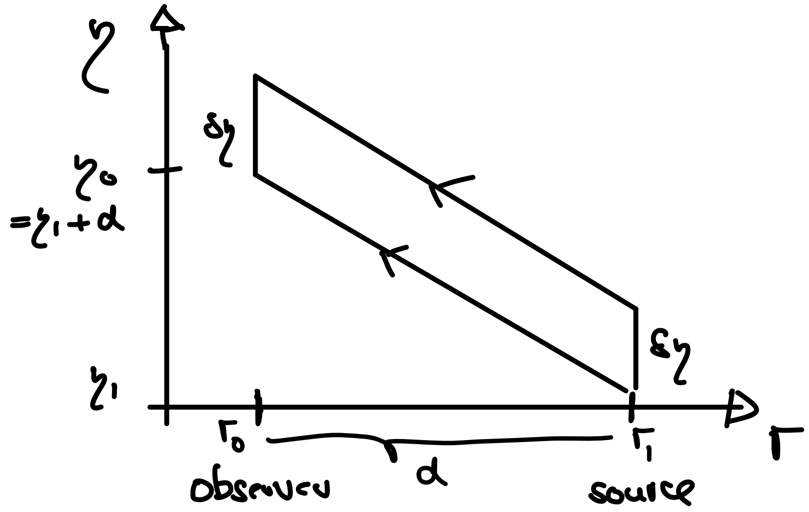
* conformal time $\eta = \int \frac{dt}{a(t)}$ $\left[\frac{d}{d\eta} f \equiv f' = a \frac{df}{dt} = a \dot{f} \right]$

$\Rightarrow$ $ds^2 = a^2(\eta)\,(-d\eta^2 + \delta_{ij}\, dx^i dx^j) = a^2(\eta)$ Minkowski

-> useful for studying light

* Hubble: $H = \frac{1}{a}\frac{da}{dt} = \frac{\dot{a}}{a}$ * obs. $H>0 \Rightarrow$ space expands!   * today $H_0 \equiv H(t_0)$

A.2 REDSHIFT



* light trajectory: $ds^2=0 \Rightarrow \frac{dr}{d\eta} = \pm 1 \Rightarrow \Delta r(\eta) = \pm \Delta\eta$

period: $\delta\eta = \delta\eta_1 = \delta\eta_0$

physical duration: $\delta t_1 = a(\eta_1)\,\delta\eta$; $\delta t_0 = a(\eta_0)\,\delta\eta$

$$\Rightarrow \lambda_1 = \delta t_1 = a(\eta_1)\,\delta\eta < \lambda_0 = \delta t_0 = a(\eta_0)\,\delta\eta$$

redshift: $z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(\eta_0) - a(\eta_1)}{a(\eta_1)}$ w.l.o.g. $a(\eta_0) = 1$ $\Rightarrow$ $\boxed{z = \frac{1-a}{a}}$ convenient time variable in cosmology

$\rightarrow$ problem!

* Hubble law: $a(t_1) = a(t_0) + \dot{a}(t_0)(t_1 - t_0) + \ldots = a(t_0)\left(1 + (t_1 - t_0)H_0 + \ldots\right)$

distance $d = t_0 - t_1$ (meaningful only for $z \ll 1$)

$$\Rightarrow z = \frac{1}{a(t_1)} - 1 = \frac{1}{1 - dH_0} - 1 = dH_0 \Rightarrow \boxed{d = H_0^{-1} z}$$

distance $\leftrightarrow$ redshift

* observations:
  SNe : $H_0 = 73 \pm 1 \frac{km}{sec\,Mpc}$ ($z \lesssim 1$) / SH0ES   systematics?
  CMB : $H_0 = 67 \pm 0.5 \frac{km}{sec\,Mpc}$ ($z \simeq 1100$) / Planck

  H_0 tension

E. Hubble, 1927, $H_0 = 500 \frac{km}{sec\,Mpc}$
recession speed: $v = cz$

A.3 DYNAMICS

* Einstein equations $\underset{\text{curvature}}{G_{\alpha\beta}[g_{\mu\nu}]} = 8\pi G \underset{\text{energy + momentum}}{T_{\alpha\beta}[g_{\mu\nu}]}$

* EMT: $T_{\alpha\beta} = \left(\begin{array}{c|c} T_{00} & T_{0i} \\ T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline & \text{stress tensor} \end{array}\right)$

isotropy: $T_{0i} = \Pi_i = 0$ & $T_{ij} \propto g_{ij}$

homogeneity: $T_{00} = \rho(t)$; $T_{ij} = P(t)\, g_{ij}$

→ FRW ansatz: $T^{\mu}_{\ \nu} = g^{\mu\lambda} T_{\lambda\nu} = \mathrm{diag}(-\rho, P, P, P)$ ← perfect fluid (rest frame)

→ general form: $T^{\mu}_{\ \nu} = (\rho + P) U^{\mu} U_{\nu} + P\, \delta^{\mu}_{\nu}$ $\left[U^{\mu} = \frac{dx^{\mu}}{d\tau} ;\ \text{co-moving observer } U^{\mu} = (1,0,0,0) \right]$

→ conservation:

Minkowski:

$$\left.\begin{array}{l} \dot{\rho} = -\partial_i \pi^i \\ \dot{\pi}^i = \partial_i P \end{array}\right\} \Leftrightarrow \partial^{\mu} T_{\mu}^{\ \nu} = 0 \longrightarrow$$

Curved spacetime [covariant derivative ensures that $\nabla_{\mu} T^{\mu}_{\ \nu}$ transforms as tensor]

$$\nabla_{\mu} T^{\mu}_{\ \nu} = \partial_{\mu} T^{\mu}_{\ \nu} + \Gamma^{\mu}_{\lambda\mu} T^{\lambda}_{\ \nu} - \Gamma^{\lambda}_{\nu\mu} T^{\mu}_{\ \lambda} \overset{!}{=} 0$$

where $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\nu})$

→ problem 2: $\nabla^{\mu} T_{\mu}^{\ \nu} = 0 \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) \Longleftrightarrow \frac{d(\rho a^3)}{dt} = -P \frac{d(a^3)}{dt}$

"$dU = -P\,dV$"

→ Def: equation of state ω $\quad P = \omega\rho$: $\boxed{\rho \propto a^{-3(1+\omega)}}$ (e.g. radiation $\omega = 1/3$)

(4)

* Friedmann eqs: $G_{00} = 3\left(\frac{\dot a}{a}\right)^2 + \frac{3K}{a^2 R_0^2}$; $G_{ij} = \delta_{ij}\left(-2\frac{\ddot a}{a} + 3\frac{\dot a^2}{a^2}\right)$

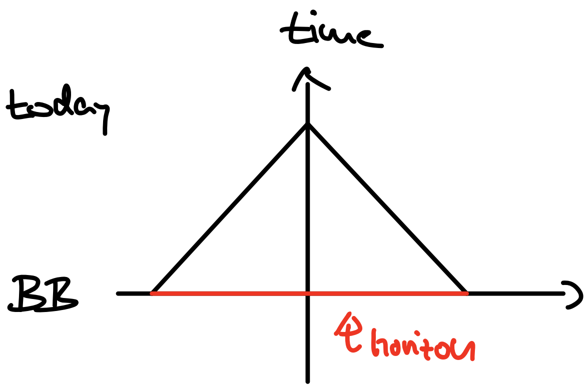
$$\left.\begin{array}{ll} \text{"00":} & \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2 R_0^2} \\ \text{"ij"} & \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) \end{array}\right\} \Longrightarrow \dot\rho + 3H(\rho + P) = 0$$

(Raychaudhuri equation)

* single fluid ($w > -\frac{1}{3}, K=0$) : $a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}\frac{1}{(1+w)}} = \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{1+3w}}$ → problem 1

→ singularity : $a \to 0$ as $\eta \to 0$ (if $w > -\frac{1}{3}$)

→ particle horizon (causal region) : $\chi_p(\eta) = \eta - \eta_{ini} \simeq \eta = \frac{2}{1+3w}\mathcal{H}^{-1}$; $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = \frac{a'}{a}$
$= aH$



A.4 ΛCDM MODEL

$$\rho = \underset{①}{\rho_r} + \underset{②}{\rho_m} + \underset{③}{\rho_\Lambda}$$

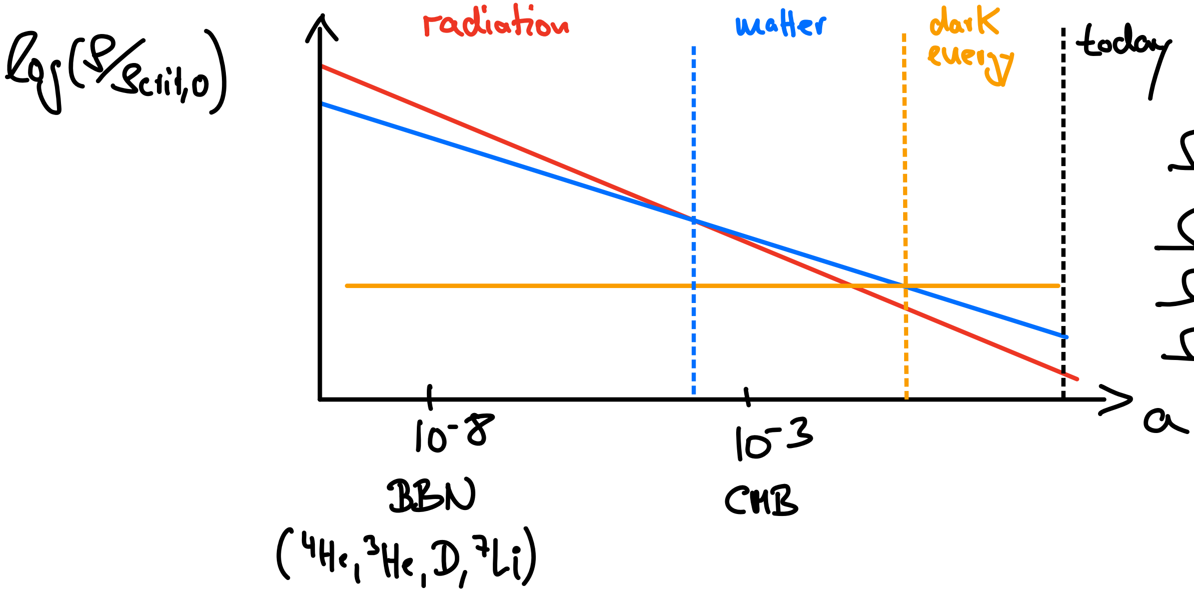
① rel. matter ($\gamma, \nu, \dots$): $\omega = \frac{1}{3} \Rightarrow \rho_r \propto \frac{1}{a^4} \propto \frac{1}{Vol}\,\frac{1}{a}$ if $\rho \simeq \rho_r$: $a \propto t^{\frac{1}{2}} \propto \eta$

② non-rel. matter ($e^-, p^+, \text{CDM} \dots$); $\omega = 0 \Rightarrow \rho_m \propto \frac{1}{a^3} \propto \frac{1}{Vol}$ if $\rho \simeq \rho_m$: $a \propto t^{\frac{2}{3}} \propto \eta^2$
(Einstein-dS)

③ dark energy (Λ): $\omega = -1 \Rightarrow \rho_\Lambda = \text{const}$ if $\rho \simeq \rho_\Lambda$: $a \propto e^{Ht} \propto \frac{1}{\eta}$

* Def: fractional densities $\Omega_I = \frac{\rho_{I,0}}{\rho_{crit,0}}$, where $\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda + \Omega_K a^{-2} \quad [\text{observation: } \Omega_K < 0.01;\ \text{set } \Omega_K = 0]$$



$$\left.\begin{array}{l} \Omega_\Lambda \simeq 70\% \\ \Omega_{CDM} \simeq 25\% \\ \Omega_b \simeq 5\% \\ \Omega_r \simeq 0\% \end{array}\right\} \text{budget today}$$