B. INHOMOGENEOUS UNIVERSE

[refs: Bardeen, Kodama, Sasaki, Ma, Bertschinger, ...]

B.1. METRIC + EMT PERTURBATIONS

* ansatz: $ds^2 = a^2(\eta)\left[-(1+2A(\eta,\vec{x}))d\eta^2 + 2B_i(\eta,\vec{x})\,d\eta\,dx^i + (\delta_{ij}+2E_{ij}(\eta,\vec{x}))dx^i dx^j\right]$

-> focus on scalars: A; $B_i = \partial_i B$, $E_{ij} = C\,\delta_{ij} + \partial_i\partial_j E$

-> invariance : $x^\mu \to \tilde{x}^\mu(x) \Rightarrow g_{\mu\nu}(x) = \frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}(\tilde{x})$

infinitesimal coord. trafo: $x^\mu \longmapsto \tilde{x}^\mu = x^\mu + \xi^\mu$; $\xi^\mu = (\xi^0, \partial^i\xi)$

problem 4: $\exists\ \xi^0$ & ξ s.t. $B = E = 0$; $A \equiv \Psi$; $C \equiv -\Phi$

-> Newtonian gauge: $ds^2 = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right]$

$T^0_{\ 0} = -(\bar{\rho}+\delta\rho) \equiv -\bar{\rho}(1+\delta)$; density contrast $\delta = \frac{\delta\rho}{\bar{\rho}}$ and $\delta\rho = \sum_I \delta\rho_I$; e.g. $I \in \{\gamma, \nu, \mathrm{CDM}, \Lambda, \dots\}$

$T_0^{\ i} = -(\bar{\rho}+\bar{P})v^i = -(\bar{\rho}+\bar{P})\partial^i v$ (bulk velocity)

$T^i_{\ j} = (\bar{P}+\delta P)\delta^i_j + \pi^i_{\ j}$ (we set $\pi^i_{\ j} = 0$; satisfied in early universe except for ν's)

B.2. EMT CONSERVATION

$\delta_{\mathrm{lin}}(\nabla_\mu T^\mu_{\ \nu}) \overset{!}{=} 0$

momentum space:

$\delta(\eta,\vec{x}) \to \delta(\eta,\vec{k}) = \int d^3x\, e^{i\vec{k}\vec{x}}\delta(\eta,\vec{x})$

$\partial_i \to i k_i$; $k = |\vec{k}|$

$\nu = 0$: $\delta' \equiv \frac{d\delta}{d\eta} = +\left(1+\frac{\bar{P}}{\bar{\rho}}\right)\left(\underset{\text{fluid flow}}{k^2 v} + \underset{\substack{\text{local exp.}\\ a_{\mathrm{eff}} = a(1-\Phi)}}{3\Phi'}\right) - \underset{\text{bg expansion}}{3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta}$

$\nu = i$: $v' = -\underset{\text{expansion}}{\left(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho}+\bar{P}}\right)}v - \underbrace{\frac{1}{\bar{\rho}+\bar{P}}\delta P - \Psi}_{\text{"force terms"}}$

* example I: radiation $w=\frac{1}{3}$, $\delta P=\frac{\bar P'}{\bar\rho'}\delta\rho=\frac{1}{3}\delta\rho$ (adiabatic!)

$$\left.\begin{aligned}\delta' &= +\frac{4}{3}\left(k^2 v+3\Phi'\right)\\ v' &= -\left(\mathcal{H}+\frac{1}{3}\frac{\bar\rho'}{\bar\rho+\bar P}\right)v-\frac{3}{4}\frac{1}{3}\delta-\Psi\\ v' &= -\frac{1}{4}\delta-\Psi\end{aligned}\right\}\quad \begin{aligned}\delta'' &= +\frac{4}{3}\left(k^2 v'+3\Phi''\right)\\ &= -\frac{1}{3}k^2\delta-\frac{4}{3}k^2\Psi+4\Phi''\end{aligned}$$

$$\Rightarrow \underbrace{\delta''+\frac{1}{3}k^2\delta}_{\text{sound waves}} = -\frac{4}{3}\underset{\uparrow\ \text{grav. sourcing}}{k^2\Psi}+4\Phi''$$

$\Rightarrow$ oscillatory: CMB anisotropies

* example II: matter $w=0$, $\delta P=0$

$$\left.\begin{aligned}\delta' &= k^2 v+3\Phi'\\ v' &= -\mathcal{H}v-\Psi\end{aligned}\right\}\quad\begin{aligned}\delta'' &= k^2 v'+3\Phi''\\ &= -k^2\mathcal{H}v-k^2\Psi+3\Phi''\end{aligned}$$

$$\Rightarrow \delta''+\underset{\uparrow\ \text{friction}}{\mathcal{H}\delta'} = -\underset{\uparrow\ \text{gravity}}{k^2\Psi}+\underset{\text{bg expansion}}{3\left(\mathcal{H}\Phi'+\Phi''\right)}$$

$\Rightarrow$ non-oscillatory: structure formation

B.3 EINSTEIN EQUATIONS

$$\delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu} \rightarrow$$

"00" $\quad -K^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 \delta\rho \quad$ (Poisson eq)

"0i" $\quad -(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 (\bar\rho + \bar p) v$

$\Phi - \Psi = 0 \quad$ (if $\Pi_{ij} = 0$)

"ij" $\quad \Phi'' + \mathcal{H}\Psi' + 2\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi = 4\pi G a^2 \delta P$

* example : w=const fluid: $\mathcal{H} = \frac{2}{(1+3w)}\frac{1}{\eta}$; $\delta P = w\,\delta\rho$ (adiabatic condition)

"00" in "ij": $\quad \Phi'' + 3\mathcal{H}\Phi' + 2\mathcal{H}'\Phi + \mathcal{H}^2\Phi = -wK^2\Phi - 3w\mathcal{H}\Phi' - 3w\mathcal{H}^2\Phi$

$$\rightarrow \Phi'' + 3(1+w)\mathcal{H}\Phi' + wK^2\Phi = 0$$

rad. era $(w=\frac{1}{3})$: $\quad \Phi = 3\Phi_{ini}\dfrac{\sin\varphi - \varphi\cos\varphi}{\varphi^3} = \begin{cases} \Phi_{ini}(\vec K) = \text{const} & ;\ K \ll \mathcal{H} = \frac{1}{\eta} \text{ (superhorizon)} \\ -9\Phi_{ini}\dfrac{\cos(\frac{1}{\sqrt3}K\eta)}{(K\eta)^2} & ;\ K \gg \mathcal{H} = \frac{1}{\eta} \text{ (subhorizon)} \end{cases}$

$\varphi \equiv \frac{K\eta}{\sqrt3}$ $\quad$ (regular solution)

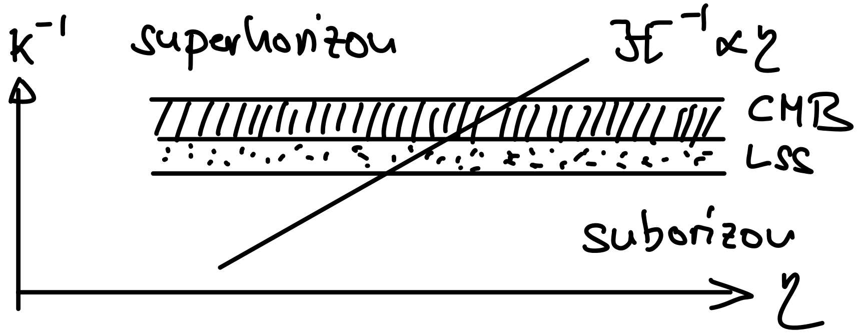
matter era $(w=0)$: $\quad \Phi = \frac{9}{10}\Phi_{ini} + \Phi_1 a^{-5/2} \simeq \text{const} \quad$ (all K)

* Grav. potential frozen on superhorizon scales if $w = \text{const.}$

→ problem 5: show that comoving curvature pert. $\boxed{\mathcal{R} = \Phi - \mathcal{H} v}$ conserved for $k \ll \mathcal{H}$

(determined by inflation!)

B.4 INITIAL CONDITIONS



* at early times observed modes were superhorizon ($k \ll \mathcal{H} = \frac{1}{\eta}$)

* small scales (large k) enter horizon first!

* adiabatic perturbations created by local time shift $\eta \to \eta + \pi(\vec{x})$ [natural prediction of inflation]

$$\bar{\rho}(\eta) \longrightarrow \rho(\eta, \vec{x}) = \bar{\rho}(\eta + \pi(\vec{x})) = \bar{\rho}(\eta) + \bar{\rho}'\pi(\vec{x}) \Rightarrow \delta\rho = \bar{\rho}'\pi(\vec{x}) \Rightarrow \boxed{\delta P = \bar{P}'\pi = \frac{\bar{P}'}{\bar{\rho}'}\delta\rho}$$

universality: $$\pi = \frac{\delta\rho_r}{\bar{\rho}_r'} = \frac{\delta\rho_m}{\bar{\rho}_m'} \Rightarrow \frac{\delta_r}{1+w_r} = \frac{\delta_m}{1+w_m} \Rightarrow \boxed{\delta_r = \frac{4}{3}\delta_m}$$

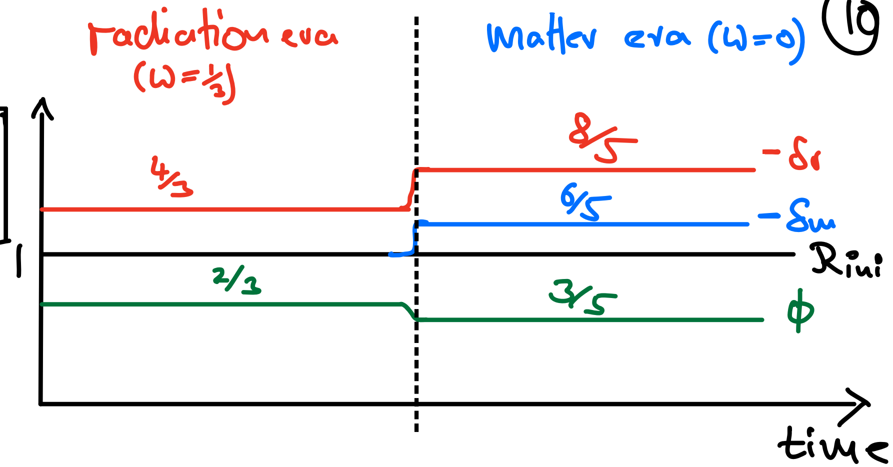
corollary: $$\delta\rho_m = \bar{\rho}_m \delta_m = \bar{\rho}_m \frac{3}{4}\delta_r \ll \delta\rho_r = \bar{\rho}_r \delta_r \quad \text{iff } \bar{\rho}_m \ll \bar{\rho}_r$$

def: $$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \simeq \begin{cases} \delta_r & \text{rad. era} \\ \delta_m & \text{matter era} \end{cases}$$

* superhorizon evolution

$$\boxed{\mathcal{R}_{ini} = \Phi - \mathcal{H}v \underset{\text{"00"},\ \text{"0i"}}{=} \Phi - \frac{\delta\rho}{3(\bar\rho+\bar P)} + \mathcal{O}\left(\frac{k^2}{\mathcal{H}^2}\right) = \Phi - \frac{1}{3(1+w)}\,\delta + \mathcal{O}\left(\frac{k^2}{\mathcal{H}^2}\right)}$$

$$\text{"00"}:\ -\underbrace{k^2\Phi}_{\to 0} - 3\mathcal{H}(\underbrace{\Phi'}_{\simeq 0} + \mathcal{H}\Psi) = 4\pi G a^2\,\delta\rho \Rightarrow \boxed{-2\Phi = \delta + \mathcal{O}\left(\frac{k^2}{\mathcal{H}^2}\right)}$$



→ cosmological preservation mechanism [CMB perturbations insensitive to subhorizon physics in early universe]

* statistics: → 2-point function: $\langle \mathcal{R}(\vec x)\,\mathcal{R}(\vec x')\rangle_{ini} = \xi_{\mathcal{R}}(|\vec x - \vec x'|)$

↑ensemble average of stochastic process ↑homogeneity ↙isotropy

[Gaussian initial conditions completely fixed by $\xi_{\mathcal{R}}$]

→ power spectrum: $\langle \mathcal{R}(\vec k)\,\mathcal{R}(\vec k')\rangle_{ini} \equiv P_{\mathcal{R}}(k)\,(2\pi)^3\,\delta(\vec k - \vec k') \equiv \frac{2\pi^2}{k^3}\,\Delta^2_{\mathcal{R}}\,\underset{\wedge}{\delta}(\vec k - \vec k')$ $(2\pi)^3$

↑predicted by inflation

→ ensemble average: $\xi_{\mathcal{R}} = \int \mathcal{D}\mathcal{R}\,\mathbb{P}[\mathcal{R}]\,\mathcal{R}(\vec x,\eta)\,\mathcal{R}(\vec x',\eta)$

↑probability

→ ergodicity: ensemble average $\simeq$ volume average (for $V\to\infty$)