G. COSMIC MICROWAVE BACKGROUND

* thermal history:

90% protons

$T > 0.3\,eV$: plasma of free e^- + nuclei + photons (thermal equilibrium)

Thomson scattering: $e^- + \gamma \leftrightarrow e^- + \gamma$ $\quad (\Gamma_\gamma = n_e \sigma_T \gg H)$

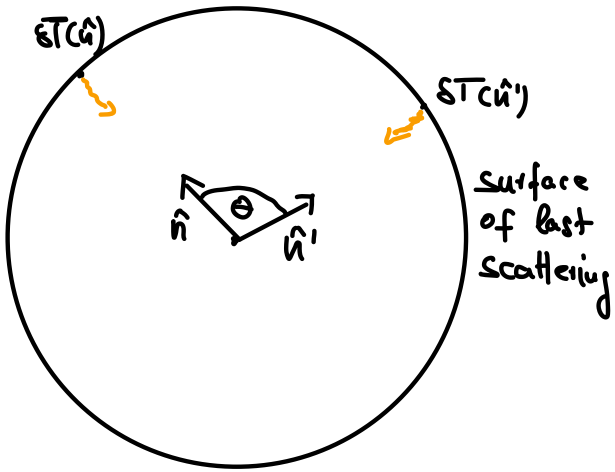
Coulomb scattering: $p^+ + e^- \leftrightarrow p^+ + e^-$ "photon-baryon fluid"

$T < 0.3\,eV$: recombination: $e^- + p^+ \leftrightarrow H + \gamma$ $\quad (\Rightarrow n_e \downarrow \Rightarrow \Gamma_\gamma \downarrow)$

$T < 0.25\,eV$: decoupling: $\Gamma_\gamma < H \Rightarrow$ Universe transparent $\Rightarrow$ geodesic motion of γ's

* we observe thermal spectrum: $\bar{T}_0 = 2.7\,K \quad \left(f(\vec{p}) = \left[e^{\frac{E(\vec{p})}{T}} - 1 \right]^{-1} \right)$

* anisotropies: $\Theta(\hat{n}) := \left. \frac{\delta T(\hat{n})}{\bar{T}} \right|_0 \sim 10^{-5}$ [dipole removed]



$$\Theta(\hat{n}) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$$

Legendre polynomial

$$C(\theta) = \langle \Theta(\hat{n})\, \Theta(\hat{n}') \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta);$$

ensemble average

angular powerspectrum: $\langle a_{\ell m} a^*_{\ell' m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$

statistical isotropy

* irreducible error: $\frac{\Delta C_\ell}{C_\ell} = \sqrt{\frac{2}{2\ell+1}}$ [cosmic variance, limiting factor up to $\ell \sim 2000$; for fixed ℓ: $2\ell+1$ m's
we only measure one Universe: ensemble average $\rightarrow$ m average]

$*$ goal: $\Delta^2_{\mathcal{R}} \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) \longrightarrow C_\ell = 4\pi \int d\ln k \, \Theta_\ell^2 \, \Delta^2_{\mathcal{R}}(k)$

Θ_ℓ: transfer function $\to$ fluid evolution $\eta < \eta_* \equiv \eta_{\rm dec}$; $\to$ geodesic motion $\eta_* < \eta < \eta_0$; $\to$ projection $\eta = \eta_0$

1. geodesic motion: $\frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$; $P^\alpha = \frac{dx^\alpha}{d\lambda}$

$$\to \frac{d}{d\eta} \ln(aE) = -\frac{d}{d\eta}\Psi(\eta, \vec{x}(\eta)) + \Phi'(\eta,\vec{x}) + \Psi'(\eta,\vec{x}) \Rightarrow \ln(aE)_0 = \ln(aE)_* - (\Psi_0 - \Psi_*) + \int d\eta\,(\Phi' + \Psi')$$

$$\left.\begin{array}{l} f(\vec{p}) = f\left(\frac{E}{T}\right) \Rightarrow E \propto T = \bar{T}\left(1 + \frac{\delta T}{\bar T}\right) \\ \delta_\gamma \propto T^4 \propto a^{-4} \Rightarrow \left\{\begin{array}{l} (aT)_0 = (aT)_* \\ \frac{\delta T}{\bar T} = \frac{1}{4}\delta_\gamma + \dots \end{array}\right. \end{array}\right\} \quad \left.\frac{\delta T}{\bar T}\right|_0 = \left.\frac{\delta T}{\bar T}\right|_* + \Psi_* + \underbrace{\int d\eta\,(\Phi' + \Psi')}_{\xrightarrow{k \ll \mathcal{H}} 0} \underbrace{(-\Psi_0)}_{\leftarrow \text{ contributes to } \ell = 0}$$

$$\Rightarrow \Theta(\hat{n}) = \underbrace{\left(\tfrac{1}{4}\delta_\gamma + \Psi\right)_*}_{\text{Sachs-Wolfe}} \quad (\text{"intrinsic temp. fluct. + grav. redshift"})$$

[we neglect Doppler effect: 

$\frac{\delta T}{T} = -(\hat{n}\cdot\vec{v}_b)_* \to 0 \; (k \ll \mathcal{H})$]

2. projection: (plane waves $\sim$ Fourier mode $\longrightarrow$ angular mode) $\left[e^{i\vec{k}\cdot\vec{x}} = \sum_\ell i^\ell (2\ell+1)\, j_\ell(k|\vec{x}|)\, P_\ell(\hat{k}\cdot\hat{n})\ ;\ \vec{x} = |\vec{x}|\hat{n}\right]$

$$\Theta(\hat{n}) = \sum_\ell i^\ell (2\ell+1) \int \frac{d^3k}{(2\pi)^3}\, \Theta_\ell(k)\, \mathcal{R}_{\rm ini}(k)\, P_\ell(\hat{k}\cdot\hat{n}), \text{ where } \Theta_\ell(k) = \frac{1}{\mathcal{R}_{\rm ini}}\left(\tfrac{1}{4}\delta_\gamma + \Psi\right)_* \underbrace{j_\ell(k|\vec{x}_*|)}_{\text{spherical Bessel function}}$$

$$\langle \Theta(\hat{n})\, \Theta(\hat{n}')\rangle \longrightarrow C_\ell = 4\pi \int_0^\infty d\ln k\, \Theta_\ell^2\, \Delta^2_{\mathcal{R}}(k)$$

$$\boxed{\begin{array}{l} P_\ell(-\hat{k}\cdot\hat{n}) = (-1)^\ell P_\ell(\hat{k}\cdot\hat{n}) \\ \int d\hat{k}\, P_\ell(\hat{k}\cdot\hat{n})\, P_{\ell'}(\hat{k}\cdot\hat{n}') = \frac{4\pi}{2\ell+1} P_\ell(\hat{n}\cdot\hat{n}')\,\delta_{\ell\ell'} \end{array}}$$

3. fluid evolution:

$\to$ large scales (low ℓ, $k \ll \mathcal{H}_*$): $\quad \left(\tfrac{1}{4}\delta_\gamma + \Phi\right)_* \overset{\text{superhorizon}}{=} \frac{1}{5}(-2+3)\,\mathcal{R}_{\rm ini} \quad (= \Theta(\hat{n}))$

Overdensity ($\delta_{\gamma *} > 0$) $\Rightarrow$ cold spot ($\Theta(\hat{n}) < 0$)

projection: $\Theta_\ell = \frac{1}{5} j_\ell(K|\vec{X}_*|)$

introduce: $\Delta^2_{\mathcal{R}}(K) = A_s \left(\frac{K}{K_0}\right)^{n_s - 1}$ (tilt: n_s-1; amplitude: A_s)

$n_s = 1 \Rightarrow$ $$\boxed{\frac{\ell(\ell+1)}{2\pi} C_\ell^{SW} = \frac{A_s}{25}}$$

→ oscillatory regime ($\ell \gg 1$): recall $\delta_r'' + c_s^2 K^2 \delta_r = -\frac{4}{3}K^2 \Phi + 4\Phi''$; $c_s^2 = \frac{1}{3}$ (toy model!)

$\left(\bar{\rho}_m \gg \bar{\rho}_r \Rightarrow \Phi = \text{const}\right) \Rightarrow \quad \delta_r = A\cos\left(\frac{1}{\sqrt{3}}K\eta\right) + B\sin\left(\frac{1}{\sqrt{3}}K\eta\right) - 4\Phi$

→ $K\eta \ll 1$: $\delta_r = -\frac{8}{5}\mathcal{R}_{ini} + \mathcal{O}(K^2\eta^2) \Rightarrow A = \frac{4}{5}\mathcal{R}_{ini}$; $B = 0$

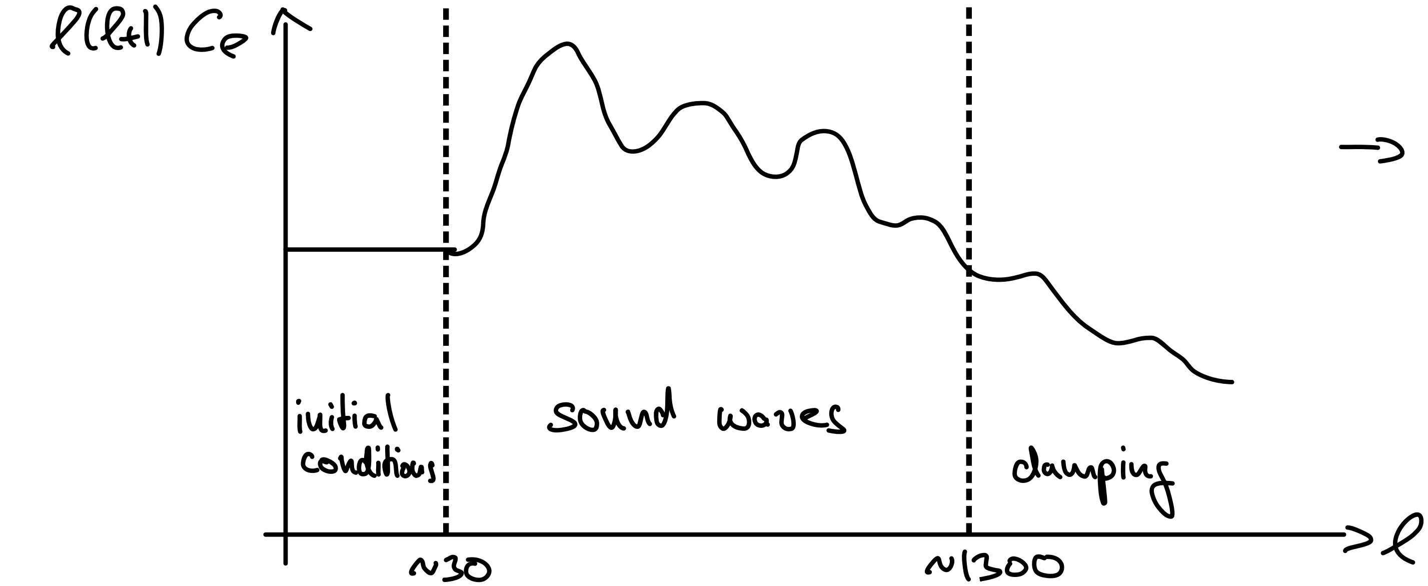
sound horizon: $$r_s \equiv \int_0^{\eta_*} d\tilde{\eta}\, c_s(\tilde{\eta}) = \begin{cases} \frac{1}{\sqrt{3}}\eta_* & ; \text{ negligible baryonic effects } (c_s = \frac{1}{\sqrt{3}}) \\ 145\text{ Mpc} & ; \text{ more realistic} \end{cases}$$

[tight-coupling $\vec{v}_b = \vec{v}_\gamma$
one fluid approximation
$c_s^2 = \frac{1}{3}\frac{1}{1+R}$; $R = \frac{3}{4}\frac{\bar{\rho}_b}{\bar{\rho}_\gamma}$]

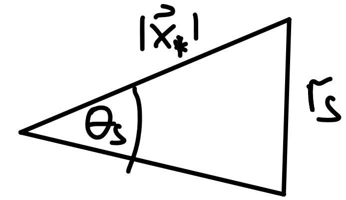
$\delta_\gamma \propto \cos(K r_s) \Rightarrow K = n\pi / r_s$

$$j_\ell(K|\vec{X}_*|) \underset{\ell \gg 1}{\Rightarrow} \ell = K|\vec{X}_*| = n\pi \frac{|\vec{X}_*|}{r_s} = n\pi \frac{14\text{ Gpc}}{145\text{ Mpc}} \sim 100\, n\pi$$

$\Rightarrow$ $\ell < 100$ multipoles of CMB measure initial conditions of Universe (A_s, n_s)!



→ position peaks → $r_s / |\vec{X}_*|$



$|\vec{X}_*|$ comoving angular diameter distance

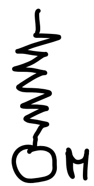
$$r_s = \int_{z_*}^{\infty} dz \, \frac{c_s(z)}{H(z)}$$ ← early universe standard ruler ;

$$|\vec{X}|_* = \int_{0}^{z_*} dz \, \frac{1}{H(z)} \propto \frac{1}{H_0}$$ ← late universe

constrains geometry: e.g. $H_0 \uparrow \underset{r_s \text{ fixed}}{\Rightarrow} \theta_s \uparrow \Rightarrow$ peaks move to left

→ for fixed pos:

$\Omega_{cdm} \downarrow \Rightarrow$ peaks $\uparrow$ (radiation driving) →



$\dot{\Phi} < 0 \Rightarrow$ amplitude grows

$\Omega_b \uparrow \Rightarrow$ 1. peak $\uparrow$ 2. peak $\downarrow$ (baryon loading) →

$$\left[(1+R)\vec{v}_\gamma\right]' = -\frac{1}{4}\vec{\nabla}\delta_\gamma - (1+R)\vec{\nabla}\Phi \; ; \; R = \frac{3}{4}\frac{\bar{\rho}_b}{\bar{\rho}_\gamma}$$

→ change M but not spring constant

→ shift: $\cos(kr_s) \to \cos(kr_s) + c$

$N_\nu \uparrow \Rightarrow \left.\frac{\theta_D}{\theta_s}\right|_* \propto H_*^{1/2} \uparrow \Rightarrow$ more damping

D. BONUS: MATTER POWERSPECTRUM

recall that $\delta_m'' + \mathcal{H}\delta_m' = -k^2\Phi + 3(\Phi'' + \mathcal{H}\Phi')$

requires more work

a posteriori

* subhorizon solution ($k > \mathcal{H}$): $\Phi = \underset{\text{fast}}{\phi_1} + \underset{\text{slow}}{\Phi_m} \longrightarrow \Phi_m = \text{const}$ (DM can't react to fast changes in Φ)

$$\delta_m'' + \mathcal{H}\delta_m' = -k^2\Phi = 4\pi G a^2 \bar\rho_m \delta_m$$

→ rad. era ($\bar\rho_r \gg \bar\rho_m$): $4\pi G a^2 \bar\rho_m \ll 4\pi G a^2 \bar\rho_r \simeq \frac{3}{2}\mathcal{H}^2$

$\Rightarrow \delta_m'' + \mathcal{H}\delta_m' = 0 \xRightarrow{\mathcal{H}=\frac{1}{\eta}} \boxed{\delta_m \propto \ln(\eta) \propto \ln a}$ Mészáros effect

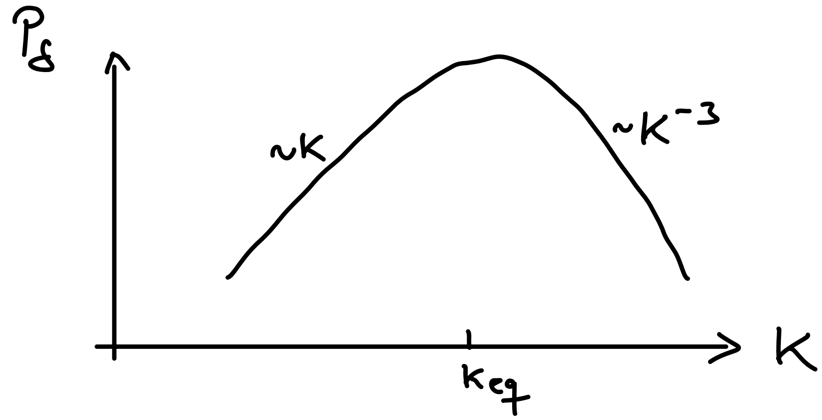
→ matter era ($\bar\rho_m \gg \bar\rho_r$): $4\pi G a^2 \bar\rho_m = \frac{3}{2}\mathcal{H}^2$

$\Rightarrow \delta_m'' + \mathcal{H}\delta_m' - \frac{3}{2}\mathcal{H}\delta_m = 0 \xRightarrow{\mathcal{H}=\frac{2}{\eta}} \boxed{\delta \propto \eta^2 \propto a}$ → Clustering!

* matter powerspectrum: $\langle \delta_m(k)\,\delta_m^*(k')\rangle = (2\pi)^3\,\delta(\vec k - \vec k')\,P_m(k)$

$$\delta_m(\eta,\vec k) = T(\eta,k)\times \mathcal{R}_{ini}(\vec k) \Rightarrow P_m(k) = |T(\eta,k)|^2 \times P_\mathcal{R} \propto |T(\eta,k)|^2 \frac{1}{k^3}$$

for $k > \mathcal{H}$: $|T(\eta,k)|^2 \propto \begin{cases} \left[\log\frac{a_{eq}}{a_k}\right]^2 \left(\frac{a_0}{a_{eq}}\right)^2; & k > k_{eq} \longrightarrow k = \mathcal{H} = \frac{1}{a_k} \Rightarrow |T|^2 \propto [\log k]^2 \\ \frac{a_0^2}{a_k^2}; & k < k_{eq} \longrightarrow k = \mathcal{H} \propto \frac{1}{\sqrt{a}} \rightarrow |T|^2 \propto k^4 \end{cases}$



P_δ
$\sim k$
$\sim k^{-3}$
k
k_{eq}