Lecture 2:

Linear perturbation theory

As hinted at in the previous section discussing the curvaton, perturbations are important for understanding and constraining models of inflation.

In flat space the only propagating modes of gravity are the two polarizations of the graviton.

The e.o.m. of the graviton in flat space is obtained in linearized gravity by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1$$

and only working to linear order in $h_{\mu\nu}$.

Now $h_{\mu\nu}$ is a symmetric 4×4 matrix with 10 components with 10 two derivative equations of motion from the Einstein equation.

However they are not all independent. The covariant conservation of the Einstein tensor $\nabla^\mu G_{\mu\nu} = 0$ (the contracted Bianchi identities) provides 4 constraints, yielding $10 - 4 = 6$ independent eq.

Now consider an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

such that derivatives of ξ are no large than $h_{\mu\nu}$. Using that the metric transforms like a tensor

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

we have

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

and the smallness of $h_{\mu\nu}$ is preserved, so therefore these types of infinitesimal coordinate transformations are symmetries of the linearized theory.

This means that we can use this gauge redundancy to gauge away (gauge fix) 4 more d.o.f leaving $((10-4)-4 = 6-4 = 2$ dynamical and physical d.o.f.

This allows us to go to the trace-less and transverse gauge

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$$

where

$$h^i{}_i = 0 \quad \text{and} \quad \partial^j h_{ij} = 0$$

de Sitter is also a vacuum solution with no new d.o.f. as the cosmological constant can't fluctuate.

Therefore the graviton in de Sitter again has two degrees of freedom and we can write the linearized gravitational fluctuations of de Sitter as

$$ds^2 = -dt + a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j$$

again with $h^i{}_i = 0$, $\partial^j h_{ij} = 0$, and $a \propto e^{Ht}$ with $H \sim$ const.

In slow-roll inflation we are no-longer in vacuum and while at the background level H is no-longer constant, also at the level of linearized perturbations the inflaton fluctuations carry an extra perturbative degree of freedom.

$$\varphi(t,x) = \varphi(t) + \delta\varphi(t,\bar{x})$$

where $\varphi(t)$ is the homogenous background inflaton field satisfying the slow-roll equations, and $\delta\varphi(t,\bar{x})$ is the linear perturbation.

The gauge where $\delta\varphi \neq 0$ and the linearized metric takes the form

$$ds^2 = -dt + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j$$

is called the flat-gauge because there is no scalar curvature perturbation in this gauge.

However by a time reparametrization

$$t \to \tilde{t} = t + \delta t$$

then $\varphi(\tilde{t}) = \varphi(t) + \dot{\varphi}(t)\delta t$

$$\Rightarrow \varphi(\tilde{t}, x) = \varphi(t) + \dot{\varphi}(t)\delta t + \delta\varphi(t, \bar{x})$$

so clearly, if we choose

$$\delta t = -\frac{\delta\varphi}{\dot{\varphi}}$$

then at linear order we have

$$\varphi(\tilde{t}, \bar{x}) = \varphi_c(t)$$

and so the field is homogeneous

This is called the comoving gauge, because in this gauge the time slices are slices of constant φ and therefor comoving with φ. In this gauge it is clear that φ is acting as a clock.

In the comoving gauge the scalar fluctuation appears in the metric as fluctuations of the scale factor

$$a(t) \to a(\tilde{t}) = a(t) + \dot{a}(t)\,\delta t$$

Defining the scalar curvature perturbation

$$\zeta = \frac{\dot{a}}{a}\delta t = H\,\delta t \quad \left(= -\frac{H}{\dot{\varphi}}\delta\varphi\right)$$

$$\Rightarrow \; ds^2 = -dt^2 + a^2\left(\delta_{ij}(1+2\zeta) + \gamma_{ij}\right)dx^i dx^j$$

to leading order in perturbations.

To leading order this is equivalent to the another convenient form

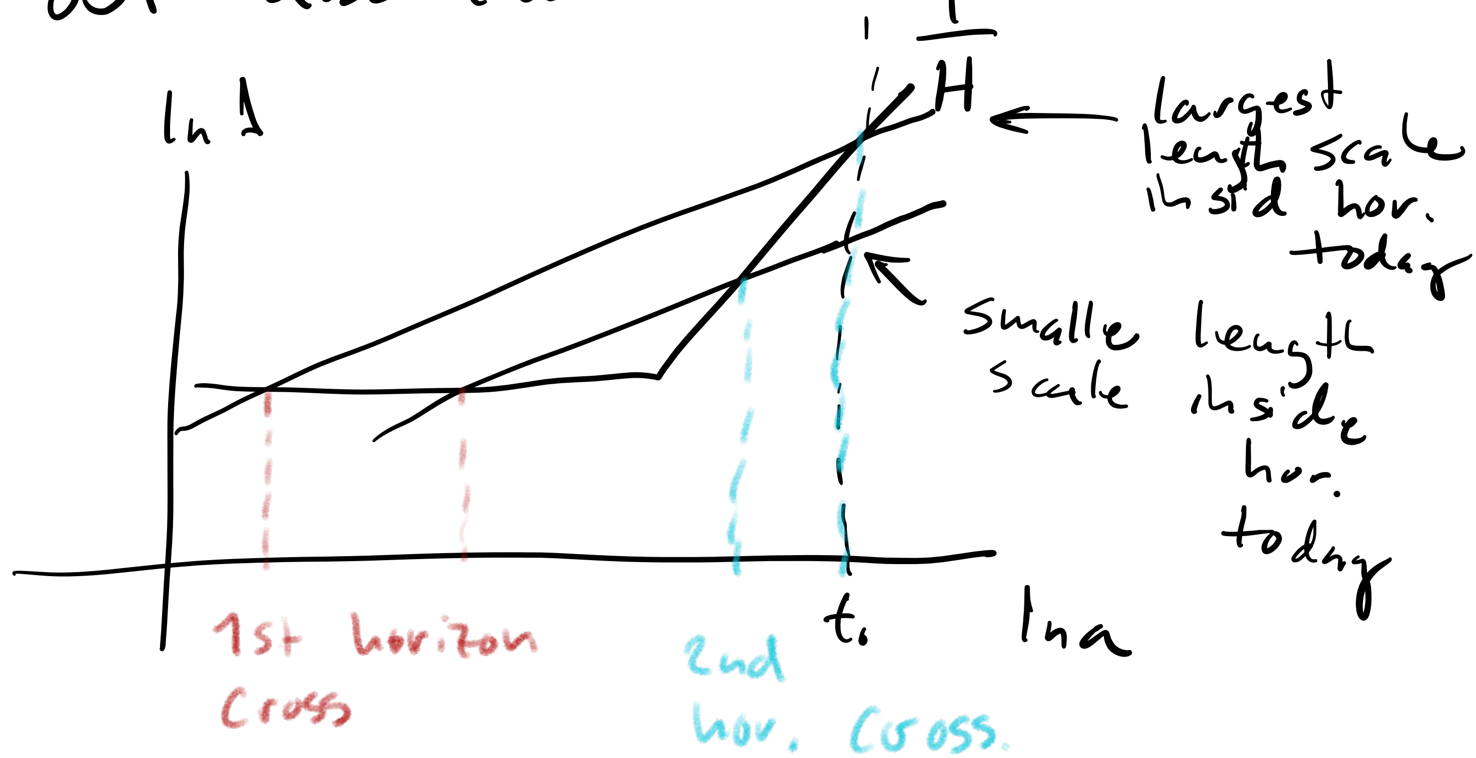
$$ds^2 = -dt^2 + a^2 \left(e^{2\zeta} \delta_{ij} + \gamma_{ij} \right) dx^i dx^j$$

During inflation $a(t)$ has the approximative form

$$a \propto e^{Ht} \rightarrow a(\tilde{t}) \propto e^{H(t+\delta t)} = e^{Ht+\zeta}$$

So during inflation, if ζ is constant, we can just view it as a shift in the normalization of the scale factor. This observation is going to be important later because ζ in fact is conserved and const. on super-horizon scales.

The fact that ζ is conserved on super-horizon scales is the main motivation for working with this variable. Remember the largest scales we observe today are the ones that exited the horizon earliest during inflation in an unfair "first out also last in".



This means that the perturbations corresponding to the largest length scales observed today are insensitive to most of the evolution of the universe in between the early phase of inflation and today. Good for making predictions of inflation.

To see why they are conserved let's derive their e.o.m. from their action. To find the action we start by perturbing

$$S = S_{grav} + S_\varphi = \frac{1}{2}\int d^4x \sqrt{-g}\,[R - \partial_\mu\varphi\,\partial^\mu\varphi - 2V(\varphi)]$$

The simplest way to proceed is to compare with the ADM formalism, and write the metric on the ADM form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

with the ADM ansatz for the metric, the action becomes

$$S = \frac{1}{2}\int d^4x \sqrt{h}\,\Big[N R^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot\varphi - N^i\partial_i\varphi) - N h^{ij}\partial_i\varphi\,\partial_j\varphi\Big]$$

where

$$E_{ij} = \frac{1}{2}(\dot h_{ij} - \nabla_i N_j - \nabla_j N_i)$$

$$E = E^i{}_i$$

and related to the extrinsic curvature

by $$K_{ij} = N^{-1} E_{ij}$$

Defined as
$$\sqrt{-g}\, R = \sqrt{h}\, N\left(R^{(3)} + K_{ij}K^{ij} - K^2 \right)$$

when using the ADM formalism one needs to be a bit careful as it is not fully coovariant, but only explicitly invariant under spatial coordinate transformations, and the invariance under the kind of time reparametrization we did to change gauge is enforced by thinking of N and N^i as lagrange multipliers whose equations of motion becomes constraint equations enforcing the invariance. These equations are for N^i
$$\nabla_i \left[N^{-1} \left(E^i{}_j - \delta^i{}_j E \right) \right] = 0$$

and N
$$R^{(3)} - 2V - N^{-2}\left(E_{ij}E^{ij} - E^2 \right) - N^{-2}\dot{\varphi}^2 = 0$$

which are also referred to as the momentum and Hamiltonian constraints.

For the physical variables h_{ij} and φ, the two gauges discussed above

1) Comoving gauge

$$\delta\varphi = 0, \quad h_{ij} = a^2[e^{2\zeta}\delta_{ij} + \gamma_{ij}],$$

$$\gamma_{ii} = 0, \quad \partial_i \gamma_{ij} = 0$$

2) Flat gauge

$$\delta\varphi \neq 0 \quad h_{ij} = a^2[\delta_{ij} + \gamma_{ij}]$$

$$\gamma_{ii} = 0, \quad \partial_i \gamma_{ij} = 0$$

Now solving the constraint equations using the comoving gauge yields to first order

$$N = 1 + \frac{1}{H}\dot{\zeta} \quad , \quad N_i = \partial_i\left(-\frac{\zeta}{H} + \frac{\dot{\varphi}^2}{2H}\partial^{-2}\dot{\zeta}\right)$$

To get the quadratic action for ζ we only need N, N_i to first order.

Inserting this into the action gives after integration by parts and using the background equation of motion for φ

$$S_\zeta = \frac{1}{2}\int dt\, d^3x\, \frac{\dot{\varphi}^2}{H^2}\left[a^3\dot{\zeta}^2 - a(\partial\zeta)^2\right]$$

Not that no slow-roll approximation has yet been made.

Also note that the action is proportional to $\epsilon \propto \frac{\dot{\varphi}^2}{H^2}$, which means that ζ becomes unphysical in pure dS when $\epsilon \to 0$. The suppression by ϵ is only apparent after integration by parts.

[To compare with Mukhanov, Feldman, Brandenberger use $v = -2\zeta$] [see https://arxiv.org/abs/0709.2708 for a more systematic understanding of slow-roll hierarchy]

Now the e.o.m. for ζ is just the Euler-Lagrange eq. obtained from

$$\delta S = 0$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \zeta} - \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\zeta}} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \zeta} = 0$$

$$-\partial_t \left(a^3 \frac{\dot{\varphi}^2}{H^2} \dot{\zeta} \right) + a \frac{\dot{\varphi}^2}{H^2} \partial^2 \zeta = 0$$

Now let's analyse this equation in Fourier space

$$\zeta(t,x) = \int \frac{d^3x}{(2\pi)^3} \zeta_k(t) e^{i\bar{k}\cdot\bar{x}}$$

so $\partial^2 \to -k^2$

$$\Rightarrow \quad -\partial_t \left(a^3 \frac{\dot{\varphi}^2}{H^2} \dot{\zeta}_k \right) - a \frac{\dot{\varphi}^2}{H^2} k^2 \zeta_k = 0$$

we see that on superhorizon scales $k \ll aH$ obtained by $k \to 0$ in eq. above

$$k \to 0 \Rightarrow \partial_t \left(\tfrac{1}{2} a^3 \epsilon \dot{\zeta} \right) = 0$$

$$\Rightarrow \dot{\zeta} = 0 \text{ or } a^3 \epsilon \dot{\zeta} \propto \text{const}$$

$$\Rightarrow \zeta = \text{constant} + \text{fast decaying term!}$$

The most elegant way to analyze this is to redefine the field

$$\chi = a\frac{\dot{\varphi}}{H}\zeta = a\frac{\varphi'}{\mathcal{H}}\zeta \equiv z\zeta$$

where prime denotes derivative with respect to conformal time, τ, and $\mathcal{H} = a'/a$.

$$\Rightarrow \dot{\zeta} = \frac{1}{a}\zeta' = \frac{1}{a}\left(\frac{1}{z}\chi\right)' = \frac{1}{a}\left(\frac{\chi'}{z} - \frac{z'}{z}\frac{\chi}{z}\right)$$

$$\Rightarrow a^2z^2\dot{\zeta}^2 = a^2z^2\frac{1}{a^2}\left(\frac{\chi'^2}{z^2} - 2\frac{z'}{z}\frac{\chi'\chi}{z^2} + \left(\frac{z'}{z}\right)^2\frac{\chi^2}{z^2}\right)$$

$$= a^2z^2\frac{1}{a^2}\left(\frac{\chi'^2}{z^2} - \frac{z'}{z}\frac{(\chi^2)'}{z^2} + \left(\frac{z'}{z}\right)^2\frac{\chi^2}{z^2}\right)$$

$$= \chi'^2 - \frac{z'}{z}(\chi^2)' + \left(\frac{z'}{z}\right)^2\chi^2$$

Integration by parts

$$= \chi'^2 + \left(\frac{z''}{z} - \cancel{\frac{z'^2}{z^2}}\right)\chi^2 + \cancel{\left(\frac{z'}{z}\right)^2\chi^2} + \text{total derivative}$$

$$= \chi'^2 + \frac{z''}{z}\chi^2$$

So in terms of χ, after an integration by part and neglecting a total time derivative, the action becomes that of a minimally coupled scalar field with time dependent mass, $m^2(\eta) = \frac{z''}{z}$

$$S_\chi = \frac{1}{2} \int d\tau\, d^4x \left[\chi'^2 - (\partial \chi)^2 + \frac{z''}{z} \chi^2 \right]$$

Defining the Fourier modes as

$$\chi(t,x) = \int \frac{d^3x}{(2\pi)^3} \chi_k(t) e^{i\bar{k}\cdot\bar{x}}$$

one finds

$$S_\chi = \frac{1}{2} \int d\tau\, d^4x \left[\chi'_{\bar{k}} \chi'_{-\bar{k}} - k^2 \chi_{\bar{k}} \chi_{-\bar{k}} + \frac{z''}{z} \chi_{\bar{k}} \chi_{-\bar{k}} \right]$$

the E. L. eq.

$$\frac{\partial L}{\partial \chi} - \partial_\tau \frac{\partial L}{\partial \chi'} = 0$$

becomes

$$\chi''_k + \left(k^2 - \frac{z''}{z}\right) \chi_k = 0$$

To leading order in slow-roll we have

$$z = a\sqrt{2\epsilon} \qquad a\,d\tau = dt \qquad \left(\epsilon = \frac{1}{2}\frac{\dot{\varphi}^2}{H^2}\right)$$

$$\Rightarrow \quad z'' = (a\sqrt{2\epsilon})'' \qquad \left(\dot{\epsilon} \propto O(\epsilon^2)\right)$$

$$= a\left(a\left(a\sqrt{2\epsilon}\right)^{\cdot}\right)^{\cdot}$$

$$\Rightarrow \quad z'' = z\,2a^2H^2\left(1 + \tfrac{5}{2}\epsilon - \tfrac{3}{2}\eta\right)$$

$$\Rightarrow \quad \frac{z''}{z} = 2a^2H^2\left(1 + \tfrac{5}{2}\epsilon - \tfrac{3}{2}\eta\right)$$

<u>Exercise 3:</u> Show that this is true

Rewriting the scale factor in terms of conformal time gives further

$$a = \frac{-1}{H\tau(1-\epsilon)} \quad \text{or} \quad H^2 = \frac{1}{a^2\tau^2(1-\epsilon)^2}$$

$$\Rightarrow a^2H^2 \approx \frac{1}{\tau^2}(1+2\epsilon)$$

$$\Rightarrow \quad \frac{z''}{z} = \frac{1}{\tau^2}(2 + 9\epsilon - 3\eta)$$

Defining

$$\frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right) = \frac{z''}{z}$$

$$\Rightarrow \nu^2 = \frac{9}{4} + 9\epsilon - 3\eta$$

$$\Rightarrow \nu = \frac{3}{2} + 3\epsilon - \eta$$

which means that the e.o.m. becomes

$$\chi_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right]\chi_k = 0$$

This is the defining function for Hankel functions (linear combinations of spherical Bessel functions)

$$\Rightarrow \chi_k(\tau) = \sqrt{\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right]$$

In order to fix the integration constants, we need some physical boundary condition. We are going to impose that the field was initially in vacuum. In order to understand the initial vacuum state, we need to quantize the field.

Canonical quantization

In time dependent perturbation theory, it is often convenient to adopt the interaction picture. In the interaction picture the free-field theory is evolved in the Heisenberg picture. In general

$$H = H_0 + H_I$$

with H_0 the Hamiltonian of the free theory and H_I that of the interactions, so

$$i\hbar \frac{d}{dt} A_I(t) = [A_I(t), H_0]$$

for some operator in the interaction picture. While interactions are evolved in Schrödinger picture

$$i\hbar \frac{d}{dt} |\psi_I(t)\rangle = H_I |\psi_I(t)\rangle$$

for some interaction picture quantum state $|\psi_I(t)\rangle$. In free theory $H_I = 0$ and states are time independent.

Now at linear pert. theory with our H given by the quadratic Lagrangian of χ above, there are no interactions, so we just have H_0, and quantize χ in the Heisenberg picture. Later when discussing non-Gaussianity and loop effects, we will have to go to higher order and include H_I.

Quantizing χ in the Heisenberg picture, using canonical quantization, we start with promoting χ and it canonical conjugate field

$$\Pi_k = \frac{\partial \mathcal{L}}{\partial \chi_k'} = \chi_{-k}'$$

to operators $\hat{\chi}$, $\hat{\Pi}$, and impose the equal time canonical commutation relations

$$[\hat{\chi}(\tau,\bar{x}), \hat{\chi}(\tau,\bar{x}')] = [\hat{\Pi}(\tau,\bar{x}), \hat{\Pi}(\tau,\bar{x}')] = 0$$

$$[\hat{\chi}(\tau,\bar{x}), \hat{\Pi}(\tau,\bar{x}')] = i\,\delta(\bar{x}-\bar{x}') \qquad [\hbar = 1]$$

The equation of χ_k is that of an harmonic oscillator

$$\chi''_k + \omega_k^2(\tau) = 0$$

with time dependent frequency

$$\omega_k^2(\tau) = k^2 - \frac{z''}{z}$$

that only depends on $|\bar{k}| = k$

Since χ is a real scalar field, it is hermitian $\chi^\dagger(\tau, \bar{x}) = \chi(\tau, \bar{x})$, which imply $\chi^\dagger_{\bar{k}} = \chi_{-\bar{k}}$, so when quantizing we can write $\hat{\chi}_k$ in terms of raising and lowering operators

$$\hat{\chi}_{\bar{k}}(\tau) = \frac{1}{\sqrt{2k}}\left(\hat{c}_{\bar{k}}(\tau) + \hat{c}^\dagger_{-\bar{k}}(\tau)\right)$$

with

$$[\hat{c}_{\bar{k}_1}, \hat{c}^\dagger_{\bar{k}_2}] = \delta^{(3)}(\bar{k}_1 - \bar{k}_2)$$

where the vacuum is the state

$$\hat{c}_k(\tau)|0\rangle_\tau = 0$$

Now clearly the definition of the vacuum is time-dependent. This follows from the fact that the Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3x \left[\hat{\Pi}^2 + (\partial\hat{\chi})^2 + \frac{z''}{z}\hat{\chi}^2\right]$$

has an explit time-dependence through $z(\tau)$, and so energy is not conserved. This is how a rich Universe can be created out of vacuum.

The standard way of dealing with this phenomena is by means of a Bogolivbov transformation

$$\hat{C}_{\bar{u}}(\tau) = \alpha_u(\tau)\hat{C}_{\bar{u}}(\tau_0) + \beta_u(\tau)\hat{C}^{+}_{-\bar{u}}(\tau_0)$$

$$\hat{C}^{+}_{\bar{u}}(\tau) = \alpha^{*}_u(\tau)\hat{C}^{+}_{\bar{u}}(\tau_0) + \beta^{*}_u(\tau)\hat{C}_{-\bar{u}}(\tau_0)$$

where α_u, β_u are the Bogolivbov coefficients, which has to satisfy

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

for the commutation relation to be preserved in time.

Note that the number of particles at time τ, if we are initially in the vacuum $|0\rangle_{\tau_0}$ is

$$_{\tau_0}\langle 0|\hat{N}_k|0\rangle_{\tau_0} = |\beta_k|^2$$

where

$$\hat{N}_k = \hat{C}^{+}_{\bar{k}}(\tau)\,\hat{C}_{\bar{k}}(\tau)$$

The solution to the dynamical equation can be obtaind through

$$f_k(\tau) = \frac{1}{\sqrt{2k}}\left(\alpha_k(\tau) + \beta_k^*(\tau)\right)$$

with

$$f_k(\tau) f_k^{*\prime}(\tau) - f_k^{\prime}(\tau) f_k^*(\tau) = i$$

which is called the Wronskian condition and ensures that the canonical commutation relation is consisten with the one of $\hat{c}_k, c_k^+$.

We then have that

$$\hat{x}_k(\tau) = f_k(\tau)\hat{c}_{\bar{k}}(\tau_0) + f_k^*(\tau) c_{-\bar{k}}^+(\tau_0)$$

$$\Rightarrow \hat{x}(\tau, \bar{x}) = \int \frac{d^3\bar{k}}{(2\pi)^3}\left[f_k(\tau)\hat{c}_k(\tau_0)e^{i\bar{k}\cdot\bar{x}} + f_k(\tau) c_{\bar{k}}^+(\tau_0)e^{-i\bar{k}\cdot\bar{x}}\right]$$

Inserting into Heisenbergs equation of motion

$$i\partial_\tau \hat{\Pi} = [\hat{\Pi}, \hat{H}]$$

one can verify that $f_k(\tau)$ satisfy the classical equation of motion with the solution

$$f_k(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right]$$

Since we saw that

$${}_{\tau_0}\langle 0|\hat{N}_k|0\rangle_{\tau_0} = |\beta_k|^2$$

and $|\alpha_k|^2 - |\beta_k|^2 = 1$, as well as $f_k = \frac{1}{\sqrt{2k}}(\alpha_k + \beta_k^*)$, we see that in the vacuum (no particle state), $|\beta_k|^2 = 0 \Rightarrow |\alpha_k|^2 = 1 \Rightarrow f_k = \frac{1}{\sqrt{2k}}\alpha_k \Rightarrow$ $|f_k|^2 = \frac{1}{2k} \Rightarrow f_k = \frac{1}{\sqrt{2k}} e^{\mp i F(k,\tau)}$, where $F(k,\tau)$ is some real function of k and τ.

Now as $\tau \to -\infty$, the physical wavelength $\lambda = \frac{a}{k} \to 0$, so the wavelength $\lambda \ll \frac{1}{H} \Leftrightarrow k \ll aH$, so far inside the horizon at early times the wavelength corresponding to the k-mode is tiny compared to the horizon, or the curvature of the spacetime. Hence, the modes are effectively in flat space, just like a tiny flat-earther doesn't realize the curvature of the earth becacause earth is much bigger - too big for him/she/it to understand...

Now for $\tau \to -\infty$

$$H_\nu^{(1)}(-k\tau) \to \frac{\sqrt{2/\pi}}{\sqrt{-\tau k}} e^{-ik\tau}, \quad H_\nu^{(2)}(-k\tau) \to \frac{\sqrt{2/\pi}}{\sqrt{-\tau k}} e^{ik\tau}$$

Thus we choose the constants of proportionality, such that our definition of the vacuum agrees with the Minkowski vacuum at $\tau \to -\infty$

$$C_1 = \frac{\sqrt{\pi}}{2}, \quad C_2 = 0$$

In this way the raising operator corresponds to creating only a positive frequency mode in the vacuum, as creating negative frequency modes would be unphysical.

[Since formally the norm is $(\chi;\chi) = i\int_{-\infty}^{\infty} dx\,(\chi^* \pi - \chi \pi^*)$ with $\pi = \chi'$ for positive freq. $(\chi_k, \chi_{k'}) = \delta(k-k')$ while for neg. freq. $(\chi_k, \chi_{k'}) = -\delta(k-k') \Rightarrow$ instabilities]

we then have

$$\hat{\chi}_k(\tau) \to \frac{1}{\sqrt{2k}} e^{-ik\tau} \hat{c}_{\bar{k}}(\tau_0) + \frac{1}{\sqrt{2k}} e^{+ik\tau} \hat{c}^{\dagger}_{-\bar{k}}(\tau_0)$$

for $\tau_0 \to \infty$.

The observables of a quantum field is of course things like expectation values and correlation functions. For $\hat{\chi}$, the two-point correlation function is

$${}_{\tau_0}\langle 0 | \hat{\chi}^{\dagger}_{\bar{k}_1} \hat{\chi}_{\bar{k}_2} | 0 \rangle_{\tau_0} \equiv \delta^{(3)}(\bar{k}_1 - \bar{k}_2) \frac{2\pi^2}{k^3} \mathcal{P}_\chi(k)$$

$k = |k_1|$

Inserting or normalized solution we obtain

$$\mathcal{P}_\chi(k) = \frac{k^3}{2\pi^2} |f_k|^2$$

Using that $\zeta_k = \chi_k / z$ and that on super-horizon scales, for $-k\tau \to 0$, we have

$$H_\nu^{(1)}(-k\tau) \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (-k\tau)^{-\nu}$$

we obtain on super-horizon scales

$$\mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2} \frac{1}{z^2} |f_k|^2$$

$$= \frac{2^{2\nu-3}}{(2\pi)^2} \left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2 \left(\frac{H}{a\dot{\varphi}}\right) (-k\tau)^{3-2\nu} (-\tau)^{-2}$$

$$\propto k^{3-2\nu}$$

Defining the spectral tilt to be

$$n_s - 1 = \frac{d \ln P_\zeta(k)}{d \ln(k)} = 3 - 2\nu$$

$\nu \simeq \frac{3}{2} + 3\epsilon - \eta$ $\qquad = 2\eta - 6\epsilon$

so

$$\boxed{n_s - 1 = 2\eta - 6\epsilon}$$

is on of the major predictions of inflation. Another important observable is the amplitude of perturbations. Since they are conserved on super horizon scales, we can evaluate the amplitude of perturbations, by avaluating the power spectrum at horizon exit

$$P_\zeta(k) = 2^{2\nu-3}(1-\epsilon)^{2\nu-1}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2 \left(\frac{H}{\dot\varphi}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH}$$

$\nu \simeq 3/2$
$\Rightarrow$

$$\boxed{P_\zeta(k) \simeq \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot\varphi}\right)^2 \Bigg|_{k=aH}}$$

Tensor modes

So far we calculated the spectrum of the scalar perturbations, ζ. But we also have two tensor modes in γ_{ij} (remember $h_{ij} = a^2(e^{2\zeta}\delta_{ij} + \gamma_{ij})$ with $\gamma_{ii} = \partial_i \gamma_{ij} = 0$)

From the ADM action we obtain at quadratic order

$$S_\gamma = \frac{1}{8}\int dt\, d^3x \left[a^3 \dot{\gamma}_{ij}\dot{\gamma}_{ij} - a\,\partial_l \gamma_{ij}\partial_l \gamma_{ij}\right]$$

which in conformal time yields

$$S_\gamma = \frac{1}{8}\int d\tau\, d^3x\, a^2\left[\dot{\gamma}_{ij}\dot{\gamma}_{ij} - \partial_l \gamma_{ij}\partial_l \gamma_{ij}\right]$$

Now expanding in plane waves for the two polarization modes

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}(k)\, \gamma^s_{\bar{k}}(\tau)\, e^{i\bar{k}\cdot\bar{x}}$$

$$\epsilon_{ii} = k^i \epsilon_{ij} = 0 \quad \text{and} \quad \epsilon^s_{ij}(k)\epsilon^{s'}_{ij}(k) = 2\delta_{ss'}.$$

$$\Rightarrow S_\gamma = \frac{1}{4}\sum_{s=+,-} \int d\tau d^3x\, a^2\left[\gamma'^s_{\vec{k}}\gamma'^s_{-\vec{k}} - k^2\gamma^s_{\vec{k}}\gamma^s_{-\vec{k}}\right]$$

defining $h^s_k = \frac{1}{\sqrt{2}} a \gamma^s_{\vec{k}}$

$$\Rightarrow S_\gamma = \frac{1}{4}\sum_{s=+,-} \int d\tau d^3x \left[h'^s_{\vec{k}} h'^s_{-\vec{k}} - \left(k^2 - \frac{a''}{a}\right) h_{\vec{k}} h_{-\vec{k}}\right]$$

This is the same action as for χ_k except now $z \to a$ and so $\frac{z''}{z} \to \frac{a''}{a} \simeq -\frac{1}{\tau^2}(2-3\epsilon)$

$$\Rightarrow \nu_T = \frac{3}{2} - \epsilon$$

So for tensor modes we find

$$n_T = \frac{d\ln P_\gamma}{d\ln k} = 3 - 2\nu_T$$

$$\Rightarrow \boxed{n_T = -2\epsilon}$$

and $\boxed{P_\gamma \simeq 8\left(\frac{H}{2\pi}\right)^2\Big|_{k=aH}}$

Factor $8 = 2\times 2\times 2$ where

$$\sum_{ss'} \langle \gamma^s \gamma^{s'} \rangle = \sum_{ss'} 2 \langle h^s h^{s'} \rangle = 4(\langle h^+ h^+ + h^- h^- \rangle$$

$$= 8 \langle h^s h^s \rangle .$$

Observational tests of inflation

We haven't measured the tensor modes from inflation yet. But from the non-observation we get important constraints. Let's consider the ratio, called the tensor-to-scalar ratio

$$\boxed{r = \frac{P_T}{P_\zeta} = \frac{8}{H^2/\dot{\varphi}^2} = 16\epsilon = -8n_T}$$

This also called the single fiel consistency relation, and is an important prediction.

In R^2-model (Starobinsky model)

$$n_s - 1 \simeq -\frac{2}{N} \quad , \quad r \simeq \frac{12}{N^2}$$

for $N \sim 60$

$$\Rightarrow \quad n_s \simeq 0.967 \quad , \quad r \simeq 0.0033$$

Curvaton

In the simplest curvaton model the is an inflaton field φ and a curvaton field σ with just the simplest possible potential

$$V(\varphi, \sigma) = \frac{1}{2} M^2 \varphi^2 + \frac{1}{2} m^2 \sigma^2$$

The curvaton is very light an subdominat during inflation, and some curvature pert. is created by the inflaton like before. But imagine the amplitude is too small to fit the data, for istance if H during inflation is small!

The all the observed perturbations can instead come from the curvaton field, which is just a spectator during inflation, if after the inflaton has decayed into radiation with energy density $\rho_r \propto 1/a^3$, and H start decreasing as $H \propto 1/a^2$, the curvaton mass will become heavy compared to H, $m \gg H$, at which point the curvaton will start to oscillate in its potential with an energy density $\rho_\sigma \propto \frac{1}{a^3}$ and soon dominate the energy density. At that point the density perturbations can easily be computed, by computing the $\delta\rho_\sigma/\rho_\sigma$ during inflation and noting it will stay frozen for super-horizon modes

$$\frac{\delta\rho_\sigma}{\rho_\sigma} = \frac{m^2\sigma\,\delta\sigma}{\frac{1}{2}m^2\sigma^2} = 2\,\frac{\delta\sigma}{\sigma}$$

which by a gauge transformation from this flat gauge into comoving gauge yields

$$\delta\rho_\sigma(\tau') = \delta\rho_\sigma(\tau) + \dot{\rho}_\sigma\,\delta t = 0$$

$$\Rightarrow \quad \delta t = -\frac{\delta\rho_\sigma}{\dot{\rho}_\sigma}$$

$$\Rightarrow \quad \zeta_\sigma = \frac{\dot a}{a}\delta t = H\,\delta t = -\frac{H}{\dot\rho_\sigma}\delta\rho_\sigma$$

by an argument very similar to how we found ζ_φ for the inflaton.

For the curvaton, the action is just

$$S = \frac{1}{2}\int d^4x\, d\tau\; a^2\left[\sigma'^2 - (\partial\sigma)^2 + m^2 a^2 \sigma^2\right]$$

So doing the field redefinition

$$\chi_\sigma = a\,\sigma$$

we find

$$S = \frac{1}{2}\int d^4x\, d\tau\; a^2\left[\chi_\sigma'^2 - (\partial\chi_\sigma)^2 - \left(m^2a^2 - \frac{a''}{a}\right)\chi^2\right]$$

$$\Rightarrow \frac{z''}{z} \rightarrow \frac{a''}{a} - \frac{m^2}{H^2} = \frac{1}{\tau^2}\left((2+3\epsilon) - 3\eta_{\cancel{\sigma}}\right)$$

with $\eta_\sigma \equiv \dfrac{m^2}{3H^2}$

$$\left[\frac{a''}{a} = \frac{1}{a}\,a(a\dot a)^{\cdot} = \dot a^2 + a\ddot a = \cancel{a^2H^2} + a^2\left(\frac{\dot a}{a}\right)^{\cdot} + a^2H^2 = (2-\epsilon)a^2H^2\right]$$

$$a^2H^2 = \frac{1}{\tau^2(1-\epsilon)^2} = \frac{1}{\tau^2}(1+2\epsilon)$$

$$\Rightarrow \frac{a''}{a} = \frac{1}{\tau^2}(2+3\epsilon)$$

$$\Rightarrow \quad \nu_\sigma^2 - \frac{1}{4} = 2 + 3\epsilon - 3\eta_\sigma$$

$$\Rightarrow \quad \nu_\sigma^2 = \frac{9}{4} + 3\epsilon - 3\eta_\sigma = \frac{9}{4}\left(1 + \frac{4}{3}\epsilon - \frac{4}{3}\eta_\sigma\right)$$

$$\Rightarrow \quad \nu_\sigma = \frac{3}{2}\left(1 + \frac{2}{3}\epsilon - \frac{2}{3}\eta_\sigma\right) + \ldots$$

$$= \frac{2}{3} + \epsilon - \eta_\sigma$$

So for the simplest curvaton we find

$$n_s - 1 = \frac{d \ln \mathcal{P}_{\zeta}(k)}{d \ln(k)} = 3 - 2\nu_\sigma$$

$$\Rightarrow \quad \boxed{n_s - 1 = -2\epsilon + 2\eta_\sigma \geq -2\epsilon}$$

During φ^2 inflation $\epsilon = \frac{1}{2N}$ and since we need 60 number of e-folds

$$n_s \gtrsim 0.99$$

In a mixed model n_s could have a contribution from the inflaton also

$$R = \frac{P_{\zeta_\sigma}}{P_{\zeta_\varphi}}$$

$\Downarrow$

$$n_s = 1 - \frac{1}{1+R}\frac{8}{4N+2} + \frac{R}{1+R}\left[-2\epsilon + 2\eta_\sigma\right]$$

$$r = \frac{16\epsilon}{1+R}$$

Also pre-big bang and Ekpyrotic scenarios require a curvaton as the contracting phases need not lead to a flat spectrum. In fact for this reason Andrei Linde called New Ekpyrotic scenario for Pre-big bang with a curvaton heart.

Observations

Most important contraints on n_s-r comes from Planck. Assuming Λ-CDM they found clearly that Starobinsky inflation is favoured

Planck Collaboration: Constraints on Inflation

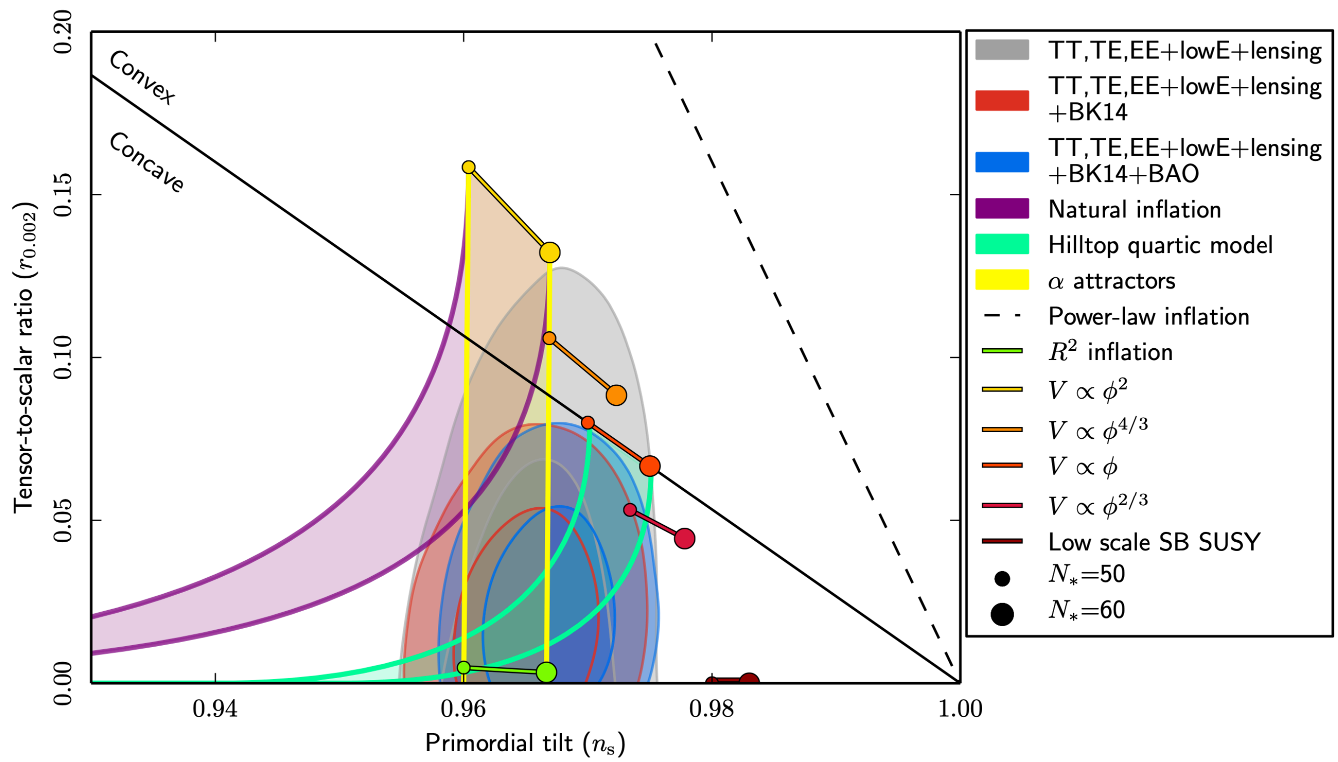


Fig. 8. Marginalized joint 68 % and 95 % CL regions for n_s and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from *Planck* alone and in combination with BK14 or BK14 plus BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions assume $dn_s/d\ln k = 0$.

https://arxiv.org/abs/1807.06211

Most people thought this result is very robust since perturbations conserved on horizon scales, so insensitive to early-universe physics before creation of CMB and late time evolution of universe also very constrained.

However, the Hubble tension is sending another message. The Hubble tension a 5-σ disagreement in the measuement of the Hubble constant today when assuming ΛCMB and using CMB and when measuring it directly using supernovae.

New Early Dark Energy (NEDE) is at the time of writing the best theory for adressing the tension by adding new physics to ΛCDM. It involves a 1st order phase transition in Dark Energy just before recombination.

NEDE however implies that $n_s \simeq 0.99$, which rules out Starobinsky inflation, but favours the simplest curvaton model. see fig.

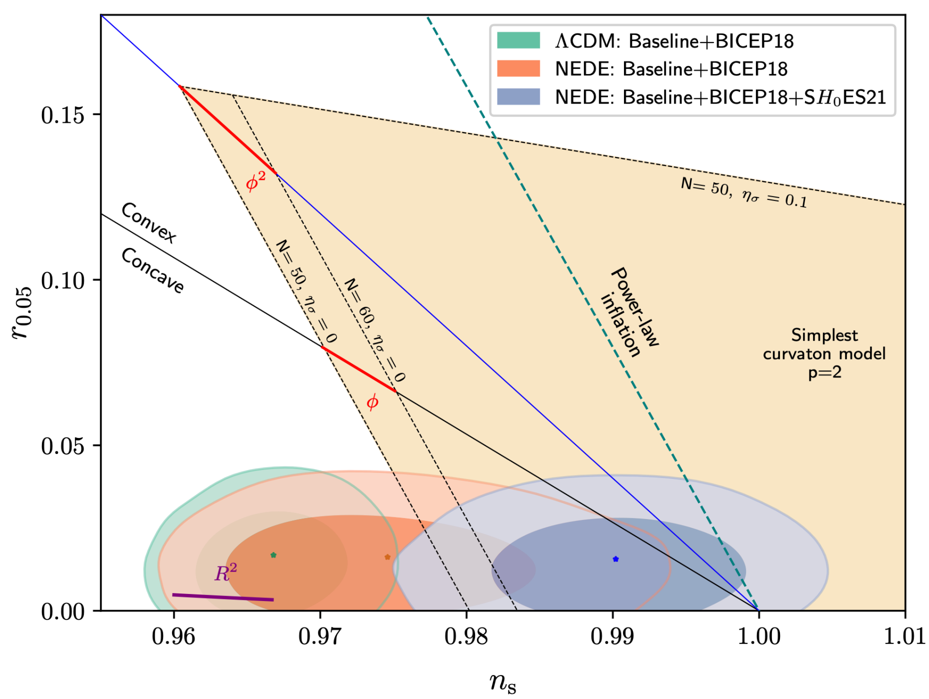


Figure 6. Results for the 68% and 95% C.L. contours relating n_s and r at a pivot scale of $k_* = 0.05$, for the ΛCDM and NEDE models alternating the baseline datasets with SH$_0$ES while including BICEP18. The small asterisks represent the mean posterior value of the corresponding contours.

https://arxiv.org/abs/2209.02708