# Physical properties of the $CP^{N-1}$ model in the large $N$ expansion

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Please keep in mind that these are very rough notes

# Contents



# 1 General properties of Quantum Chromodynamics (QCD)

QCD is described by the following Lagrangian:

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \theta q(x) \tag{1.1}$$

with

$$D_\mu = \partial_\mu - igA_\mu \ ; \ (A_\mu)^\alpha{}_\beta = A^a_\mu (\tau^a)^\alpha{}_\beta \ ; \ \psi^{A;\alpha} \ ; \ q(x) = \frac{g^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \ ; \ \tilde{F}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma} \tag{1.2}$$

where $\tau^a$ are the generators of the gauge group $SU(3)$. The quark field $\psi$ has two indices: $\alpha$ is a colour index of the fundamental representation of $SU(3)$ and $A$ is a flavour index that runs from 1 to $N_f$. $q(x)$ is the topological charge density.

$L$ has the following properties:

- It is conformal invariant at the classical level. Both $g$ and $\theta$ are dimensionless quantities.

- Dimensional transmutation: at the quantum level $g$ is replaced by an energy scale $\Lambda$ and the coupling constant is now running

$$\alpha_s(Q^2) \equiv \frac{g^2}{4\pi} = \frac{4\pi}{\beta_0 \log \frac{Q^2}{\Lambda^2}} \quad ; \quad \beta(x) = -\left(\frac{x}{4\pi}\right)^2 \beta_0 + \dots \quad ; \quad \Lambda^2 = \mu^2 e^{-\frac{4\pi}{\beta_0 \alpha_s(\mu^2)}} \tag{1.3}$$

where $\beta_0 = 11 - -\frac{2}{3}N_f$ and $\Lambda^2$ is independent on the choice of $\mu^2$.

- Asymptotic freedom at short distances ($Q^2 \to \infty$) and perturbation theory can be used.

- The running coupling constant becomes large at large distances and one needs non-perturbative methods to study the theory.

- Non-perturbative methods are the lattice gauge theory and the large $N$ (number of colours) expansion.

- QCD is a matrix theory, the large $N$ expansion corresponds to sum all planar diagrams and such a sum cannot be performed explicitly.

- It turns out that the quarks and gluons are confined and the only states that one can produce are colour singlet states as the mesons and the baryons

- For massless quarks $L$ is invariant at the classical level under the transformations $U(N_f)_L \times U(N_f)_R$. They act as follows

$$\psi_L \to B\psi_L \quad ; \quad \psi_R \to A\psi_R \tag{1.4}$$

where $\psi_L = \frac{1-\gamma_5}{2}\psi$ and $\psi_R = \frac{1+\gamma_5}{2}\psi$. $A$ and $B$ are matrices of $U(N_f)$.

- The $U(1)_A$ is broken in the quantum theory by the anomaly:

$$\partial^\mu \left(\bar{\psi}\gamma_\mu\gamma_5\psi\right) = 2N_f q(x) \tag{1.5}$$

- Only $SU(N_f)_V \times SU(N_f)_A \times U(1)_B$ is the symmetry in the quantum theory.

- Instanton solutions in Euclidean space:

$$F^{\mu\nu} = \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{\rho\sigma} \quad ; \quad D_\mu z = i\epsilon_{\mu\nu}D_\nu z \tag{1.6}$$

In these lectures I will be discussing the large $N$ expansion in the $CP^{N-1}$ model that, unlike in QCD, can be explicitly performed obtaining very explicit results. The $CP^{N-1}$ model is a two-dimensional model that shares many of the properties of QCD and has been very useful to understand non-perturbative properties of QCD as for instance the solution of the $U(1)$ problem. If I have time in the last lecture, following what we have learned from the $CP^{N-1}$ model, I will introduce the low-energy effective Lagrangian in QCD for a large number of colours and I will discuss its physical implications both for QCD and for its extension including QCD axions.

# 2 The $CP^{N-1}$ model

The manifold $CP^{N-1}$ is a complex manifold described by $N$ complex coordinates $z_1, z_2 \dots z_N$ with the identification

$$z_1, z_2 \dots z_N \equiv \lambda(z_1, z_2 \dots z_N) \quad ; \quad \lambda \in C \tag{2.1}$$

It depends on $N-1$ complex coordinates that are invariant under the previous rescaling. They are the projective coordinates:

$$Z_1 = \frac{z_1}{z_N}, Z_2 = \frac{z_2}{z_N} \dots Z_{N-1} = \frac{z_{N-1}}{z_N} \tag{2.2}$$

It can be regarded as a quotient of the $2N-1$ sphere in $C^N$ under the action of $U(1)$: $CP^{N-1} = \frac{S^{2N-1}}{U(1)}$.

The Lagrangian of the $CP^{N-1}$ model is constructed in terms of $N$ complex variables $z^i (i = 1 \dots N)$ with the constraint $\sum_{i=1}^N |z^i|^2 = 1$ that eliminates the absolute value of the extra variable and imposing $U(1)$ gauge invariance that eliminates the extra phase:

$$L = \frac{1}{2f} \overline{D_\mu z} D_\mu z \quad ; \quad D_\mu = \partial_\mu + i A_\mu \quad ; \quad \sum_{i=1}^N |z_i|^2 = 1 \tag{2.3}$$

After a convenient rescaling of the variables the Lagrangian is given by:

$$L = \overline{D_\mu z} D_\mu z \quad ; \quad D_\mu = \partial_\mu + \frac{i}{\sqrt{N}} A_\mu \quad ; \quad \sum_{i=1}^N |z_i|^2 = \frac{N}{2f} \tag{2.4}$$

where $f$ is a dimensionless coupling constant that is kept fixed for large $N$. We work in euclidean space.

As in QCD the complete Lagrangian contains also the fermions and the $\theta$-angle:

$$L = \overline{D_\mu z} D_\mu z + \bar{\psi}(\gamma^\mu D_\mu - M_B)\psi - \frac{g}{2N_F}\left( (\bar{\psi}\lambda^i \psi)^2 + (\bar{\psi}\gamma_5 \lambda^i \psi)^2 \right) + iq\theta \tag{2.5}$$

where the fermion field has two indices $\psi^{\alpha;A}$ where $\alpha = 1 \dots N_f$ is a flavour index and $A = 1 \dots N_F$ is a colour index. $g$ is another dimensionless coupling constant and $q$ is the topological charge density:

$$q(x) = \frac{1}{2\pi\sqrt{N}} \epsilon_{\mu\nu} \partial_\mu A_\nu \tag{2.6}$$

$\lambda^i$ are the generators of $U(N_f)$ normalised as follows:

$$Tr(\lambda^i\lambda^j) = \delta^{ij} \tag{2.7}$$

and the Dirac $\gamma$ matrices are chosen to be

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \; ; \; \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; i\gamma_5 = i\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

Some properties of the previous Lagrangian are:

- No dimensional coupling constant: conformal invariance at the classical level
- Conformal invariance is broken at the quantum level: non-zero $\beta$ function, running coupling constant and asymptotic freedom as in QCD.
- $U(N_f) \times U(N_f)$ chiral invariance:

$$\psi_L = \frac{1 - i\gamma_5}{2}\psi \to B\psi_L \quad ; \quad \psi_R = \frac{1 + i\gamma_5}{2}\psi \to A\psi_R \tag{2.9}$$

  with $A$ and $B$ being matrices of flavour $U(N_f)$.

- $U(1)$ axial vector current has an anomaly:

$$\partial_\mu \left(\bar{\psi}\gamma_5\gamma^\mu\psi\right) = 2N_f q(x) \quad ; \quad A = B^\dagger = e^{i\alpha} \tag{2.10}$$

# 3 Integrating the fundamental degrees of freedom: the generating functional

We have to compute the generating functional:

$$Z(J, \bar{J}; \eta, \bar{\eta}) = \int DzD\bar{z}D\bar{\psi}D\psi DA_\mu \delta(|z|^2 - \frac{N}{2f}) \times \exp\left[ - S + \int d^2x \left( \bar{J}z + J\bar{z} + \bar{\eta}\psi + \bar{\psi}\eta \right)\right] \tag{3.1}$$

By using the following identities:

$$\delta(|z|^2 - \frac{N}{2f}) = \int D\alpha e^{i\frac{\alpha}{\sqrt{N}}(|z|^2 - \frac{N}{2f})}$$

$$e^{\int d^2x \frac{g}{2N_F}(\bar{\psi}\lambda^i\psi)^2} = \int d\phi^i e^{\int d^2x \left( \bar{\psi}\lambda^i\psi\frac{\phi^i}{\sqrt{N_F}} - \frac{1}{2g}(\phi^i)^2 \right)} \tag{3.2}$$

we can rewrite the Lagrangian as follows:

$$L = \overline{D_\mu z} D_\mu z + m^2|z|^2 + \bar{\psi}(\gamma^\mu D_\mu - M_B - \frac{\lambda^i}{\sqrt{N_F}}(\phi^i + \gamma_5 \phi_5^i))\psi + \frac{1}{2g}((\phi^i)^2 + (\phi_5^i)^2)$$
$$- i\frac{\alpha}{\sqrt{N}}(|z|^2 - \frac{N}{2f}) \tag{3.3}$$

The action is now quadratic in the fundamental fields. We can integrate them and obtain an effective Lagrangian for the composite fields. We get:

$$Z(J,\bar{J};\eta,\bar{\eta}) = \int DA_\mu D\alpha D\phi^i D\phi_5^i$$
$$\times \exp\Bigg\{ - S_{eff} + \int d^2x \int d^2y \Bigg[ \bar{J}(x)\Delta^{-1}(x;y)J(y)\Bigg) + \bar{\eta}(x)\Delta_F^{-1}(x;y)\eta(y)\Bigg]\Bigg\} \tag{3.4}$$

where

$$\Delta_B = -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{N}}\alpha \ \ ; \ \ \Delta_F = \gamma_\mu D_\mu - M_B - \frac{\lambda^i}{\sqrt{N_F}}(\phi^i + \gamma_5 \phi_5^i) \tag{3.5}$$

and

$$S_{eff} = NTr\log\Delta_B - N_F Tr\log\Delta_F + \int d^2x \left[ i\frac{\sqrt{N}}{2f}\alpha + \frac{1}{2g}\left(\phi^i\phi^i + \phi_5^i\phi_5^i\right)\right] \tag{3.6}$$

Notice that in $\Delta_B$ we have introduced a term with $m^2$ that is a constant since $|z|^2$ is a constant. This term will be important to regularise the infrared divergences occurring in the case of a massless propagator.

From Eq. (3.2) we get the relation::

$$\frac{\bar{\psi}\psi}{\sqrt{N_F N_f}} = \frac{\phi^0}{g} \Longrightarrow N_f N_F \frac{\langle\bar{\psi}\psi\rangle}{\sqrt{N_F N_f}} = \langle\frac{\phi^0}{g}\rangle = \sqrt{N_F N_f} M_S \Longrightarrow \langle\bar{\psi}\psi\rangle = \frac{M_S}{g} \tag{3.7}$$

where we have assumed that all the components of $\langle\bar{\psi}\psi\rangle$ have the same value.

# 4 Large $N$ and $N_F$ expansion

The leading bosonic terms for large $N$ are given by:

$$NTr\log\left( - \partial^2 + m^2 - \frac{2i}{\sqrt{N}}A_\mu\partial_\mu + \frac{1}{N}A^2 - \frac{i\alpha}{\sqrt{N}}\right) + \int d^2x \frac{i\sqrt{N}\alpha}{2f}$$
$$\sim NTr\log(-\partial^2 + m^2) - NTr\left((-\partial^2+m^2)^{-1}\frac{i\alpha}{\sqrt{N}} + \ldots\right) + \int d^2x \frac{i\sqrt{N}\alpha}{2f}$$
$$= NTr\log(-\partial^2 + m^2) + i\sqrt{N}\int d^2x\alpha\left(\frac{1}{2f} - \int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2}\right) + \ldots \tag{4.1}$$

where we have used

$$(-\partial^2 - m^2)^{-1}(x,y) = \int \frac{d^2p}{(2\pi)^2} e^{ip(x-y)} \frac{1}{p^2+m^2} \tag{4.2}$$

Using Pauli-Villars regularisation:

$$\int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+m^2} \Longrightarrow \int \frac{d^2p}{(2\pi)^2} \left( \frac{1}{p^2+m^2} - \frac{1}{p^2+\Lambda^2} \right) = \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2} \tag{4.3}$$

the saddle point equation implies:

$$\frac{1}{2f(\Lambda)} = \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2} \tag{4.4}$$

One can introduce a renormalised coupling constant $f_R(\mu^2)$ function of the renormalisation scale $\mu^2$ through the relation:

$$\frac{2\pi}{f(\Lambda)} = \frac{2\pi}{f_R(\mu)} + \log \frac{\Lambda^2}{\mu^2} \tag{4.5}$$

and one can use it to eliminate the cutoff $\Lambda$ getting:

$$\frac{2\pi}{f_R(\mu)} = \log \frac{\mu^2}{m^2} \Longrightarrow m^2 = \mu^2 e^{-\frac{2\pi}{f_R(\mu)}} \tag{4.6}$$

It turns out that the renormalised coupling constant is running and the quantity $m^2$ is the dimensional renormalisation invariant scale that replaces the dimensionless coupling constant $f$. This can be seen by changing the scale $\mu$ to $\mu'$. The coupling constants at the two different scales are related through the following relation:

$$\frac{2\pi}{f(\Lambda)} = \frac{2\pi}{f_R(\mu)} + \log \frac{\Lambda^2}{\mu^2} = \frac{2\pi}{f_R(\mu')} + \log \frac{\Lambda^2}{(\mu')^2} \Longrightarrow (\mu')^2 e^{-\frac{2\pi}{f_R(\mu')}} = \mu^2 e^{-\frac{2\pi}{f_R(\mu)}} \tag{4.7}$$

that implies that $m^2$ is independent on the value of $\mu$ chosen. From Eq. (4.5) we can compute the $\beta$-function:

$$\beta(f_R) = \mu^2 \frac{\partial f_R(\mu)}{\partial \mu^2} = -\frac{f_R^2(\mu)}{2\pi} \tag{4.8}$$

that is negative. This theory is asymptotically free as QCD.

In order to get the cancellation of the term of order $\sqrt{N_F}$ in the fermionic part of the effective Lagrangian we need to give a vacuum expectation value to the flavour singlet field $\phi^0$ given by $\langle\phi^0\rangle = M_S\sqrt{N_F N_f}$ and then the leading term is given by

$$
\begin{aligned}
&- N_F Tr\log(\gamma_\mu\partial_\mu - M) - N_F Tr\log\left(1 - (\gamma_\mu\partial_\mu - M)^{-1}\left(\frac{\lambda^i}{\sqrt{N_F}}(\phi^i + \gamma_5\phi_5^i) + \frac{\gamma^\mu A_\mu}{\sqrt{N}}\right)\right) \\
&+ \frac{1}{2g}\int d^2x\left(2\phi^0 M_S\sqrt{N_F N_f} + (\phi^i)^2 + (\phi_5^i)^2\right) = -N_F Tr\log(\gamma_\mu\partial_\mu - M) \\
&+ N_F Tr\left((\gamma_\mu\partial_\mu - M)^{-1}\left(\frac{\lambda^i}{\sqrt{N_F}}(\phi^i + \gamma_5\phi_5^i) + \frac{\gamma^\mu A_\mu}{\sqrt{N}}\right)\right) \\
&+ \frac{1}{2g}\int d^2x\left(2\phi^0 M_S\sqrt{N_F N_f} + (\phi^i)^2 + (\phi_5^i)^2\right)
\end{aligned}
\tag{4.9}
$$

where $M = M_B + M_S$ and the trace involves also a trace over the flavour. The trace over the flavour and over the Dirac matrices eliminates the terms with $\phi^i$ with $i \neq 0$ and the terms with $\phi_5^i$ and $A_\mu$. One is left with

$$
\begin{aligned}
&- N_F Tr\log(\gamma_\mu\partial_\mu - M) + \sqrt{N_F N_f}\,Tr\left((\gamma_\mu\partial_\mu - M)^{-1}\phi^0\right) \\
&+ \frac{1}{2g}\int d^2x\left(2\phi^0 M_S\sqrt{N_F N_f} + (\phi^i)^2 + (\phi_5^i)^2\right)
\end{aligned}
\tag{4.10}
$$

The saddle point equation is then given by

$$
2\sqrt{N_F N_f}\int d^2x\phi^0\left(\frac{M_S}{2g} - M\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2 + M^2}\right) = 0 \tag{4.11}
$$

where we have used the following fermion propagator

$$
(\gamma_\mu\partial_\mu - M)^{-1}(x - y) = \int\frac{d^2p}{(2\pi)^2}\frac{-i\gamma_\mu p_\mu - M}{p^2 + M^2}e^{ip(x-y)} \tag{4.12}
$$

Using again Pauli-Villars regularisation we get

$$
\frac{M_S}{2g(\Lambda)} = \frac{M}{4\pi}\log\frac{\Lambda^2}{M^2} \quad ; \quad M = M_S + M_B \tag{4.13}
$$

The previous equation can be satisfied if we impose that

$$
M_B \sim \epsilon\frac{2\pi M_S}{\log\frac{\Lambda^2}{M^2}} \tag{4.14}
$$

Then the previous equation becomes:

$$
\frac{2\pi}{g(\Lambda)} - \log\frac{\Lambda^2}{M^2} - 2\pi\epsilon = 0 \tag{4.15}
$$

Also in this case we can introduce a renormalised coupling constant through the relation:

$$\frac{2\pi}{g(\Lambda)} = \log\frac{\Lambda^2}{\mu^2} + \frac{2\pi}{g(\mu)} \Longrightarrow M^2 = \mu^2 e^{-\frac{2\pi}{f_R(\mu)}+2\pi\epsilon} \tag{4.16}$$

Having eliminated the terms proportional to $\sqrt{N}$ and $\sqrt{N_F}$ by means of the two saddle point equations, we can in the next section compute the terms of order 1 as $N, N_f \to \infty$.

The vacuum expectation value given to the scalar singlet field $\phi^0$ breaks spontaneously chiral invariance and one is left with only the vector symmetry $U(N_f)$ broken explicitly by a small mass term. This appears in the spectrum as we are going to show in the next section.

# 5 Term of order 1 for large $N$ and $N_F$

Here we compute the terms of order 1 for large $N$ and $N_F$ that give the kinetic terms of the composite fields. We get:

$$S^{(2)}_{eff} = \frac{1}{2}\int d^2x \int d^2y \Big\{ \alpha(x)\Gamma^\alpha(x-y)\alpha(y) + A_\mu(x)\Gamma^A_{\mu\nu}(x-y)A_\nu(y) + \phi^i(x)\Gamma^\phi(x-y)\phi^i(y) + \phi^i_5(x)\Gamma^{\phi_5}(x-y)\phi^i_5(y) + 2A_\mu\Gamma^{A\phi}(x-y)\phi^0_5 \Big\} \tag{5.1}$$

where

$$\tilde{\Gamma}^\alpha = A(p;m^2) = \frac{1}{2\pi\sqrt{p^2(p^2+m^2)}}\log\frac{\sqrt{p^2}+\sqrt{p^2+m^2}}{\sqrt{p^2+m^2}-\sqrt{p^2}} \to \frac{1}{4\pi m^2}\left(1 - \frac{2p^2}{12m^2} + \dots\right) \tag{5.2}$$

$$\tilde{\Gamma}^A_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left[(p^2+4m^2)A(p;m^2) - \frac{1}{\pi} - \frac{N_F N_f}{N}\left(4M^2A(p;M^2) - \frac{1}{\pi}\right)\right] \tag{5.3}$$

$$\tilde{\Gamma}^\phi_{ij} = \delta_{ij}\left(\epsilon + (p^2+4M^2)A(p;M^2)\right) \tag{5.4}$$

$$\tilde{\Gamma}^{\phi_5}_{ij} = \delta_{ij}\left(\epsilon + p^2 A(p;M^2)\right) \tag{5.5}$$

$$\tilde{\Gamma}^{A\phi}_\mu = -\epsilon_{\mu\nu}p_\nu 2\sqrt{N_f}M\sqrt{\frac{N_F}{N}}A(p;M^2) \tag{5.6}$$

At the end of this section we give some detail on how to reach some of the previous results. The terms of order 1 in the bosonic theory are given by:

$$Tr((-\partial^2+m^2)^{-1}A^2)-\frac{1}{2}Tr\Bigg((-\partial^2+m^2)^{-1}\left(2iA_\mu\partial_\mu+i\partial_\mu A_\mu+i\alpha\right)(-\partial^2+m^2)^{-1}$$
$$\times\left(2iA_\mu\partial_\mu+i\partial_\mu A_\mu+i\alpha\right)\Bigg) \tag{5.7}$$

One gets two quadratic terms. The first one contains the field $\alpha$ and is given by:

$$\frac{1}{2}\int d^2x\int d^2y\Bigg(\int\frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2+m^2}\alpha(y)\int\frac{d^2q}{(2\pi)^2}\frac{e^{iq(y-x)}}{q^2+m^2}\alpha(x)\Bigg)$$
$$=\frac{1}{2}\int d^2x\alpha(x)\int d^2y\alpha(y)\int\frac{d^2r}{(2\pi)^2}e^{ir(x-y)}\int\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m^2)((q+r)^2+m^2)}$$
$$=\frac{1}{2}\int d^2x\alpha(x)\int d^2y\alpha(y)\int\frac{d^2r}{(2\pi)^2}e^{ir(x-y)}\tilde{\Gamma}^\alpha(r) \tag{5.8}$$

where

$$\Gamma^\alpha(x-y)=\int\frac{d^2p}{(2\pi)^2}e^{ip(x-y)}\tilde{\Gamma}^\alpha(r)\quad;\quad\tilde{\Gamma}^\alpha(r)\equiv A(r) \tag{5.9}$$

where $A(r)$ is given in (5.6).

The second term is

$$Tr((-\partial^2+m^2)^{-1}A^2)+\frac{1}{2}Tr\Bigg((-\partial^2+m^2)^{-1}(2A_\mu\partial_\mu+\partial_\mu A_\mu)(-\partial^2+m^2)^{-1}$$
$$(2A_\mu\partial_\mu+\partial_\mu A_\mu)\Bigg)=Tr((-\partial^2+m^2)^{-1}A^2)+\frac{1}{2}\int d^2x\int d^2y\int\frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2+m^2}$$
$$\times(2A_\mu\partial^y_\mu+\partial^y_\mu A_\mu)(y))\int\frac{d^2q}{(2\pi)^2}\frac{e^{iq(y-x)}}{q^2+m^2}(2A_\nu\partial^x_\nu+\partial^x_\nu A_\nu)(x))\Bigg)$$
$$=Tr((-\partial^2+m^2)^{-1}A^2)+\frac{1}{2}\int d^2x\int d^2y\frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2+m^2}\left(2iq_\mu A_\mu(y)+\partial^y_\mu A_\mu(y)\right)$$
$$\times\frac{d^2q}{(2\pi)^2}\frac{e^{iq(y-x)}}{q^2+m^2}(2A_\nu(x)ip_\nu+\partial^x_\nu A_\nu(x))\Bigg)=Tr((-\partial^2+m^2)^{-1}A^2)$$
$$-\frac{1}{2}\int d^2x\int d^2y\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{e^{i(p-q)(x-y)}}{(p^2+m^2)(q^2+m^2)}(2q_\mu+(p-q)_\mu)A_\mu(y)$$
$$\times(2p_\nu-(p-q)_\nu)A_\nu(x) \tag{5.10}$$

that is equal to

$$\int d^2xA^2(x)\int\frac{d^2q}{(2\pi)^2}\frac{e^{iq(x-y)}}{q^2+m^2}$$
$$-\frac{1}{2}\int d^2x\int d^2yA_\mu(y)A_\nu(x)\int\frac{d^2r}{(2\pi)^2}e^{ir(x-y)}\int\frac{d^2q}{(2\pi)^2}\frac{(2q_\mu+r_\mu)(2q_\nu+r_\nu)}{((q+r)^2+m^2)(q^2+m^2)} \tag{5.11}$$

where $r=p-q$. Let us use the following identity:

$$\frac{4q_\mu q_\nu}{(q^2+m^2)((r+q)^2+m^2)}=\frac{4(q_\mu q_\nu-\frac{1}{2}q^2\delta_{\mu\nu})}{(q^2+m^2)((r+q)^2+m^2)}+\frac{2\delta_{\mu\nu}}{(q+r)^2+m^2}$$
$$-\frac{2m^2\delta_{\mu\nu}}{(q^2+m^2)((r+q)^2+m^2)} \tag{5.12}$$

The second term in the right-hand-side cancels the first divergent term and one is left to compute the following quantity:

$$\frac{1}{2}\int d^2x\int d^2yA_\mu(y)A_\nu(x)\int\frac{d^2r}{(2\pi)^2}e^{ir(x-y)}G_{\mu\nu}(r) \tag{5.13}$$

where

$$G_{\mu\nu}(r)=-\int\frac{d^2q}{(2\pi)^2}\frac{4(q_\mu q_\nu-\frac{1}{2}q^2\delta_{\mu\nu})+2(q_\mu r_\nu+q_\nu r_\mu)+r_\mu r_\nu-2m^2\delta_{\mu\nu}}{((q+r)^2+m^2)(q^2+m^2)} \tag{5.14}$$

This quantity is computed in App. A with the following result:

$$G_{\mu\nu}(r)=(\delta_{\mu\nu}-\frac{r_\mu r_\nu}{r^2})\left((r^2+4m^2)A(r,m^2)-\frac{1}{\pi}\right) \tag{5.15}$$

# 6 The low-energy of $S^{(2)}_{eff}$

The low-energy limit of the kinetic term of the composite fields is given by

$$L^{(2)}_{eff}=\frac{1}{2}\left((\partial_\mu\Pi^i)^2+m_\pi^2(\Pi^i)^2\right)+\frac{1}{2}\left((\partial_\mu\sigma^i)^2+(m_\pi^2+4M^2)(\sigma^i)^2\right)$$
$$+\frac{\alpha^2}{8\pi m^2}+\frac{F^2}{24\pi m^2}+i\sqrt{\frac{2N_f}{N}}\frac{1}{2\pi F_\pi}FS \tag{6.1}$$

where

$$F=\epsilon_{\mu\nu}\partial_\mu A_\nu\ \ ;\ \ \Pi^i=\frac{\phi_5^i}{\sqrt{4\pi}M}\ \ ;\ \ \sigma^i=\frac{\phi^i}{\sqrt{4\pi}M}\ \ ;\ \ \Pi^0=S$$
$$F_\pi=\frac{1}{\sqrt{2\pi}}\ \ ;\ \ m_\pi^2=4\pi\epsilon M^2 \tag{6.2}$$

We can rewrite it in terms of the topological charge in (2.6):

$$L_{eff}^{(2)} = \frac{1}{2}\sum_{i\neq 0}\left((\partial_\mu \Pi^i)^2 + m_\pi^2(\Pi^i)^2\right) + \frac{1}{2}\left((\partial_\mu \sigma^i)^2 + (m_\pi^2 + 4M^2)(\sigma^i)^2\right)$$

$$+ \frac{\alpha^2}{8\pi m^2} + \frac{1}{2}\left((\partial_\mu S)^2 + m_\pi^2 S^2\right) + \frac{1}{2\chi}q^2 + i\frac{\sqrt{2N_f}}{F_\pi}qS \tag{6.3}$$

where

$$\chi = \frac{3m^2}{\pi N} \quad ; \quad F_\pi = \frac{1}{\sqrt{2\pi}} \tag{6.4}$$

The last three terms of (6.3) can be written as follows:

$$\frac{1}{2}\left((\partial_\mu S)^2 + M_S^2 S^2\right) + \frac{1}{2\chi}\left(q + i\frac{\chi\sqrt{2N_f}}{F_\pi}S\right)^2 \tag{6.5}$$

with

$$M_S^2 = m_\pi^2 + \frac{2\chi N_f}{F_\pi^2} \tag{6.6}$$

We see that the singlet pseudoscalar meson gets an extra contribution to its mass with respect to the other pseudoscalar mesons whose mass is instead entirely given by $m_\pi$. This follows from the fact that there is a non-vanishing two-point amplitude involving $S$ and the topological charge density that is a consequence of the anomaly equation (2.10). In particular, at large $N$ both the extra term in (6.6) and the axial anomaly disappear.

From the previous quadratic Lagrangian one can compute the two-point amplitudes. One gets:

$$\langle S(x)S(y)\rangle = \int \frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2 + M_S^2} \tag{6.7}$$

Then from the vanishing of the following propagator:

$$\langle (q + i\frac{\chi\sqrt{2N_f}}{F_\pi}S)(x)S(y)\rangle = 0 \tag{6.8}$$

and from (6.7) one gets:

$$\langle q(x)S(y)\rangle = -i\frac{\chi\sqrt{2N_f}}{F_\pi}\int \frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2 + M_S^2} \tag{6.9}$$

Finally, from the following two-point amplitude:

$$\langle (q + i\frac{\chi\sqrt{2N_f}}{F_\pi}S)(x)(q + i\frac{\chi\sqrt{2N_f}}{F_\pi}S)(y)\rangle = \chi\int \frac{d^2p}{(2\pi)^2}e^{ip(x-y)} \tag{6.10}$$

and from (6.7) and (6.9) one gets:

$$\langle q(x)q(y)\rangle = \chi \int \frac{d^2p}{(2\pi)^2} e^{ip(x-y)} \frac{p^2+m_\pi^2}{p^2+M_S^2} \tag{6.11}$$

that implies the following topological susceptibility:

$$\langle q(x) \int d^2y q(y)\rangle = \chi \frac{m_\pi^2}{M_S^2} \tag{6.12}$$

that vanishes if $m_\pi = 0$, while the same quantity in the theory without quarks is given by:

$$\langle q(x) \int d^2y q(y)\rangle_{No\ quarks} = \chi = \frac{3m^2}{\pi N} \tag{6.13}$$

It goes to zero at large $N$.

From Eq. (3.2) we get the relation::

$$\frac{\bar\psi\psi}{\sqrt{N_F N_f}} = \frac{\phi^0}{g} \Longrightarrow N_f N_F \frac{\langle\bar\psi\psi\rangle}{\sqrt{N_F N_f}} = \langle\frac{\phi^0}{g}\rangle = \sqrt{N_F N_f}\frac{M_S}{g} \Longrightarrow \langle\bar\psi\psi\rangle = \frac{M_S}{g} \tag{6.14}$$

where we have assumed that all the components of $\langle\bar\psi\psi\rangle$ have the same value. The previous relation implies:

$$2M_B\langle\bar\psi\psi\rangle = \frac{2M_B M_S}{g} = \frac{2\epsilon 2\pi M_S}{\log\frac{\Lambda^2}{m^2}}\frac{M_S}{2\pi}\log\frac{\Lambda^2}{m^2} = 2\epsilon M_S^2 \sim 2\epsilon M^2 \tag{6.15}$$

On the other hand we can compute the quantity:

$$F_\pi^2 m_\pi^2 = 2\epsilon M^2 \tag{6.16}$$

The last two equations imply:

$$2M_B\langle\bar\psi\psi\rangle = F_\pi^2 m_\pi^2 \tag{6.17}$$

The left hand side contains quantities of the fundamental Lagrangian, while the right hand side those from the effective Lagrangian.

# 7 $\theta$-vacua and Witten-Veneziano relation

Let us consider the effective Lagrangian in a theory without fermions and let us add a non-zero $\theta$ angle and an external source:

$$L_{eff} = \frac{1}{2\chi}q^2 + i\theta q + Jq \tag{7.1}$$

The equation of motion of $q$ is given by

$$q = -\chi(J + i\theta) \tag{7.2}$$

Inserting it back into the Lagrangian we get:

$$L_{eff} = -\frac{\chi}{2}(J + i\theta)^2 \tag{7.3}$$

The generating functional is given by:

$$Z(\theta, J) = e^{\int d^2x \frac{\chi}{2}(J+i\theta)^2} \equiv e^{-W(\theta,J)} \tag{7.4}$$

The vacuum energy is given by

$$E(\theta) = \frac{W(\theta, J=0)}{V_2} = \frac{\chi\theta^2}{2} = N\frac{3m^2}{2\pi}\left(\frac{\theta}{N}\right)^2 \tag{7.5}$$

The one-point amplitude is given by:

$$\langle q(x)\rangle_\theta = \frac{1}{Z(\theta, J)}\frac{\delta Z}{\delta J(x)}|_{J=0} = i\chi\theta = i\frac{3m^2\theta}{\pi N} \tag{7.6}$$

The two-point amplitude is given by:

$$\langle q(x)q(y)\rangle = \frac{1}{Z(\theta, J)}\frac{\delta^2 Z(\theta, J)}{\delta J(x)\delta J(y)} = \chi\delta^2(x-y) \tag{7.7}$$

in agreement with (6.13).

From the vacuum energy we can compute the WV relation. From (7.5) we get

$$\frac{d^2E(\theta)}{\partial^2\theta}|_{\theta=0} = \chi = \frac{3m^2}{\pi N} = \frac{F_\pi^2 M_S^2}{2N_f} \tag{7.8}$$

where we have used (6.6). It implies the WV relation:

$$M_S^2 = \frac{2N_f}{F_\pi^2}\chi = \frac{2N_f}{F_\pi^2}\frac{d^2E(\theta)}{\partial^2\theta}|_{\theta=0} \tag{7.9}$$

Notice that, if we had considered the theory with fermions with $m_\pi = 0$, we would have obtained zero.

The fact that the singlet pseudoscalar meson gets an extra contribution with respect to the pseudoscalars of $SU(N_f)$ is a direct consequence of the axial anomaly. In fact, both the axial anomaly and the extra term go to zero when $N \to \infty$.

# 8 Confinement

In the classical theory there is no kinetic term for the gauge field. This term is generated in the quantum theory. From (6.13) we can extract the following two-point amplitude:

$$\langle \frac{A_\mu(x)}{\sqrt{N}} \frac{A_\nu(y)}{\sqrt{N}} \rangle = \int \frac{d^2p}{(2\pi)^2} e^{ip(x-y)} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{12\pi m^2}{Np^2} \tag{8.1}$$

after having used the relation: $\epsilon_{\rho\mu}\epsilon_{\sigma\nu} = \delta_{\rho\sigma}\delta_{\mu\nu} - \delta_{\rho\nu}\delta_{\sigma\mu}$. Taking into account that, in the non-relativistic limit, the Coulomb potential is given by:

$$V(R) = -\frac{12\pi m^2}{N} \int_{-\infty}^{+\infty} \frac{dp}{2\pi p^2} e^{ipR} = \frac{12\pi m^2}{N} R \tag{8.2}$$

A linear potential means confinement for the states that are charged with respect to the gauge field. Unlike $QCD_2$, where there is already a linear potential at the classical level, in this case confinement is a purely quantum effect.

# 9 Term responsible of the anomaly

In this section we extract the term responsible of reproducing the anomaly in the effective Lagrangian. Let us consider for simplicity the massless case. Let us introduce the following operators:

$$\Delta = \gamma^\mu D^\mu - \gamma_- B - \gamma_+ B^\dagger \quad ; \quad \Delta^\dagger = -\gamma^\mu D^\mu - \gamma_- B^\dagger - \gamma_+ B \tag{9.1}$$

where

$$B = \frac{1}{\sqrt{N_F}}(\phi^i + \gamma_5 \phi_5^i)\lambda^i \quad ; \quad B^\dagger = \frac{1}{\sqrt{N_F}}(\phi^i - \gamma_5 \phi_5^i)\lambda^i \tag{9.2}$$

and

$$\begin{aligned} \Delta\Delta^\dagger &= \gamma_-(-D^2 + BB^\dagger + F) + \gamma_+(-D^2 + B^\dagger B - F) \\ \Delta^\dagger\Delta &= \gamma_-(-D^2 + B^\dagger B + F) + \gamma_+(-D^2 + BB^\dagger - F) \end{aligned} \tag{9.3}$$

Let us compute

$$\delta Tr \log \Delta = Tr(\delta\Delta\Delta^{-1}) = \frac{1}{2} Tr[\delta\Delta\Delta^\dagger(\Delta\Delta^\dagger)^{-1} + \Delta^\dagger\delta\Delta(\Delta^\dagger\Delta)^{-1}] \tag{9.4}$$

We get

$$\frac{1}{2}Tr\Bigg((\gamma^\mu \delta D^\mu - \gamma_-\delta B - \gamma_+\delta B^\dagger)(-\gamma^\nu D^\nu - B^\dagger\gamma_- - \gamma_+ B)(\Delta\Delta^\dagger)^{-1}$$
$$+ (-\gamma^\nu D^\nu - B^\dagger\gamma_- - \gamma_+ B)(\gamma^\nu \delta D^\nu - \gamma_-\delta B - \gamma_+\delta B^\dagger)(\Delta^\dagger\Delta)^{-1}\Bigg)$$
$$= \frac{1}{2}Tr\Bigg((-\gamma^\mu\gamma^\nu \delta D_\mu D_\nu + \gamma_-\delta B B^\dagger + \gamma_+\delta B^\dagger B)(\Delta\Delta^\dagger)^{-1}$$
$$+ (-\gamma^\mu\gamma^\nu D_\mu \delta D_\nu + \gamma_- B^\dagger \delta B + \gamma_+ B\delta B^\dagger)(\Delta^\dagger\Delta)^{-1}\Bigg) \quad (9.5)$$

that is equal to

$$\frac{1}{2}Tr\Bigg( - \gamma^\mu\gamma_\nu \delta D_\mu D_\nu[\gamma_-(-D^2 + BB^\dagger + F)^{-1} + \gamma_+(-D^2 + B^\dagger B - F)^{-1}]$$
$$- \gamma^\mu\gamma^\nu D_\mu \delta D_\nu[\gamma_-(-D^2 + B^\dagger B + F)^{-1} + \gamma_+(-D^2 + BB^\dagger - F)^{-1}]$$
$$+ \gamma_-\delta B B^\dagger[(-D^2 + BB^\dagger + F)^{-1} + \gamma_+\delta B^\dagger B(-D^2 + B^\dagger B - F)^{-1}$$
$$+ \gamma_- B^\dagger \delta B(-D^2 + B^\dagger B + F)^{-1} + \gamma_+ B\delta B^\dagger(-D^2 + BB^\dagger - F)^{-1}\Bigg) \quad (9.6)$$

Let us compute the terms in the first two lines. We get:

$$\frac{1}{2}Tr\Bigg(\gamma^\mu\gamma^\nu(-\delta(D_\mu D_\nu) + D_\mu \delta D_\nu)[\gamma_-(-D^2 + BB^\dagger + F)^{-1} + \gamma_+(-D^2 + B^\dagger B - F)^{-1}]$$
$$- \gamma^\mu\gamma^\nu D_\mu \delta D_\nu[\gamma_-(-D^2 + B^\dagger B + F)^{-1} + \gamma_+(-D^2 + BB^\dagger - F)^{-1}] \quad (9.7)$$

The first term is equal to

$$\frac{1}{2}Tr\Bigg(\gamma^\mu\gamma^\nu(-\delta(D_\mu D_\nu)[\gamma_-(-D^2 + BB^\dagger + F)^{-1} + \gamma_+(-D^2 + B^\dagger B - F)^{-1}]\Bigg) \quad (9.8)$$

The remaining terms are:

$$\frac{1}{2}Tr\Bigg(\gamma^\mu\gamma^\nu D_\mu \delta D_\nu[\gamma_- \left((-D^2 + BB^\dagger + F)^{-1} - (-D^2 + B^\dagger B + F)^{-1}\right)$$
$$+ \gamma_+ \left((-D^2 + B^\dagger B - F)^{-1} - (-D^2 + BB^\dagger - F)^{-1}\right)] \quad (9.9)$$

They vanish as we show now rewriting those in the square bracket as follows:

$$\gamma_- \int_0^\infty ds Tr(e^{s(-D^2+F)})Tr\left(e^{-sBB^\dagger} - e^{-sB^\dagger B}\right) = 0 \quad (9.10)$$

In a similar way one can show that also the second term is vanishing.

We are left with:

$$\frac{1}{2}Tr\Bigg( \big(-\delta(D^2+K)\gamma_+ - \gamma_-(D^2-K)\big)\,[\gamma_-(-D^2+BB^\dagger+F)^{-1}+\gamma_+(-D^2+B^\dagger B-F)^{-1}]$$
$$+\gamma_-\delta BB^\dagger(-D^2+BB^\dagger+F)^{-1}+\gamma_+\delta B^\dagger B(-D^2+B^\dagger B-F)^{-1}$$
$$+\gamma_- B^\dagger\delta B(-D^2+B^\dagger B+F)^{-1}+\gamma_+(-D^2+BB^\dagger-F)^{-1}\Bigg) \tag{9.11}$$

that is equal to

$$\frac{1}{2}Tr\Bigg(\gamma_+\delta(-D^2-K+B^\dagger B)(-D^2+B^\dagger B-F)^{-1}+\gamma_-\delta(-D^2+K+BB^\dagger)(-D^2+BB^\dagger+F)^{-1}$$
$$+\gamma_+\Bigg(-\delta(B^\dagger B)(-D^2+B^\dagger B-F)^{-1}+\delta B^\dagger B(-D^2+B^\dagger B-F)^{-1}$$
$$+\delta BB^\dagger(-D^2+BB^\dagger-F)^{-1}\Bigg)+\gamma_-\Bigg(-\delta(BB^\dagger)(-D^2+BB^\dagger+F)^{-1}$$
$$+\delta BB^\dagger(-D^2+BB^\dagger+F)^{-1}+B^\dagger\delta B(-D^2+B^\dagger B+F)^{-1}\Bigg) \tag{9.12}$$

The two terms in the first line can be integrated getting:

$$\frac{1}{2}Tr\Bigg(\delta\log(-D^2-F+B^\dagger B)+\delta\log(-D^2+F+BB^\dagger)\Bigg) \tag{9.13}$$

After taking the trace over the $\gamma$ matrices the remaining terms are

$$\frac{1}{2}Tr\Bigg[B^\dagger\delta B\Bigg((-D^2+B^\dagger B+F)^{-1}-(-D^2+B^\dagger B-F)^{-1}\Bigg)$$
$$+B\delta B^\dagger\Bigg((-D^2+BB^\dagger-F)^{-1}-(-D^2+BB^\dagger+F)^{-1}\Bigg)\Bigg] \tag{9.14}$$

that can be written as follows:

$$\frac{1}{2}\int_0^\infty dsTr(e^{sD^2})\Bigg[(e^{-sF}-e^{sF})Tr(B^\dagger\delta Be^{-sB^\dagger})+(e^{sF}-e^{-sF})Tr(B\delta B^\dagger e^{-sBB^\dagger})\Bigg] \tag{9.15}$$

Using the identity:

$$Tr(e^{sD^2})=\frac{1}{4\pi s}\int d^2x\frac{Fs}{\sinh(sF)} \tag{9.16}$$

we get

$$\int d^2x \frac{F}{4\pi} \int_0^\infty ds Tr\left( B\delta B^\dagger e^{-sBB^\dagger} - B^\dagger \delta B e^{-sB^\dagger B} \right)$$
$$= \frac{1}{4\pi} \int d^2x F \left( \delta B^\dagger (B^\dagger)^{-1} - \delta B B^{-1} \right)$$
$$= \delta \left( \frac{1}{4\pi} F Tr(\log B^\dagger - \log B \right) \quad (9.17)$$

In conclusion we get:

$$Tr \log \Delta = \frac{1}{2} Tr \left( \log(-D^2 - F + B^\dagger B) + \log(-D^2 + F + BB^\dagger) \right)$$
$$- \frac{1}{4\pi} \int d^2x F \left( \log B - \log B^\dagger \right) \quad (9.18)$$

# 10 Computation of the $Trlog\Delta$ in the constant field approximation

In this section we compute the bosonic and fermionic effective Lagrangian in the approximation that the various fields are constant.

# 11 From the $CP^{N-1}$ model to QCD

The low-energy effective Lagrangian of QCD is the Lagrangian of the non-linear $\sigma$-model:

$$L = L_B = \frac{1}{2} \partial^\mu U \partial_\mu U^\dagger + \frac{F_\pi}{2\sqrt{2}} Tr(M(U + U^\dagger)) \;\; ; \;\; UU^\dagger = \frac{F_\pi^2}{2}$$
$$U = \frac{F_\pi}{\sqrt{2}} e^{i \frac{\sqrt{2}}{F_\pi} \Pi^i \lambda^i} \;\; ; \;\; Tr(\lambda^i \lambda^j) = \delta^{ij} \;\; ; \;\; M = \mu_i^2 \delta_{ij} \quad (11.1)$$

where the matrices $\lambda^i$ belong to the flavour $U(N_f)$. $\Pi^i$ are the fields of the pseudoscalar mesons. For $N_f = 3$ they are the nonet of pseudoscalar mesons that have the following masses:

$$m_\pi \sim 140 MeV \;\; ; \;\; m_K \sim 497 MeV \;\; ; \;\; m_\eta = 547 MeV \;\; ; \;\; m_{\eta'} = 957 MeV \quad (11.2)$$

The $U(1)$ problem consists in the fact that the splitting of the quark masses $m_u, m_d, m_s$ is not sufficient to explain the splitting of the pseudoscalar mesons.

The fundamental Lagrangian is that of QCD that we write as follows:

$$L_F = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + L_{int} \tag{11.3}$$

An important requirement for an effective Lagrangian is that it should have the same symmetries of the fundamental Lagrangian. This is the case for the Lagrangian in (11.1) as we are going to show.

Both have a vector and an axial vector current that satisfy the following equations:

$$\begin{aligned}
&\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) = 0 \quad ; \quad \partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right) = 2im\bar{\psi}\gamma_5\psi \\
&\partial_\mu\left(\bar{\psi}\gamma^\mu\frac{1+\gamma_5}{2}\psi\right) = im\bar{\psi}\frac{1+\gamma_5}{2}\psi - im\bar{\psi}\frac{1-\gamma_5}{2}\psi \\
&\partial_\mu\left(\bar{\psi}\gamma^\mu\frac{1-\gamma_5}{2}\psi\right) = im\bar{\psi}\frac{1-\gamma_5}{2}\psi - im\bar{\psi}\frac{1+\gamma_5}{2}\psi
\end{aligned} \tag{11.4}$$

and

$$\partial_\mu(U^\dagger\partial^\mu U) = \frac{F_\pi}{2\sqrt{2}}\mu^2(U^\dagger - U) \quad ; \quad \partial_\mu(U\partial^\mu U^\dagger) = \frac{F_\pi}{2\sqrt{2}}\mu^2(U - U^\dagger) \tag{11.5}$$

We assume the following correspondence between the composites of the fundamental Lagrangian and the fields of the effective Lagrangian:

$$\bar{\psi}\gamma^\mu\frac{1+\gamma_5}{2}\psi \leftrightarrow -iU\partial^\mu U^\dagger \quad ; \quad \bar{\psi}\gamma^\mu\frac{1-\gamma_5}{2}\psi \leftrightarrow -iU^\dagger\partial^\mu U \tag{11.6}$$

They imply

$$m\bar{\psi}\frac{1+\gamma_5}{2}\psi \leftrightarrow \frac{\mu^2 F_\pi}{2\sqrt{2}}U^\dagger \quad ; \quad m\bar{\psi}\frac{1-\gamma_5}{2}\psi \leftrightarrow \frac{\mu^2 F_\pi}{2\sqrt{2}}U \tag{11.7}$$

They are reminiscent of the bosonisation rules in $D = 2$. Using the GMOR relation:

$$-2m\langle\bar{\psi}\psi\rangle = \mu^2 F_\pi^2 \tag{11.8}$$

that is obtained by imposing that the vev of the two mass terms are equal:

$$-m\langle\bar{\psi}\psi\rangle = \frac{F_\pi}{2\sqrt{2}}\mu^2\langle(U + U^\dagger)\rangle = \frac{F_\pi}{2\sqrt{2}}\mu^2\frac{2F_\pi}{\sqrt{2}} = \frac{1}{2}\mu^2 F_\pi^2 \tag{11.9}$$

we get

$$2\bar{\psi}\psi \leftrightarrow -\langle\bar{\psi}\psi\rangle\frac{\sqrt{2}(U + U^\dagger)}{F_\pi} \quad ; \quad 2\bar{\psi}\gamma_5\psi \leftrightarrow -\langle\bar{\psi}\psi\rangle\frac{\sqrt{2}(U^\dagger - U)}{F_\pi} \tag{11.10}$$

QCD has also the axial anomaly that is not included in (11.1). How can we impose it?

For the rest of these lectures follow Sect. 4 of [4] and Sect. 4 of [5].

# A Computation of Eqs. (5.2) and (5.3)

Let us compute:

$$A(p,m^2)=\int\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m^2)((p+q)^2+m^2)}=\int\frac{d^2q}{(2\pi)^2}\int_0^\infty ds\int_0^\infty dte^{-(q^2+m^2)s}$$
$$\times\int_0^\infty e^{-t((r+q)^2+m^2)}=\int_0^\infty ds\int_0^\infty dte^{-m^2(s+t)}e^{-r^2\frac{ts}{t+s}}\int\frac{d^2q}{(2\pi)^2}e^{-(s+t)(q-\frac{rt}{s+t})^2}$$
$$=\int_0^\infty ds\int_0^\infty dte^{-m^2(s+t)}e^{-r^2\frac{ts}{t+s}}\frac{1}{4\pi(s+t)}\tag{A.1}$$

Changing variables $u=s+t, t=xu$ we get

$$A(r,m^2)=\int_0^\infty du\frac{e^{-m^2u}}{4\pi}\int_0^1 dxe^{-r^2ux(1-x)}=\frac{1}{4\pi}\int_0^1\frac{dx}{[m^2+r^2x(1-x)]}\tag{A.2}$$

Changing variable again $x=y+\frac{1}{2}$ we get

$$A(r,m^2)=\frac{1}{\pi}\int_{-1/2}^{1/2}\frac{dy}{r^2+4m^2-4r^2y^2}=\frac{1}{\pi}\int_{-1/2}^{1/2}\frac{dy}{(\sqrt{r^2+4m^2}-2\sqrt{r^2}y)(\sqrt{r^2+4m^2}+2\sqrt{r^2}y}$$
$$=\frac{1}{2\pi\sqrt{r^2}}\int_{-1/2}^{1/2}\frac{dy2\sqrt{r^2}}{2\sqrt{r^2+4m^2}}\left(\frac{1}{\sqrt{r^2+4m^2}-2\sqrt{r^2}y}+\frac{1}{\sqrt{r^2+4m^2}+2\sqrt{r^2}y}\right)$$
$$=\frac{1}{4\pi\sqrt{r^2(r^2+4m^2)}}\int_{-1/2}^{1/2}dx\left(\frac{1}{\sqrt{r^2+4m^2}-x}+\frac{1}{\sqrt{r^2+4m^2}+x}\right)\tag{A.3}$$

The integral can be performed and one gets:

$$A(r,m^2)=\frac{1}{2\pi\sqrt{r^2(r^2+4m^2)}}\log\frac{\sqrt{r^2}+\sqrt{r^2+4m^2}}{-\sqrt{r^2}+\sqrt{r^2+4m^2}}\tag{A.4}$$

In the following we compute the following quantity:

$$G_{\mu\nu}(r)=-\int\frac{d^2q}{(2\pi)^2}\frac{4(q_\mu q_\nu-\frac{1}{2}q^2\delta_{\mu\nu})+2(q_\mu r_\nu+q_\nu r_\mu)+r_\mu r_\nu-2m^2\delta_{\mu\nu}}{((q+r)^2+m^2)(q^2+m^2)}\tag{A.5}$$

that can be written as follows:

$$G_{\mu\nu}(r)=-\int_0^\infty dt\int_0^\infty dse^{-m^2(s+t)}\int\frac{d^2q}{(2\pi)^2}\Bigg[4\left(\frac{\partial^2}{\partial\phi_\mu\partial\phi_\nu}-\frac{1}{2}\delta_{\mu\nu}\frac{\partial^2}{\partial\phi_\rho\partial\phi_\rho}\right)$$
$$+2(r_\mu\frac{\partial}{\partial\phi_\nu}+r_\nu\frac{\partial}{\partial\phi_\mu})+r_\mu r_\nu-2m^2\delta_{\mu\nu}\Bigg]e^{-q^2(s+t)-r^2t-2tqr+\phi q}|_{\phi=0}\tag{A.6}$$

We can now compute:

$$\int \frac{d^2q}{(2\pi)^2} e^{-q^2(s+t)-r^2t-2tqr+\phi q} = e^{-r^2t} \int \frac{d^2q}{(2\pi)^2} e^{-(s+t)(q+\frac{t}{s+t}r-\frac{\phi}{2(s+t)})^2} e^{\frac{(tr-\phi/2)^2}{s+t}}$$
$$= \frac{e^{-r^2t}}{4\pi(s+t)} e^{\frac{(tr-\phi/2)^2}{s+t}} \tag{A.7}$$

Then we get

$$G_{\mu\nu}(r) = -\int_0^\infty dt \int_0^\infty ds e^{-m^2(s+t)} e^{-r^2t} \Bigg[ 4\left( \frac{\partial^2}{\partial\phi_\mu \partial\phi_\nu} - \frac{1}{2}\delta_{\mu\nu} \frac{\partial^2}{\partial\phi_\rho\partial\phi_\rho} \right)$$
$$+ 2(r_\mu \frac{\partial}{\partial\phi_\nu} + r_\nu \frac{\partial}{\partial\phi_\mu}) + r_\mu r_\nu - 2m^2\delta_{\mu\nu} \Bigg] \frac{1}{4\pi(s+t)} e^{\frac{(tr-\phi/2)^2}{s+t}} |_{\phi=0} \tag{A.8}$$

It is equal to

$$G_{\mu\nu}(r) = -\int_0^\infty dt \int_0^\infty ds e^{-m^2(s+t)} \frac{e^{-\frac{tsr^2}{s+t}}}{4\pi(s+t)} \Bigg[ \frac{4t^2}{(s+t)^2} \left( r_\mu r_\nu - \frac{1}{2} r^2 \delta_{\mu\nu} \right)$$
$$- 4r_\mu r_\nu \frac{t}{s+t} + r_\mu r_\nu - 2m^2 \delta_{\mu\nu} \Bigg] \tag{A.9}$$

that can be written as follows:

$$G_{\mu\nu}(r) = -\int_0^\infty dt \int_0^\infty ds e^{-m^2(s+t)} \frac{e^{-\frac{tsr^2}{s+t}}}{4\pi(s+t)} \left[ r_\mu r_\nu \frac{(s-t)^2}{(s+t)^2} - \frac{2t^2r^2}{(s+t)^2} \delta_{\mu\nu} - 2m^2\delta_{\mu\nu} \right] \tag{A.10}$$

Introducing the quantity $u = t + s$ we get:

$$G_{\mu\nu}(r) = -\int_0^\infty du e^{-m^2u} \int_0^u dt \frac{e^{-\frac{t(u-t)}{u}r^2}}{4\pi u} \left[ r_\mu r_\nu \frac{(u-2t)^2}{u^2} - \frac{2t^2r^2}{u^2}\delta_{\mu\nu} - 2m^2\delta_{\mu\nu} \right]$$
$$= -\frac{1}{4\pi} \int_0^\infty du \int_0^1 dx e^{-u(m^2+r^2x(1-x))} \left[ r_\mu r_\nu (1-2x)^2 - 2x^2r^2\delta_{\mu\nu} - 2m^2\delta_{\mu\nu} \right]$$
$$= -\frac{1}{4\pi} \int_0^1 dx \frac{r_\mu r_\nu (1-2x)^2 - 2x^2r^2\delta_{\mu\nu} - 2m^2\delta_{\mu\nu}}{m^2 + r^2x(1-x)}$$
$$= -\frac{1}{\pi} \int_0^1 dx \frac{r_\mu r_\nu (1-2x)^2 - 2x^2r^2\delta_{\mu\nu} - 2m^2\delta_{\mu\nu}}{r^2 + 4m^2 - 4r^2(x-\frac{1}{2})^2} \quad y = x - \frac{1}{2}$$
$$= -\frac{1}{\pi} \int_{-1/2}^{1/2} dy \frac{r_\mu r_\nu 4y^2 - 2r^2\delta_{\mu\nu}(y+\frac{1}{2})^2 - 2m^2\delta_{\mu\nu}}{r^2 + 4m^2 - 4r^2y^2} \tag{A.11}$$

The term proportional to $r_\mu r_\nu$ is equal to

$$\frac{r_\mu r_\nu}{r^2}\frac{1}{\pi}\int_{-1/2}^{1/2} dy \frac{r^2+4m^2-4r^2y^2-(r^2+4m^2)}{r^2+4m^2-4r^2y^2} = \frac{r_\mu r_\nu}{r^2}\left(\frac{1}{\pi}-(r^2+4m^2)A(r;m^2)\right) \tag{A.12}$$

The term proportional to $\delta_{\mu\nu}$ is given by:

$$\frac{\delta_{\mu\nu}}{\pi}\int_{-1/2}^{1/2} dy \frac{2r^2(y+\frac{1}{2})^2+2m^2}{r^2+4m^2-4r^2y^2} = \frac{\delta_{\mu\nu}}{\pi}\int_{-1/2}^{1/2} dy \frac{2r^2y^2+2r^2y+\frac{1}{2}(r^2+4m^2)}{r^2+4m^2-4y^2r^2}$$
$$= \delta_{\mu\nu}\frac{1}{2}(r^2+4m^2)A(r,m^2)+\frac{\delta_{\mu\nu}}{\pi}\int_{-1/2}^{1/2} dy \frac{\frac{1}{2}(4r^2y^2-r^2-4m^2)+\frac{1}{2}(r^2+4m^2)+2r^2y}{r^2+4m^2-4y^2r^2}$$
$$= \delta_{\mu\nu}(r^2+4m^2)A(r,m^2)-\delta_{\mu\nu}\frac{1}{2\pi} \tag{A.13}$$

In conclusion we get

$$G_{\mu\nu}(r) = (\delta_{\mu\nu}-\frac{r_\mu r_\nu}{r^2})(r^2+4m^2)A(r,m^2)+\frac{1}{\pi}(\frac{r_\mu r_\nu}{r^2}-\frac{\delta_{\mu\nu}}{2}) \tag{A.14}$$

In principle we should add to the previous expression a term proportional to $\delta_{\mu\nu}$ of the type $a\delta_{\mu\nu}$ because in the previous expression we have cancelled two infinities. $a$ can then be fixed by imposing gauge invariance. This is obtained by fixing $a = -\frac{1}{2\pi}$ arriving at the following expression:

$$G_{\mu\nu}(r) = (\delta_{\mu\nu}-\frac{r_\mu r_\nu}{r^2})\left((r^2+4m^2)A(r,m^2)-\frac{1}{\pi}\right) \tag{A.15}$$

# B Computation of the string tension

We start from Eq. 83 of Ref. 18 of your paper that we rewrite here putting $e^2 = N = 1$ and calling $\Lambda^2$ what in that paper is called $m^2$. We divide it with the number of colours of the bosonic field that we call $N$. Then from Eq. 83 we get

$$c = T = \frac{12\pi^2\Lambda^2}{N}\frac{\epsilon M^2}{3\Lambda^2+\pi\epsilon(M^2+2\Lambda^2)} \tag{B.1}$$

Then introducing the variables

$$x = \frac{m_f}{\Lambda} \quad ; \quad M = \Lambda + m_f \quad ; \quad \epsilon = \frac{1}{\pi}\log\frac{M}{\Lambda} = \frac{1}{\pi}\log(1+x) \tag{B.2}$$

we get

$$T = \frac{12\pi\Lambda^2}{N}\frac{(1+x)^2\log(1+x)}{3+\log(1+x)((1+x)^2+2)} \tag{B.3}$$

This is your equation 5.14 if you take away the factor $\pi$ in the denominator.

For $x$ small the previous equation gives:

$$T \sim \frac{4\pi\Lambda m_f}{N} \tag{B.4}$$

that is the susy soft breaking result, while for $x \to \infty$ one gets:

$$T \sim \frac{12\pi\Lambda}{N} \tag{B.5}$$

that is the result of the large $N$ expansion.

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