Physics of Gravitational Waves (GWs)

some history:

- 1916-18: prediction of GWs (Einstein)
- 1936/37: doubts (Einstein, Rosen)
- 1950+: theoretical understanding
- 1960+: bar detectors (Weber)
  inception of interferometer detectors
- 1970+: debate over quadrupole formula
- 1979: binary pulsar (Hulse, Taylor)
  Nobel Prize 1993
- 2000+: 1st generation detectors
  LIGO USA (2x)
  Virgo Italy
  KAGRA Japan
- 2015+: upgrade to 2nd generation detectors
- 14. Sep. 2015: detection by LIGO !

conventions: $\mu, \nu, \ldots = 0,1,2,3$

$i, j, \ldots = 1,2,3$

$\eta = \mathrm{diag}(-1,1,1,1)$

What are GWs?

- classical waves in the metric $g_{\mu\nu}$, speed $c=1$
- (bosonic) massless spin-2 particles: gravitons
  - ↳ 2 polarizations

linearized gravity

$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$

↑ flat spacetime ⇝ $\boxed{\text{Indices now pulled with } \eta_{\mu\nu}\text{!}}$

in $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$ $\quad (G=1)$

$$\leadsto \underbrace{\partial_\alpha \partial^\alpha}_{\Box} \bar{h}_{\mu\nu} - \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\alpha{}_\alpha$ (trace-reversal)

gauge/coordinate fixing: $\partial^\alpha \bar{h}_{\mu\alpha} = 0$ harmonic gauge

$$\leadsto \boxed{\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}$$

gravity (linear)	electrodyn.
harm. gauge: $\partial^\alpha \bar{h}_{\mu\alpha} = 0$	Lorenz gauge: $\partial^\alpha A_\alpha = 0$
Einstein eq.: $\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$	Maxwell eq.: $\Box A^\mu = -4\pi j^\mu$
solution: $\bar{h}_{\mu\nu} = 4\int d^3x' \frac{T_{\mu\nu}(\vec{x}', t_{ret})}{\lvert\vec{x}-\vec{x}'\rvert}$, $t_{ret} = t - \lvert\vec{x}-\vec{x}'\rvert$	$A^\mu = \int d^3x' \frac{j^\mu(\vec{x}', t_{ret})}{\lvert\vec{x}-\vec{x}'\rvert}$
quadrupolar radiation	dipolar rad.
2 polarizations $h_+, h_\times$	2 polarizations
transverse-traceless waves	transverse waves

vacuum: $T_{\mu\nu}=0$

↳ plane-wave solutions

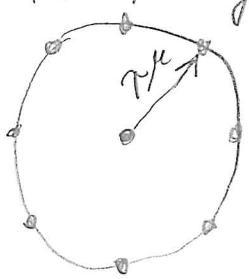
↳ transverse-traceless (TT) gauge: (makes polarizations: physical d.o.f. manifest)

e.g. wave in z-direction

$$(h^{TT}_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\mu x^\mu} + \text{c.c.} \qquad (*)$$

Observing GWs:

↳ free-falling ring of test-masses



geodesic deviation w.r.t. center

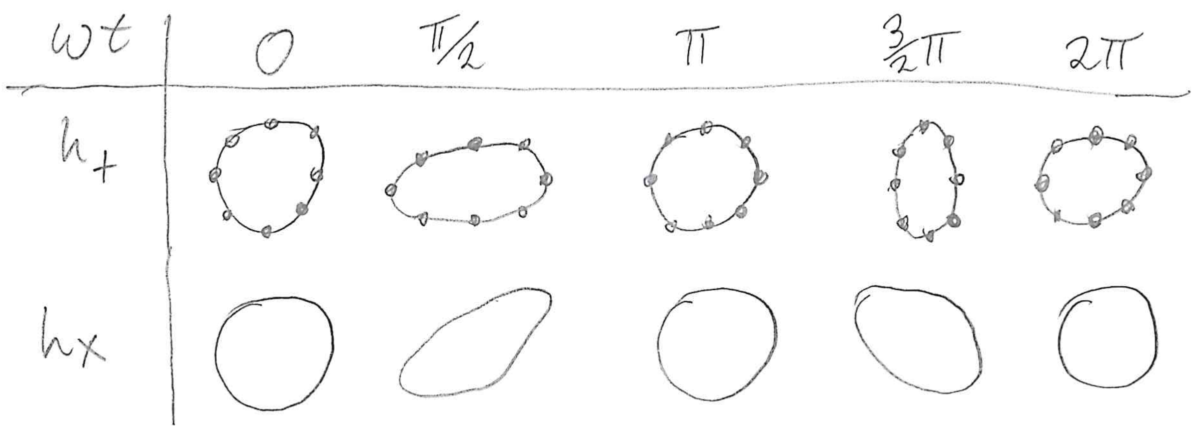
$$\frac{D^2 r^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\nu}\, u^\alpha u^\beta r^\nu \qquad \tau: \text{proper time}$$

$$\hookrightarrow \frac{d^2 r_i}{dt^2} \approx \frac{1}{2} r^j \frac{\partial^2 h^{TT}_{ij}}{\partial t^2} \qquad t: \text{coord. time}$$

ansatz: $r^i = r^i_0 + \Delta r^i$, $|\Delta \vec{r}| \ll |\vec{r}_0|$, and $(*)$

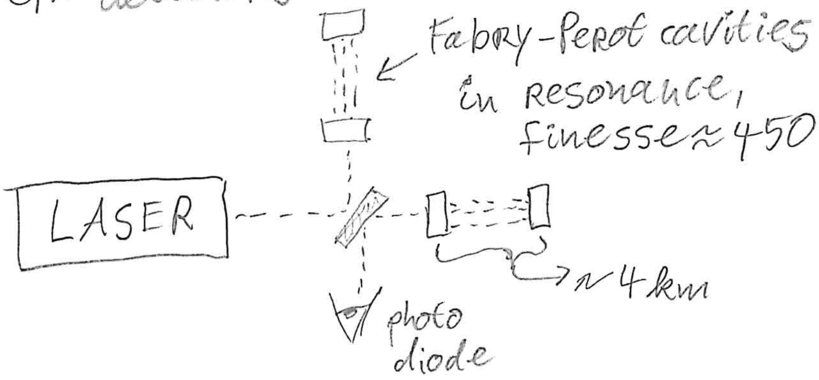
$$\hookrightarrow \Delta r_i \approx \frac{1}{2} r_0^j h^{TT}_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{r}_0$$

$\boxed{h_{+,\times} \sim \sin\omega t}$



$h = h_+ + i h_\times \sim$ dimensionless strain

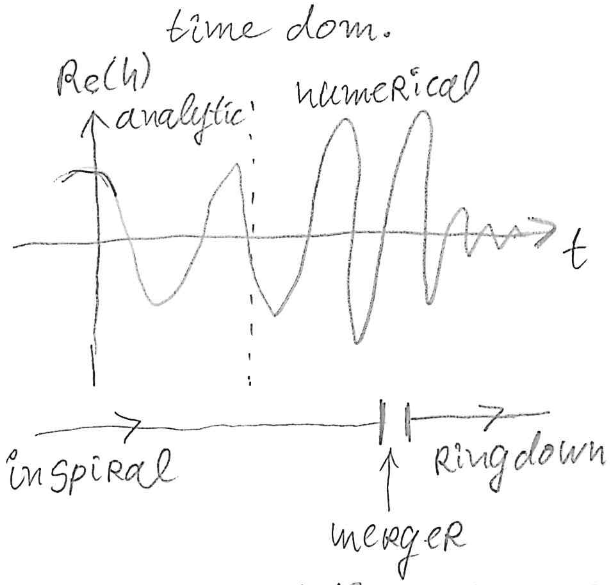
GW detectors ~ Michelson interferometer

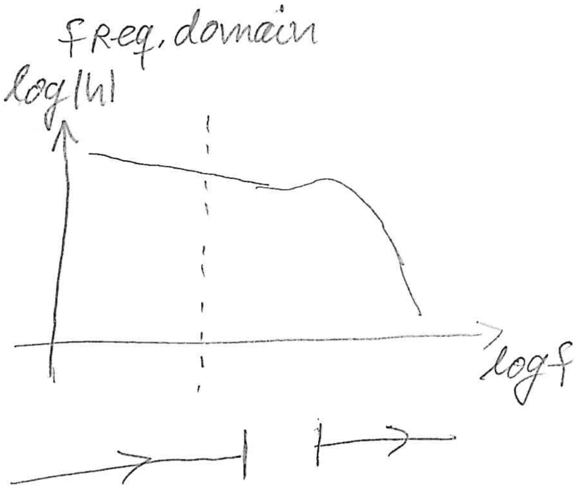


Sensitivity: $|h| \sim 10^{-21}$

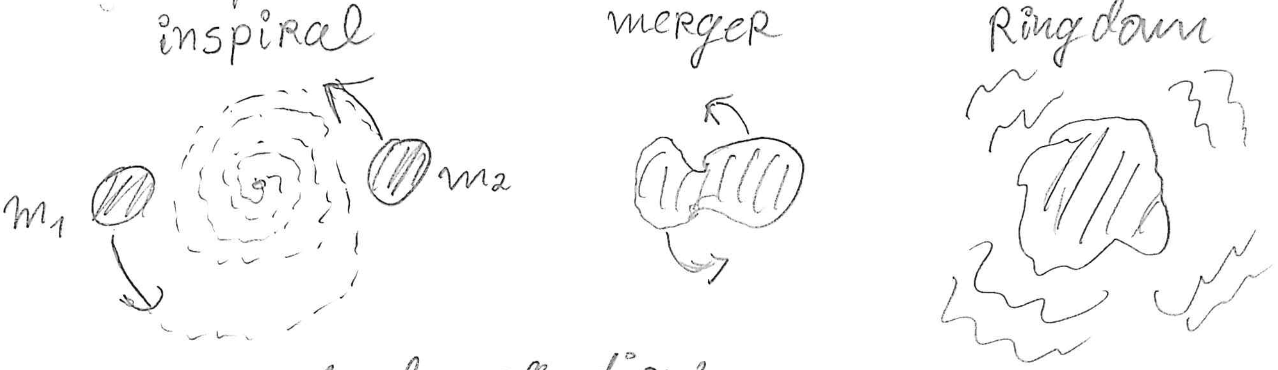
↳ $\Delta L \sim 10^{-18}\,\mathrm{m} \sim \frac{1}{1000}$ proton rad

measurement: (idealized)





interpretation: GWs from binary black holes (BHs)



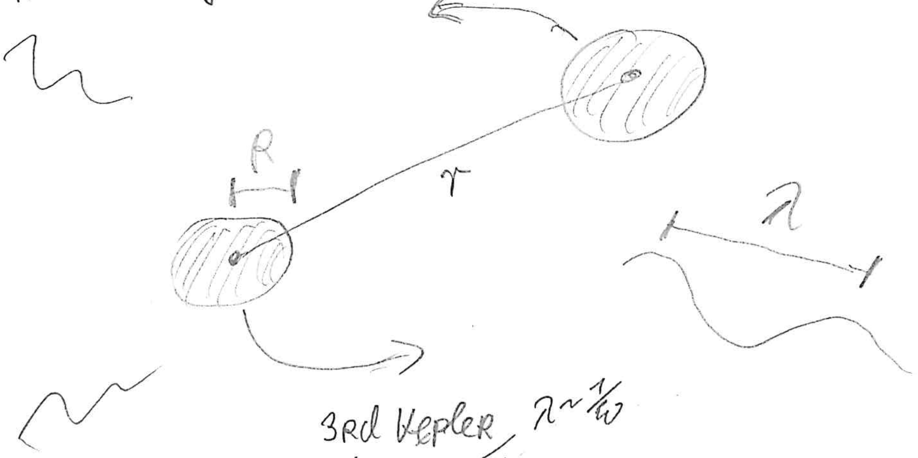
astrophysical application:

e.g. estimate source param. like m_1, m_2

↳ requires prediction for h (and noise model)

↳ Bayesian statistics

predicting the (early) inspiral



$R \ll r \ll \lambda$
↳ hierarchy of scales
↳ approximate in ratio of scales
↳ tower of effective field theories (EFTs)

$$\frac{GM}{r} \overset{\text{3rd Kepler}}{\sim} v^2 \overset{\lambda \sim 1/\omega}{\sim} \frac{r^2}{\lambda^2} \ll 1$$

↳ weak-field & slow motion approximation
↳ post-Newtonian (PN) ‖
↳ perturb. expansion can be done with Feynman diagrams & integrals

leading-order (0PN) GWs

(point-masses, from multipole expansion in $\frac{R}{r}$)

$$\bar{h}_{\mu\nu} = 4 \int d^3x' \frac{T_{\mu\nu}(\vec{x}', t_{ret})}{|\vec{x} - \vec{x}'|} + (\text{nonlin.})$$

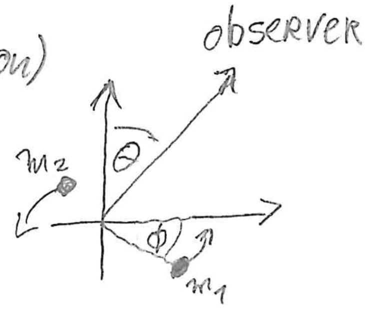
↳ far-zone approx. $\frac{r}{\lambda} \ll 1$
↳ TT-projection:

$$\boxed{h_{ij}^{TT} \approx \frac{2}{R} \underbrace{\Lambda_{ijkl}}_{\text{TT-projector}} \ddot{Q}_{kl} \Big|_{t = t_{ret}}}$$

R: distance to observer
Q_{kl}: Quadrupole

for circular orbits: (orbits circularize over time due to GW emission)

$$\boxed{\begin{aligned} h_+ &= -\underbrace{\frac{4 \mathcal{M}_c^{5/3} \omega^{2/3}}{R}}_{\text{amplitude}} \cdot \underbrace{\frac{1+\cos^2\theta}{2}}_{\text{angular pattern}} \cdot \underbrace{\cos(2\phi)}_{\text{phase}} \\ h_\times &= -\quad '' \quad \cdot \cos\theta \cdot \sin(2\phi) \end{aligned}} + \dots$$



GW frequency ~ 2× orbital freq. $\omega = \dot{\phi}$!
chirp mass $\mathcal{M}_c = \mu^{3/5} M^{2/5}$, $\mu = \frac{m_1 m_2}{M}$, $M = m_1 + m_2$

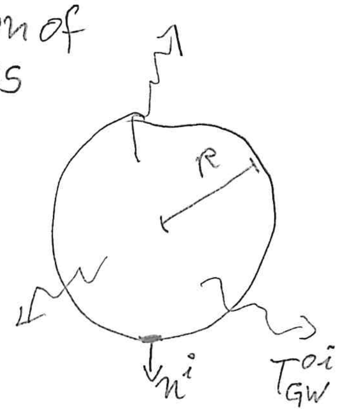
↳ system looses energy → orbit decays

energy-momentum of GWs:

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi} \langle \partial_\mu h^{TT}_{ij} \partial_\nu h^{TT\,ij} \rangle \quad \sim (\text{amplitude})^2$$

↑ average ~ separation of scales

integrate over sphere → luminosity $\mathcal{L}$



$$\mathcal{L} = \oint d\Omega \cdot R^2 n^i \cdot T^{oi}_{GW}$$

insert $T^{\mu\nu}_{GW}$ & h^{TT}_{ij}

$$\hookrightarrow \boxed{\mathcal{L} = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle \Big|_{t=t_{ret}}} \quad \text{quadrupole formula}$$

for circular orbits: $\mathcal{L} = \frac{32}{5} (M_c \omega)^{10/3} + \ldots$ (0PN)

orbital decay from energy balance:

$$\frac{dE}{dt} \stackrel{!}{=} -\mathcal{L} \quad , \quad E = \frac{1}{2}\mu v^2 - \frac{\mu M}{r} + \ldots \text{ (0PN)}$$

(adiabatic approx.) ~ separation of time scales

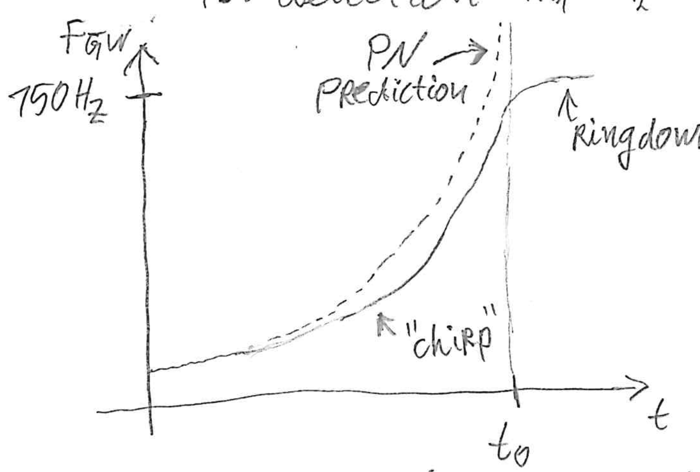
$$v = \omega r \; ; \; \omega^2 r^3 = M \quad \text{(3rd Kepler)}$$

$$\hookrightarrow \dot{\omega} \cdot \omega^{-11/3} = \frac{32}{5} \cdot 3 M_c^{5/3}$$

integrate:

$$f_{GW} \approx 2 \cdot f \approx 134\,\text{Hz} \left(\frac{1.21 M_\odot}{M_c}\right)^{5/8} \left(\frac{1s}{\underbrace{t_0 - t}_{\text{time to merger}}}\right)^{3/8}$$

1st detection: $m_1 \sim m_2 \sim 30 M_\odot$



$f_{GW} \sim 150$ Hz at peak

& $\omega^2 r^3 \sim M$

↳ separation ~ 350 km at peak

↳ BHs !

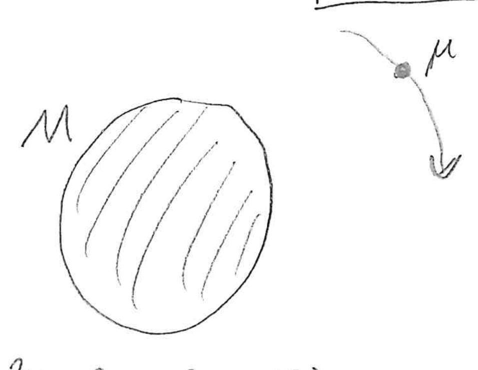
other approximations: – weak field, post-Minkowsian (PM) → scattering

– small mass ratio (SMR), self-force

Strong-field effects

↳ look at test-mass in BH spacetime

Start from mass-shell of 4-momentum p_μ:



$$g^{\mu\nu} p_\mu p_\nu = \mu^2 \qquad (*)$$

Hamiltonian/Energy: $H = -p_0$

metric: $$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = \underbrace{\left(1 - \frac{2M}{r}\right)}_{\equiv A} dt^2 - \frac{dr^2}{A} - r^2 (\underbrace{d\theta^2}_{0} + \underbrace{\sin^2\theta}_{1} d\phi^2)$$

$$(x^\nu) = \begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix} \qquad \text{choose } \theta = \frac{\pi}{2}$$

in $(*)$: $-\frac{1}{A} H^2 + A p_r^2 + \frac{L^2}{r^2} = -\mu^2$, $\quad L \equiv p_\phi$ angular mom.

↳ $$H = \sqrt{A\left(\mu^2 + A \cdot p_r^2 + \frac{L^2}{r^2}\right)}$$

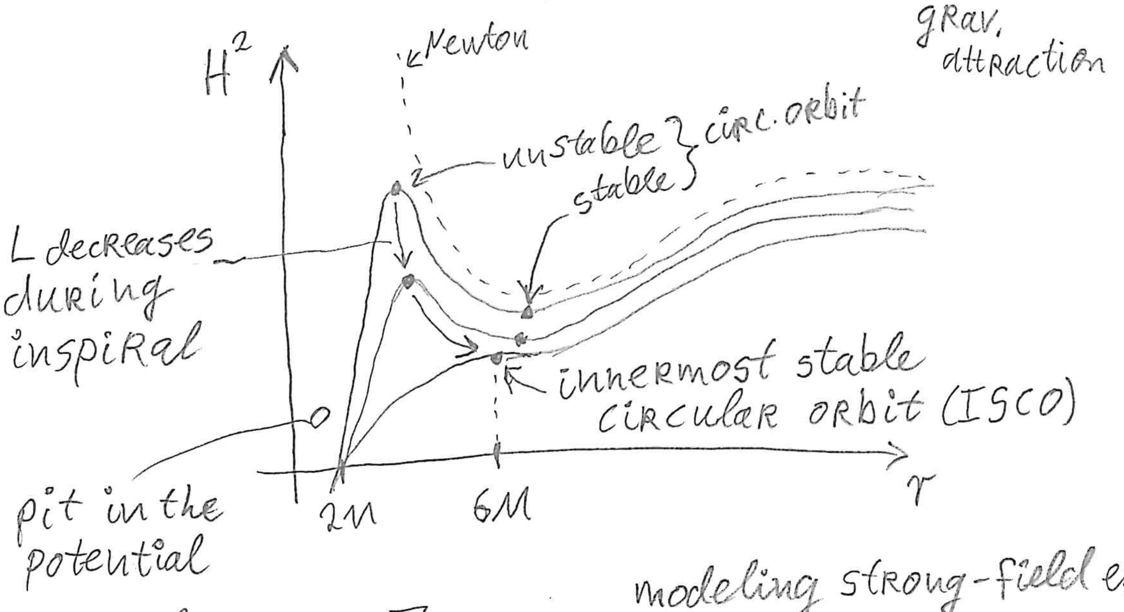
Hamilton's eqs:

$$\frac{dr}{dt} = \frac{\partial H}{\partial p_r} \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r}$$

$$\omega \equiv \dot{\phi} = \frac{\partial H}{\partial L} \qquad \frac{dL}{dt} = -\frac{\partial H}{\partial \phi} = 0 \;\Rightarrow\; L = \text{const}$$

now: circular orbits, $p_r = 0$, $r = \text{const}$

↳ $\frac{\partial H}{\partial r} = 0$ or $\frac{\partial H^2}{\partial r} = 0$ with $H^2 = \underbrace{\left(1 - \frac{2M}{r}\right)}_{\text{grav. attraction}} \underbrace{\left(\mu^2 + \frac{L^2}{r^2}\right)}_{\text{angular mom. "barrier"}}$

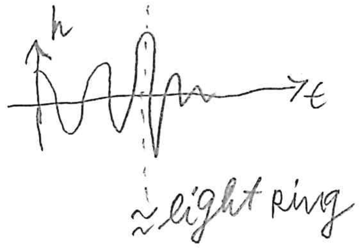


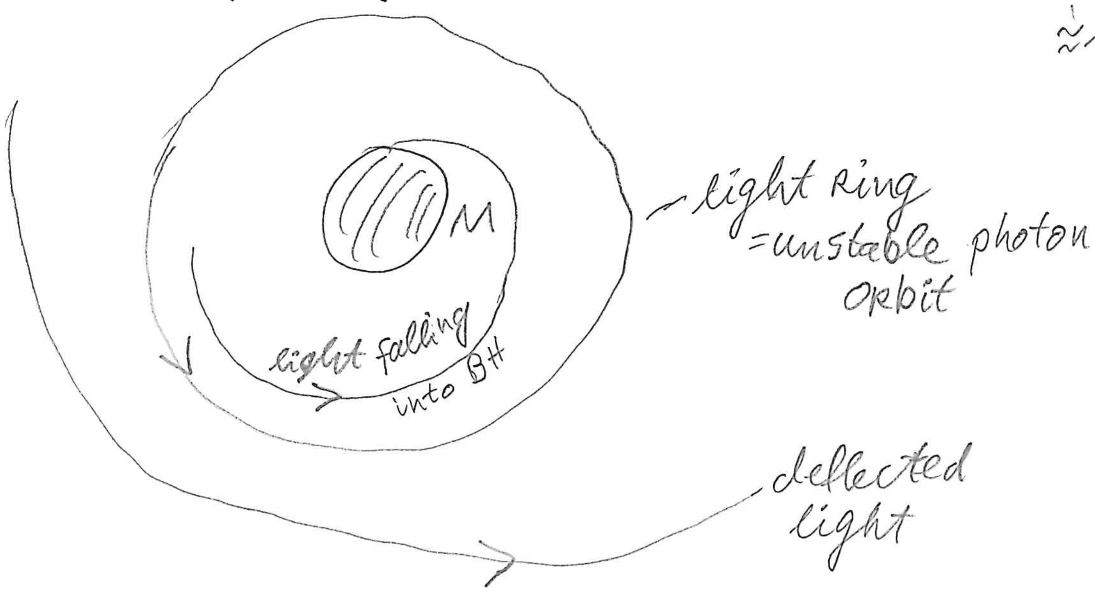
↳ **gravity wins!**

modeling strong-field effects:
- resummation of PN
- validate/calibrate against against numerical relativity
- use SMR, PM

Ringdown

↳ starts at point of max. amplitude
≈ at light ring





⇝ "half" of the emitted GW fall into the BH @ light ring

↳ max. amplitude → Ringdown!

light-ring properties:

$$\text{circular} \quad \begin{matrix} \dot{r}=0 \\ p_r=0 \end{matrix} \quad \Downarrow \quad 0=\dot{p}_r = -\frac{\partial H(\overset{\text{photon}}{\mu=0})}{\partial r} \quad \Downarrow \quad \boxed{r=3M}$$

estimate:

$$\boxed{\omega_{RD} \approx 2\omega = 2\frac{\partial H(\mu=0)}{\partial L} = \frac{2}{3\sqrt{3}M} \approx \frac{0.38\ldots}{M}}$$

matches with numerics!

(with M: mass of final BH)

(comment: Lyapunov exponent of geodesic congruence near light ring ⇝ damping time of Ringdown)