Relational Quantum Relativity

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Motivation

- Motivation: despite tremendous success, quantum field theory is not fully satisfactory as a fundamental framework for physics:
  - no rigorous formalization of interacting theories
  - dependent on fixed background space-time
  - compromised operationality and relationality
- Goal:
  - operational, completely rigorous, non-perturbative and background-less approach to relativistic (post-)quantum physics with a fully relational account of interactions, including gravity.

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Plan

1. Preliminaries
2. Reference frames
3. Frame-relative descriptions
4. Restriction and localization
5. Frame transformations
6. Further perspectives

Quantum Mechanics

We endorse the following perspective on quantum mechanics (aligned with GPT frameworks):

- States are density operators $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})^{sa}$,
- Observables are positive operator-valued measures (POVMs)

$$\mathsf{E} : \mathcal{F}(\Sigma) \to B(\mathcal{H}),$$

giving rise to probability distributions via Born rule:

$$\mu_{\omega}^{\mathsf{E}} : \mathcal{F}(\Sigma) \ni X \mapsto \mathrm{tr}[\omega \mathsf{E}(X)].$$

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Operational equivalence

Set of available operators may be constrained $\mathcal{O} \subset B(\mathcal{H})$. Then

- $\omega \sim_{\mathcal{O}} \omega'$ iff $\mathrm{tr}[\omega A] = \mathrm{tr}[\omega' A]$ for all $A \in \mathcal{O}$,
- $\mathcal{O}$-operational state space:

$$\mathcal{S}(\mathcal{H})/\sim_{\mathcal{O}} \subset \mathcal{T}(\mathcal{H})^{sa}/\sim_{\mathcal{O}}.$$

We have:

$$\left[\mathrm{span}(\mathcal{O})^{cl}\right]_* \cong \mathcal{T}(\mathcal{H})/\sim_{\mathcal{O}},$$

which extends the usual states/operators duality ($\mathcal{O} = B(\mathcal{H})$):

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What is an operational quantum reference frame?

Intuition: coordinates are *abstractions of physical systems*

1. **Reference frames can be *(re)oriented*.** Operationally speaking, they should be equipped with *group action* on system's state space, and (covariant) *frame observable* measuring orientation.
2. **Relativity of measurement/observation.** The operationally meaningful observables depend on the choice of the *reference frame = measuring instrument*. They should be defined on composite systems, (gauge-)invariant and compatible with choice of frame observable.
3. **Universality of quantum mechanics.** Physical systems are modelled by Hilbert space-based quantum mechanics.

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Quantum reference frames

Quantum reference frame (for G) is a triple $\mathcal{R} = (\mathcal{H}_\mathcal{R}, U_\mathcal{R}, \mathsf{E}_\mathcal{R})$:

- Hilbert space $\mathcal{H}_\mathcal{R}$
- group action $U_\mathcal{R} : G \to B(\mathcal{H}_\mathcal{R})^{uni}$
- *covariant* POVM $\mathsf{E}_\mathcal{R} : \mathcal{B}(G) \to B(\mathcal{H}_\mathcal{R})$, i.e. for all $X \in \mathcal{B}(G)$

$$\mathsf{E}(g.X) = U_\mathcal{R}(g).\mathsf{E}(X)U^*_\mathcal{R}(g)$$

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Frame-relative descriptions

Given $\mathcal{R} = (\mathcal{H}_\mathcal{R}, U_\mathcal{R}, \mathsf{E}_\mathcal{R}), \mathcal{S} = (\mathcal{H}_\mathcal{S}, U_\mathcal{S})$ *restrict* available effects to those respecting choice of *frame observable* (call them *framed*):

$$B(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})_{\mathsf{E}_\mathcal{R}} := \mathrm{conv}\{\mathsf{E}(X) \otimes A_\mathcal{S} \mid X \in \mathcal{B}(G), A_\mathcal{S} \in B(\mathcal{H}_\mathcal{S})\}.$$

But we also want them to be *invariant*:

$$B(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})^G_{\mathsf{E}_\mathcal{R}} := B(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})_{\mathsf{E}_\mathcal{R}} \cap B(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})^G.$$

Is this *non-empty*?

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Relativization

Relativization map is given by

$$\yen^{\mathcal{R}} : B(\mathcal{H}_{\mathcal{S}}) \ni A_{\mathcal{S}} \mapsto \int_G dE_{\mathcal{R}}(g) \otimes g.A_{\mathcal{S}} \in B(\mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{S}})^G_{\mathsf{E}_{\mathcal{R}}}.$$

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It is understood as incorporating reference explicitly into the description of $\mathcal{S}$. *Relative* description is given by

$$B(\mathcal{H}_{\mathcal{S}})^{\mathcal{R}} := \mathrm{Im}(\yen^{\mathcal{R}})^{cl} \subseteq B(\mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{S}})^G_{\mathsf{E}_{\mathcal{R}}},$$

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Restriction and localization

For $\omega \in \mathcal{S}(\mathcal{H}_\mathcal{R})$ the ω-restriction maps are given by

$$\Gamma_\omega : B(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S}) \ni A_\mathcal{R} \otimes A_\mathcal{S} \mapsto \omega(A_\mathcal{R}) A_\mathcal{S} \in B(\mathcal{H}_\mathcal{S})$$

and understood as conditioning description of composite system, upon a choice of reference's state. For frames that we call *localizable* one can find a sequence of states (ω_n) such that

$$\lim_{n\to\infty} (\Gamma_{\omega_n} \circ \yen^\mathcal{R})(A_\mathcal{S}) = A_\mathcal{S}$$

for all $A_\mathcal{S} \in B(\mathcal{H}_\mathcal{S})$. Thus the non-relational QM is recovered in a limiting sense upon externalizing localizable reference frames.

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Internal frame change maps

Consider two *internal* localizable frames

$$\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_{\mathcal{S}}$$

$$\mathsf{E}_1 : \mathcal{B}(G) \to B(\mathcal{H}_1), \quad \mathsf{E}_2 : \mathcal{B}(G) \to B(\mathcal{H}_2).$$

There is a well-defined, invertible and composable (in the context of three frames) map $\Phi^{loc}_{1\to 2}$ making the following diagrams commute

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 & \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_{\mathcal{S}})_G \arrow[dl, "\pi_{\mathsf{E}_2} \circ \yen_*^{\mathcal{R}_1}"'] \arrow[dr, "\pi_{\mathsf{E}_1} \circ \yen_*^{\mathcal{R}_2}"] & \\
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Further research directions:

- More general frames and systems
- Relational interactions
- Relational Process Theories
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Framed quantum observables

For any $\mathsf{E}_{\mathcal{R}} : \mathcal{F}(\Sigma) \to B(\mathcal{H}_{\mathcal{R}})$ and $f : \Sigma \to B(\mathcal{H}_{\mathcal{S}})$ such that

$$\Sigma \ni x \mapsto \mathrm{tr}[\rho f_{\mathcal{S}}(x)] \in \mathbb{C}$$

are *integrable* for any state $\rho \in \mathcal{S}(\mathcal{H}_{\mathcal{S}})$, the operator

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$$\mathcal{T}(\mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{S}}) \ni \omega \otimes \rho \mapsto \int_{\Sigma} d\mu_{\omega}^{\mathsf{E}_{\mathcal{R}}}(x)\, \mathrm{tr}[\rho f_{\mathcal{S}}(x)] \in \mathbb{C}.$$

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General framed observables

For any $\hat{\mathsf{E}}_{\mathcal{R}} : L^\infty(\Sigma, \mu) \to X_{\mathcal{R}}$ and $f : \Sigma \to X_{\mathcal{S}}$ such that

$$\Sigma \ni x \mapsto f(x)[t_{\mathcal{S}}] \in \mathbb{R}$$

are *integrable* for any $t_{\mathcal{S}} \in (X_{\mathcal{S}})_*$, the operator

$$\int_\Sigma d\hat{\mathsf{E}}_{\mathcal{R}}(x) \otimes f(x) \in X_{\mathcal{R}} \otimes X_{\mathcal{S}}$$

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$$(X_{\mathcal{R}})_* \odot (X_{\mathcal{S}})_* \ni t_{\mathcal{R}} \otimes t_{\mathcal{S}} \mapsto \int_\Sigma d\hat{\mathsf{E}}_{\mathcal{R}}[t_{\mathcal{R}}](x) f(x)[t_{\mathcal{S}}] d\mu \in \mathbb{R}.$$

Relational interactions

For $\Sigma \cong G/H$ and $\sigma : G/H \to G$ (equivariant, Borel) we define

$$¥^{\mathcal{R}}_{\sigma}(A) := \int_{G/H} d\mathsf{E}(x) \otimes \sigma(x).A \in B(\mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{S}})^{G}. \tag{1}$$

For $G = P$ (Poincar'e) and $H = O(1,3)$ (Lorentz) consider

$$\sigma : P/O(1,3) \cong \mathbb{M}^4 \ni x \mapsto (x, \hat{\sigma}(x)) \in \mathbb{M}^4 \rtimes O(1,3) \cong P.$$

Section $\hat{\sigma} : \mathbb{M}^4 \to O(1,3)$ can be interpreted as encoding interaction between frame and system, reflected by relativization map $¥^{\mathcal{R}}_{\sigma}$.

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Relational Process Theories

The relativization construction can be seen as a functor

$$¥ : \mathbf{Frm}_G \times \mathbf{Rep}_G \ni (\mathcal{R}, \mathcal{S}) \mapsto B(\mathcal{H}_\mathcal{S})^\mathcal{R} \subset B(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})^G \in \mathbf{Equiv}_G,$$

where category $\mathbf{Frm}_G$ is defined with $\mathcal{R} \to \mathcal{R}'$ given by

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Thank you for your attention!