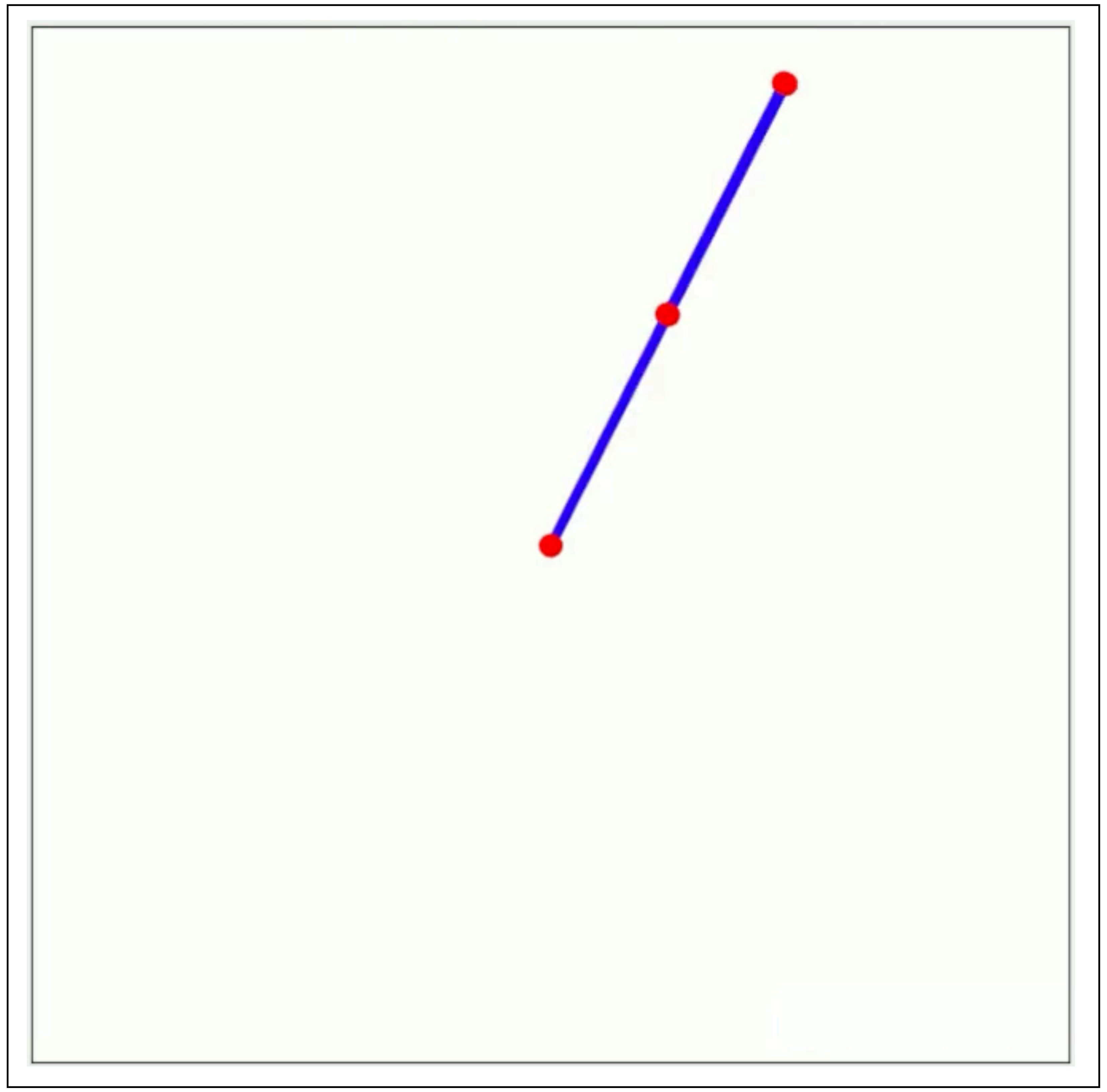
A Soft Introduction to Dynamical Systems

Nordita Day of Open Doors

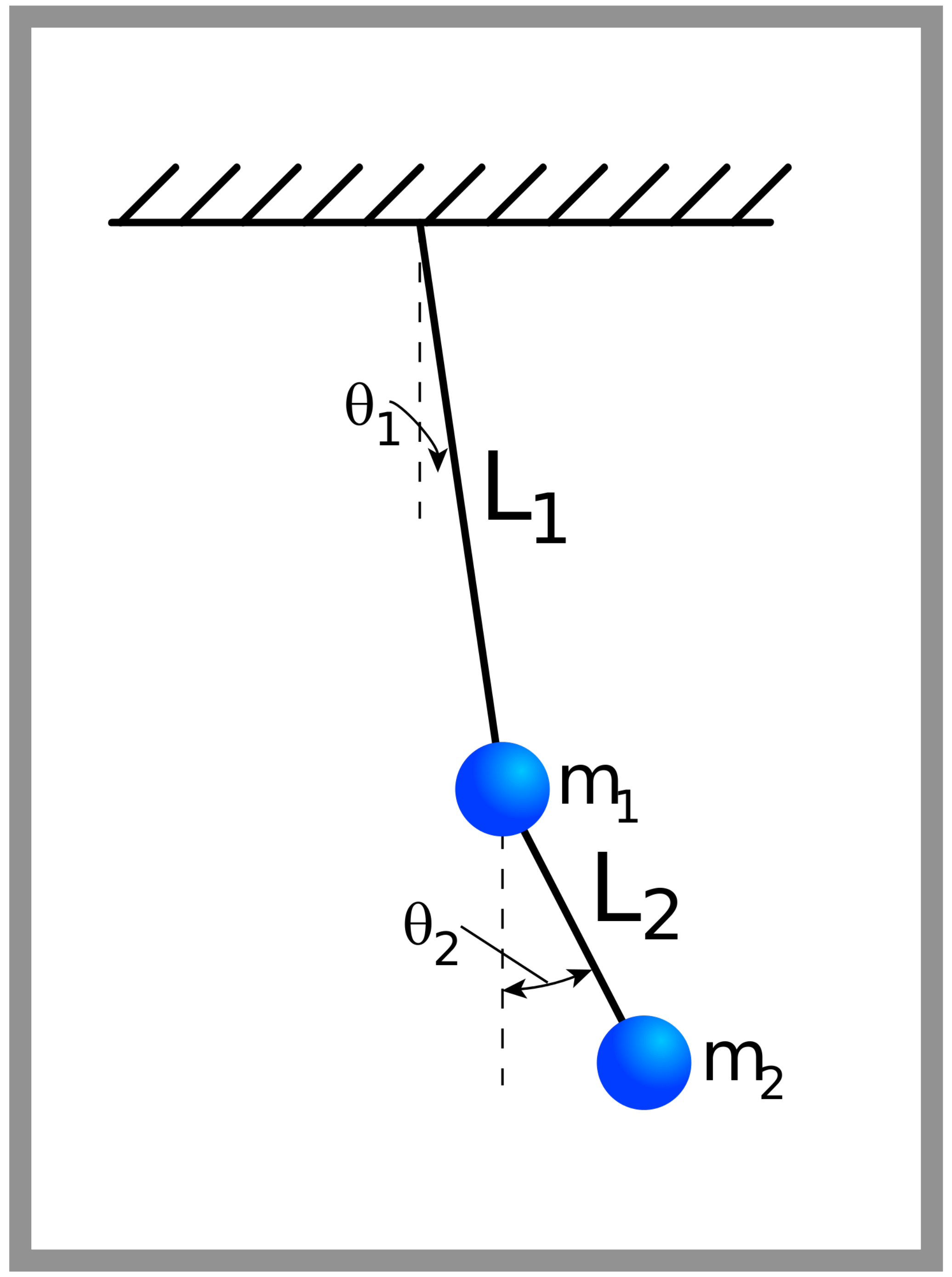
Alessia Ferraro

The Double Pendulum

The Double Pendulum



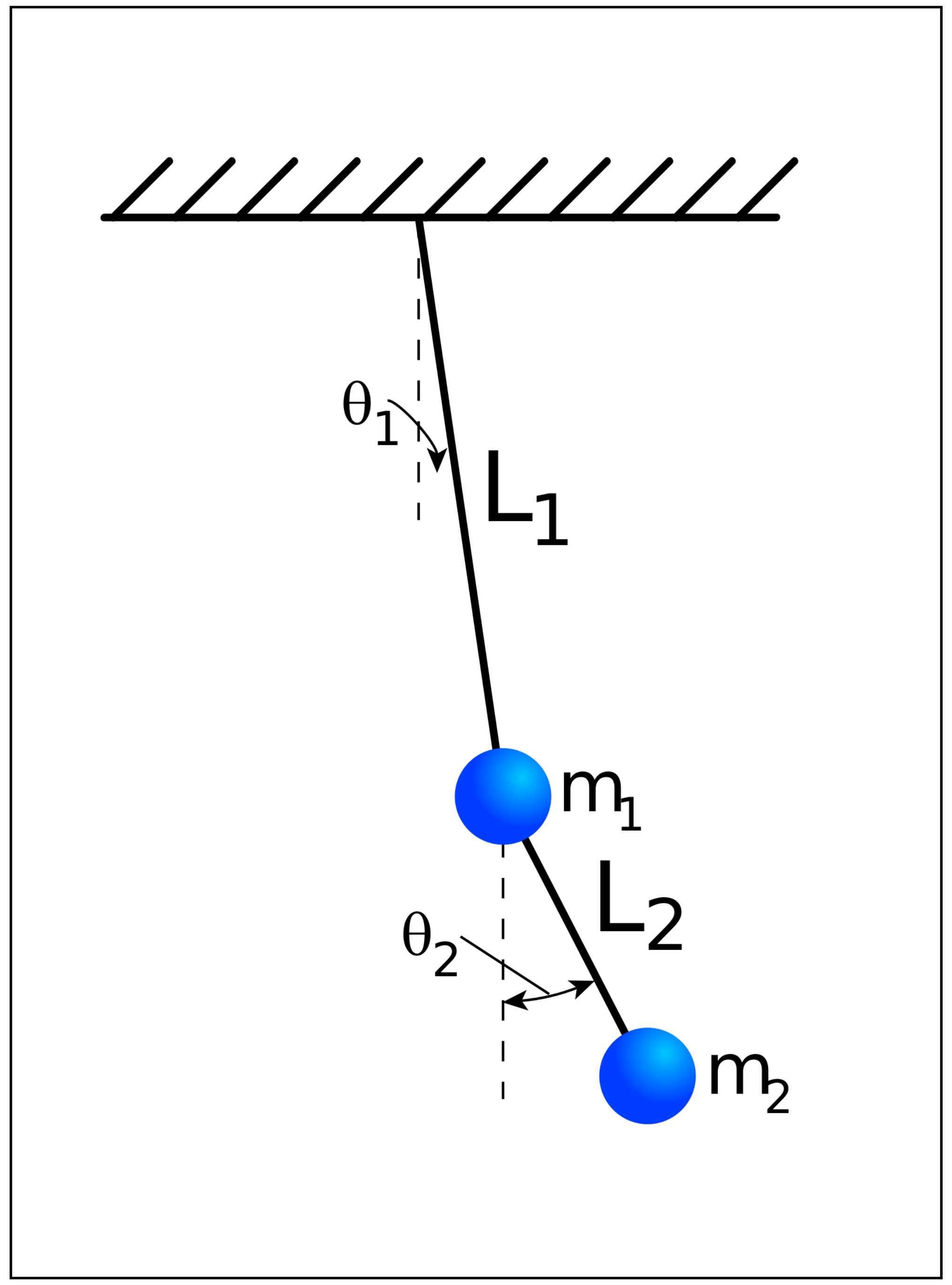
The Double Pendulum



$$
\begin{aligned}
(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) &= \\
&= m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g \sin\theta_1
\end{aligned}
$$

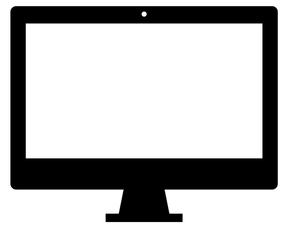
$$
l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) = -l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - g \sin\theta_2
$$

The Double Pendulum



$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) = m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g \sin\theta_1$$

$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) = -l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - g \sin\theta_2$$



https://web.mit.edu/jorloff/www/chaosTalk/double-pendulum/double-pendulum-en.html

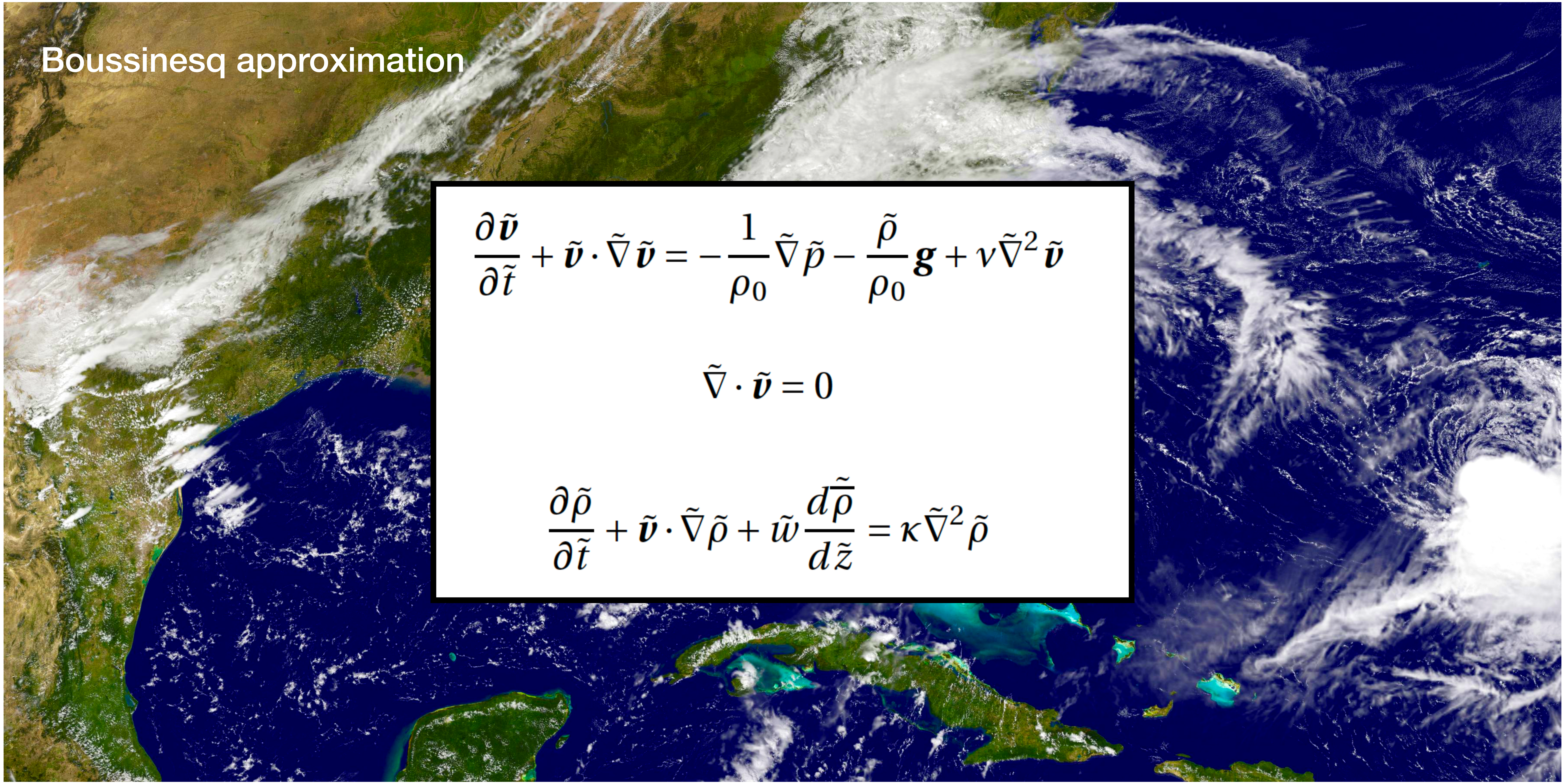
How well can we predict the future ?

How well can we predict the future ?

(Or better … for how long?)


AND NOW THE WEATHER
IT WILL BE SUNNY UNLESS IT RAINS

Boussinesq approximation



$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + \tilde{\boldsymbol{v}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}} = -\frac{1}{\rho_0} \tilde{\nabla} \tilde{p} - \frac{\tilde{\rho}}{\rho_0} \boldsymbol{g} + \nu \tilde{\nabla}^2 \tilde{\boldsymbol{v}}$$

$$\tilde{\nabla} \cdot \tilde{\boldsymbol{v}} = 0$$

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\boldsymbol{v}} \cdot \tilde{\nabla} \tilde{\rho} + \tilde{w} \frac{d\tilde{\bar{\rho}}}{d\tilde{z}} = \kappa \tilde{\nabla}^2 \tilde{\rho}$$

Lorenz System

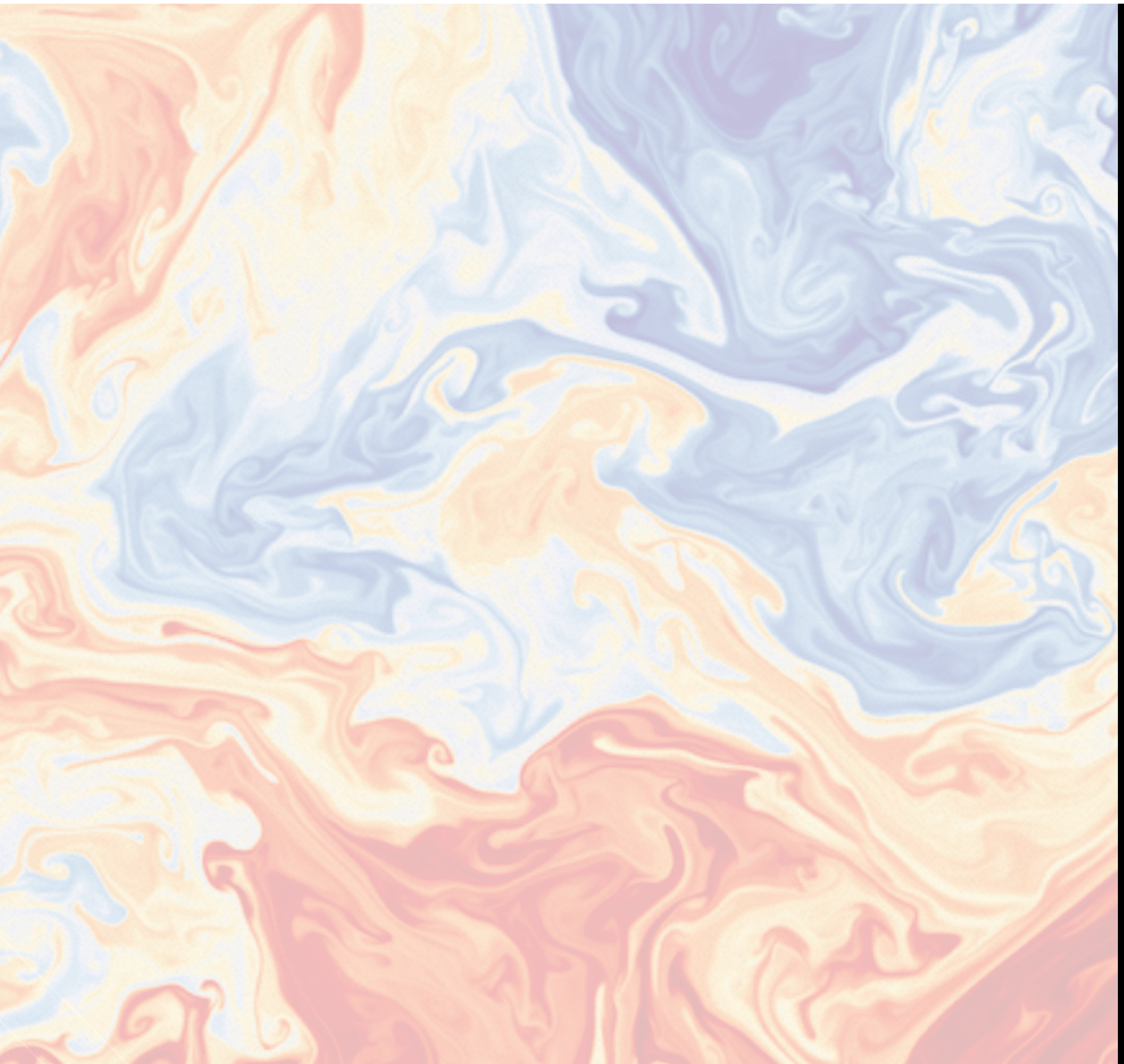
From Oberbeck–Boussinesq approximation (describing Rayleigh-Benard convection)

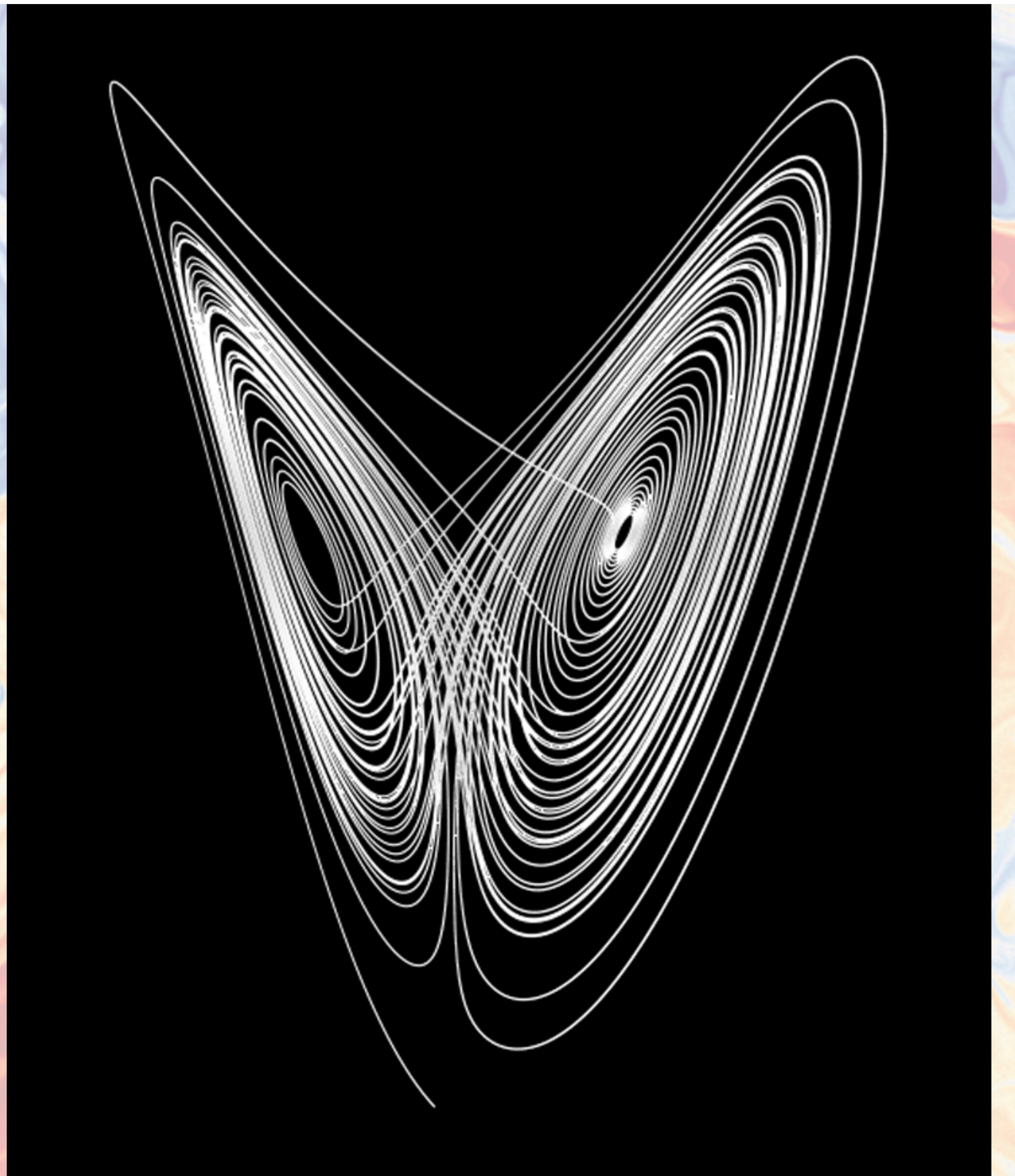
$$\dot{x} = \sigma(y - x) \qquad \dot{y} = x(\rho - z) - y \qquad \dot{z} = xy - \beta z$$

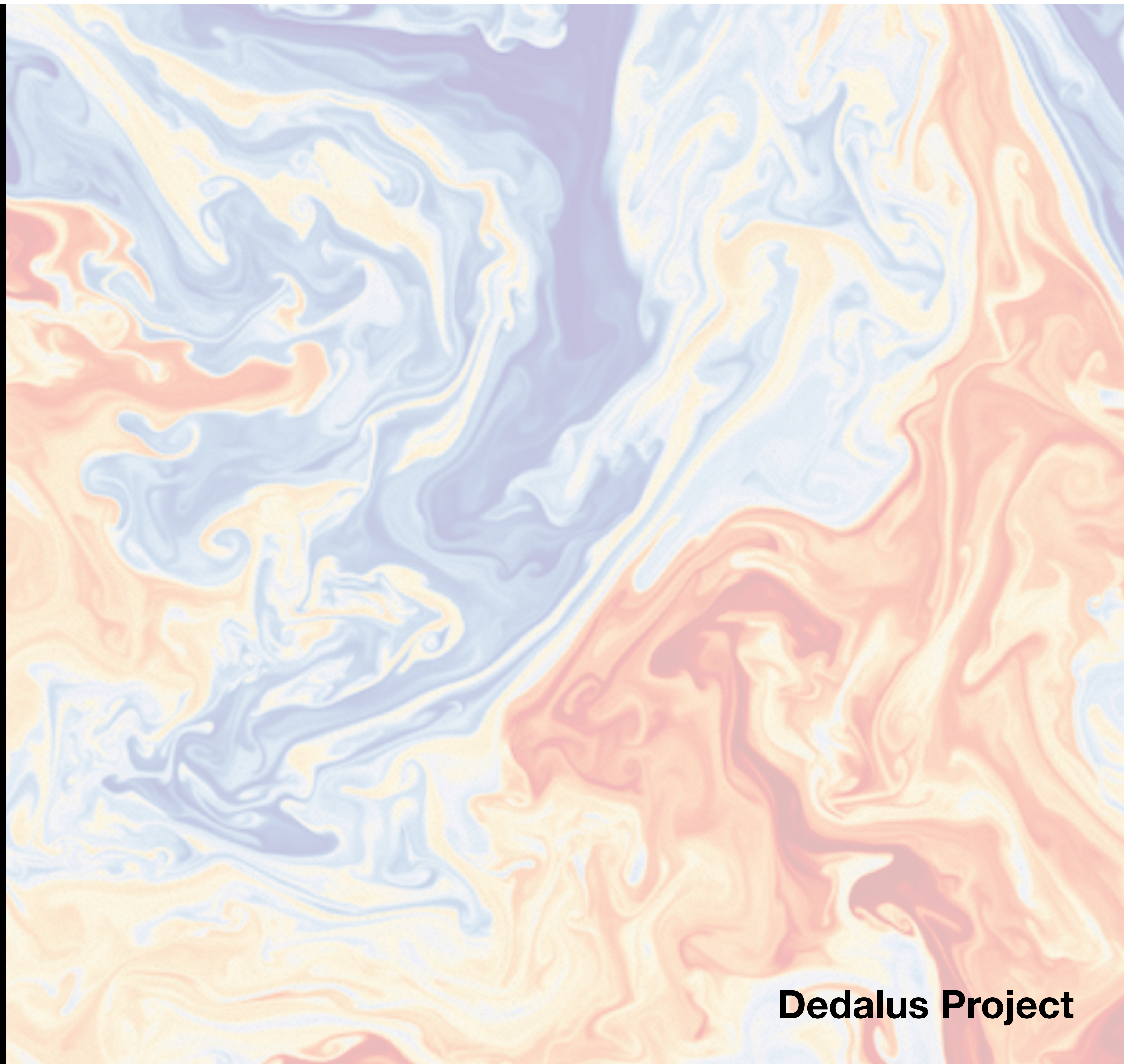
Lorenz System

From Oberbeck–Boussinesq approximation (describing Rayleigh-Benard convection)

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = x(\rho - z) - y \qquad \dot{z} = xy - \beta z$$







Dedalus Project

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector | Dynamics | Time | Control Knobs | Other variables

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector Dynamics Time Control Knobs Other variables

CHALLENGES

1 $\mathbf{f}$ is **UNKNOWN**

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector — Dynamics — Time — Control Knobs — Other variables

CHALLENGES

1. $\mathbf{f}$ is **UNKNOWN**
2. $\mathbf{f}$ is **NON-LINEAR**

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector　Dynamics　Time　Control Knobs　Other variables

CHALLENGES

1. $\mathbf{f}$ is **UNKNOWN**
2. $\mathbf{f}$ is **NON-LINEAR**
3. $\mathbf{x}$ is **HIGH-DIMENSIONAL**

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector — Dynamics — Time — Control Knobs — Other variables

CHALLENGES

1. $\mathbf{f}$ is **UNKNOWN**
2. $\mathbf{f}$ is **NON-LINEAR**
3. $\mathbf{x}$ is **HIGH-DIMENSIONAL**
4. $\mathbf{f}$ is **MULTI-SCALE**

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector | Dynamics | Time | Control Knobs | Other variables

CHALLENGES

1. $\mathbf{f}$ is **UNKNOWN**
2. $\mathbf{f}$ is **NON-LINEAR**
3. $\mathbf{x}$ is **HIGH-DIMENSIONAL**
4. $\mathbf{f}$ is **MULTI-SCALE**
5. **CHAOS** – Sensitivity to initial Conditions
6. **UNCERTANTY** – Noise, latent variables …

Anatomy of a Dynamical System

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \mathbf{q}, \mathbf{s})$$

State Vector — Dynamics — Time — Control Knobs — Other variables

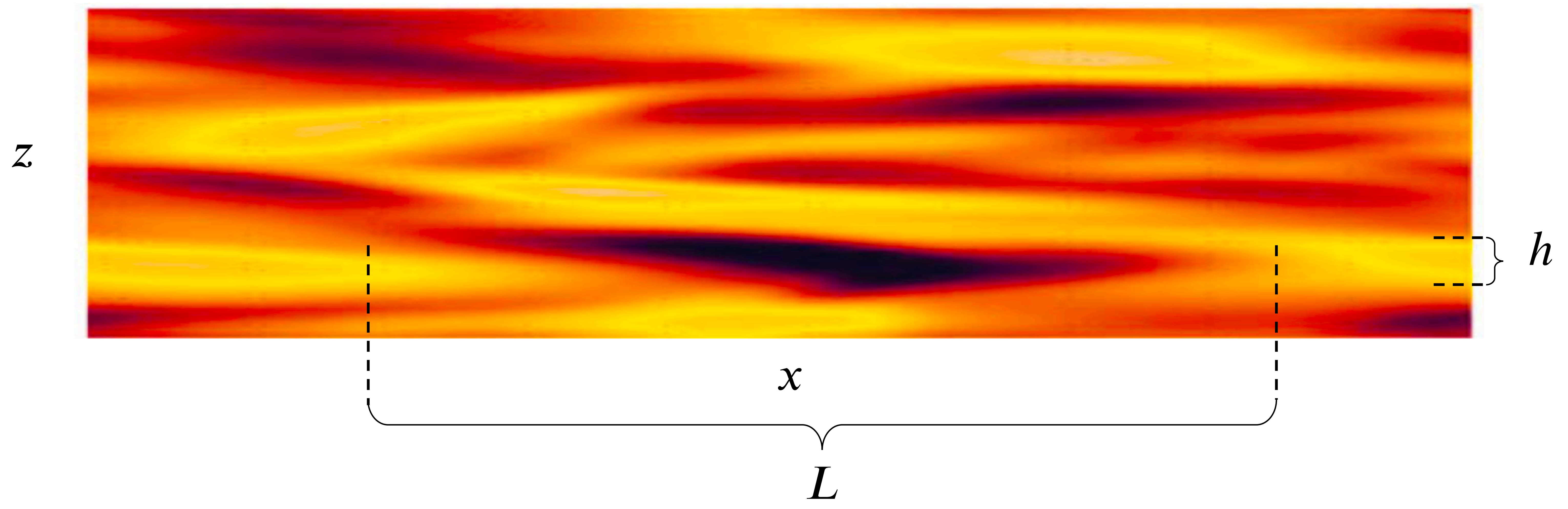
MY RESEARCH

1. **f** is **UNKNOWN**
2. **f** is **NON-LINEAR**
3. **x** is **HIGH-DIMENSIONAL**
4. **f** is **MULTI-SCALE**
5. **CHAOS** – Sensitivity to initial Conditions
6. **UNCERTANTY** – Noise, latent variables …

Strongly Stratified Flows

$Fr = \dfrac{U}{NL} \to 0$ $\qquad Re = \dfrac{UL}{\nu} \to \infty$ $\qquad Re\,Fr^2 > 1$ $\qquad$ Strongly stratified turbulence

$Fr \simeq 0.015 \qquad Re = 1.2\ 10^3$



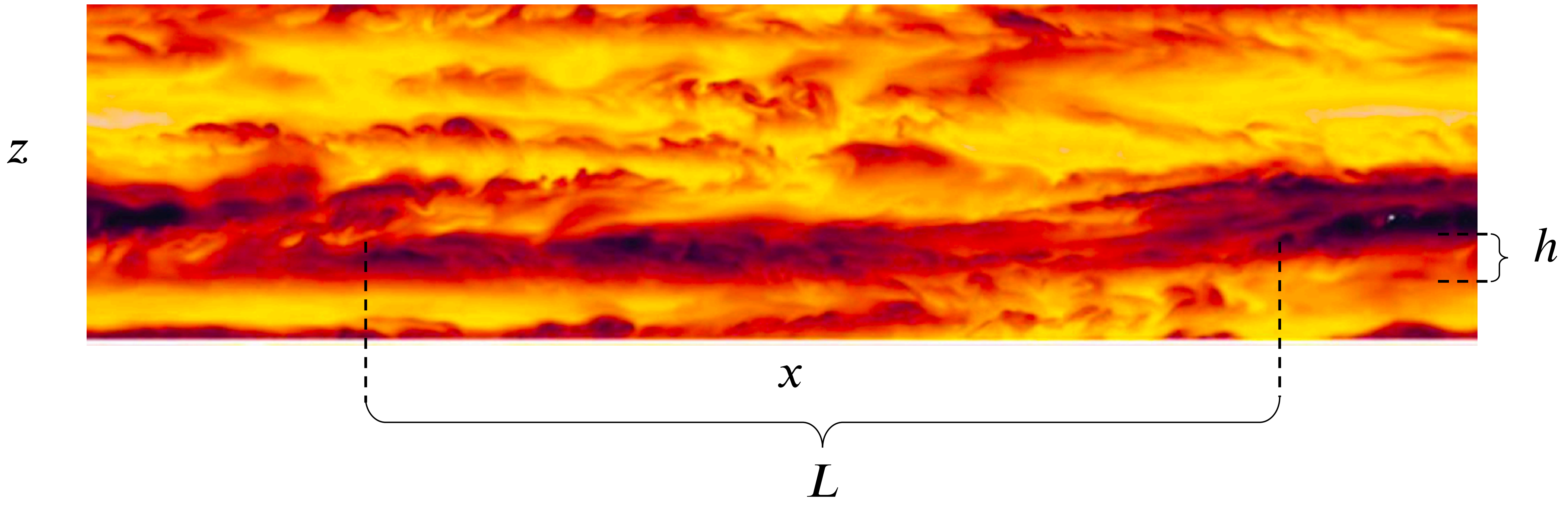
Snapshot from DNS of density fluctuations in a vertical plane (Brethouwer et al. JFM 2007)

Strongly Stratified Flows

$$Fr = \frac{U}{NL} \to 0 \qquad Re = \frac{UL}{\nu} \to \infty \qquad Re\,Fr^2 > 1$$

Strongly stratified turbulence

$$Fr \simeq 0.015 \qquad Re = 1.2\ 10^3$$

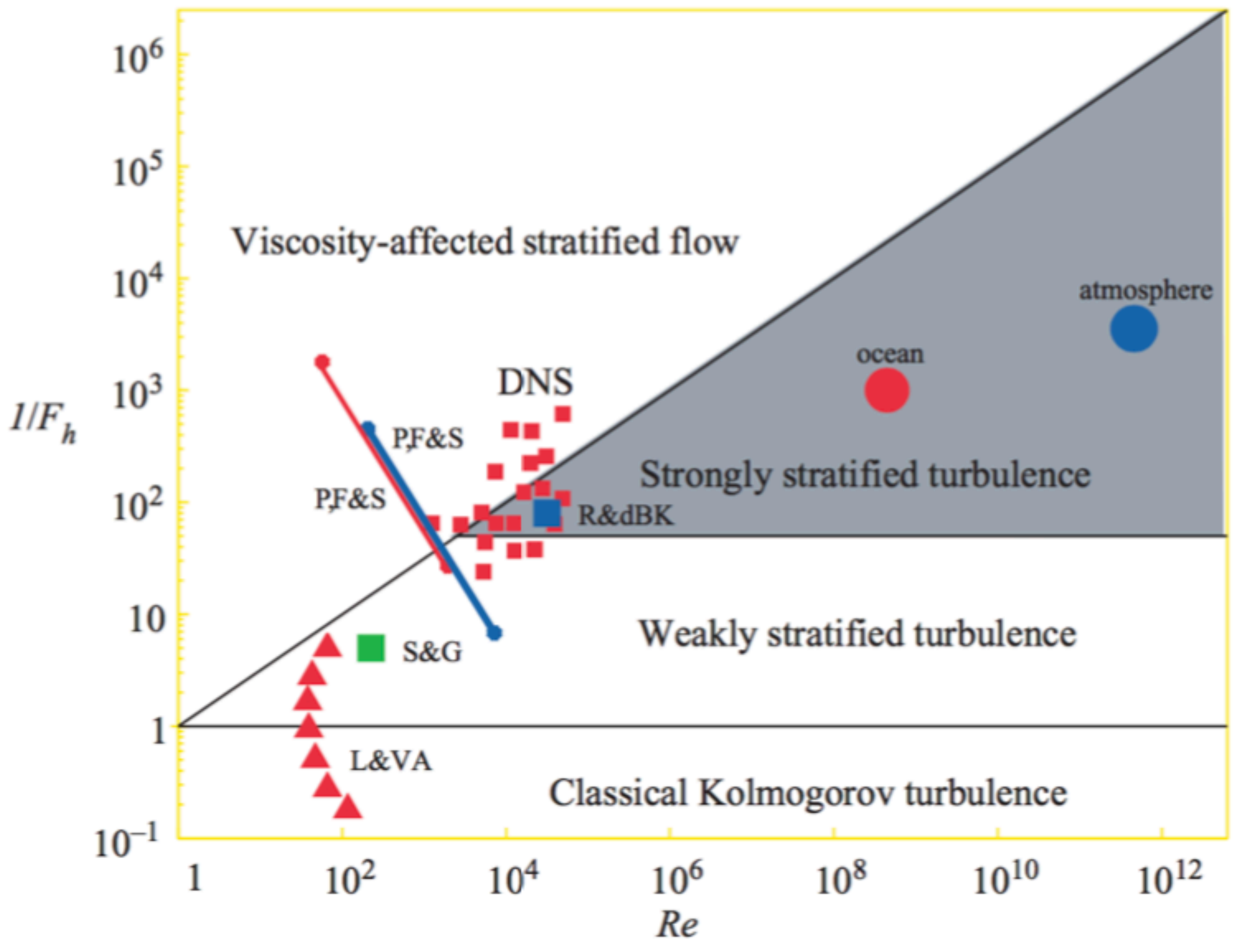


Snapshot from DNS of density fluctuations in a vertical plane (Brethouwer et al. JFM 2007)

Strongly Stratified Flows

Modelling and Computational Challenges

The different regimes of stratified flows as a function of the Reynolds number and the horizontal Froude number

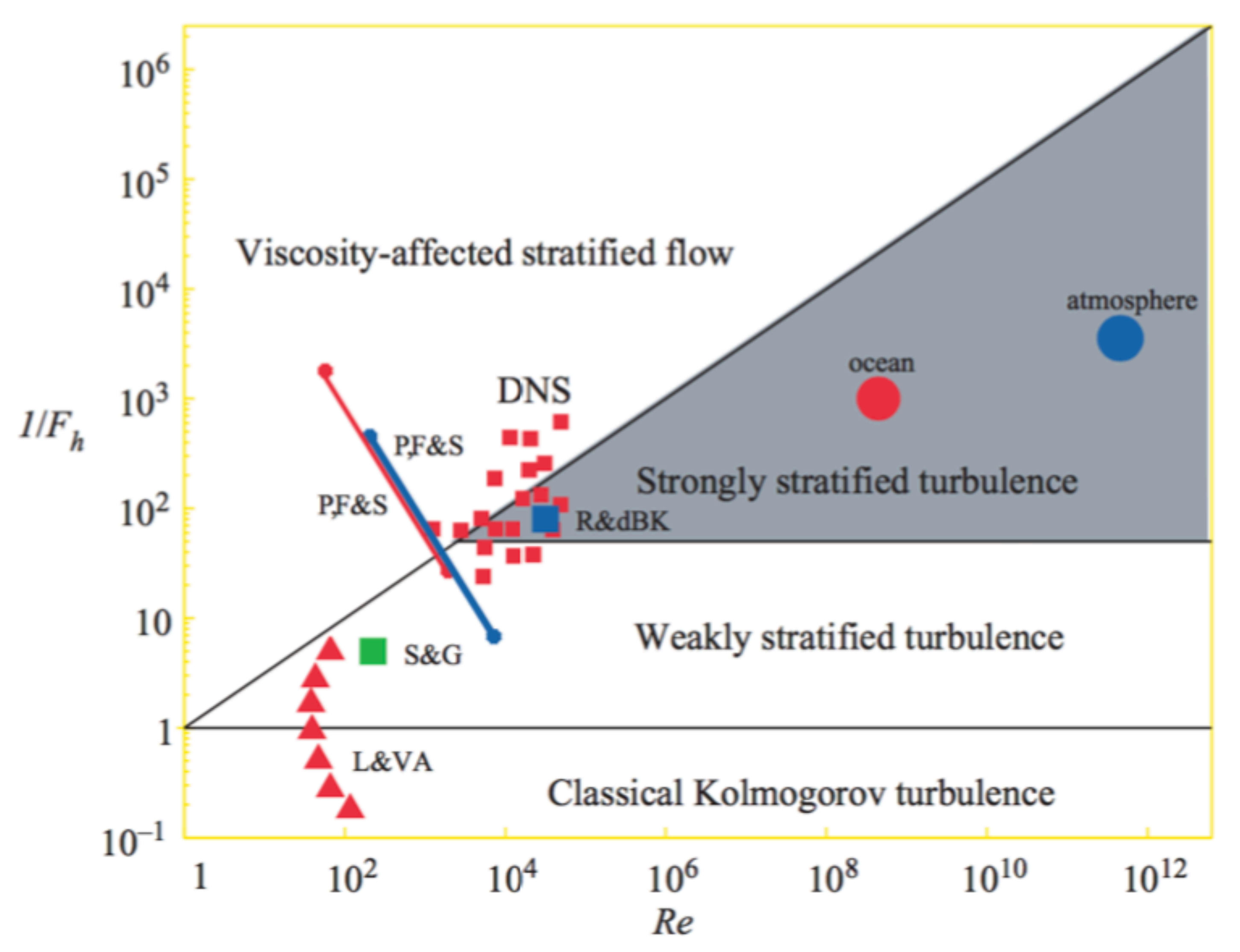


(Brethouwer et al. JFM 2007)

QL Reduced Model

Modelling and Computational Challenges

The different regimes of stratified flows as a function of the Reynolds number and the horizontal Froude number



GOAL

Exploit the scale separation
to derive

a REDUCED MODEL

to access the regime

$$Fr = \frac{U}{NL} \to 0 \quad Re = \frac{UL}{\nu} \to \infty$$

and gain a physical insight for
parametrization

Thank you for your attention