A glimpse of Network Science

Structure and Dynamic

Nordita Open day, Stockholm

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Hanlin Sun

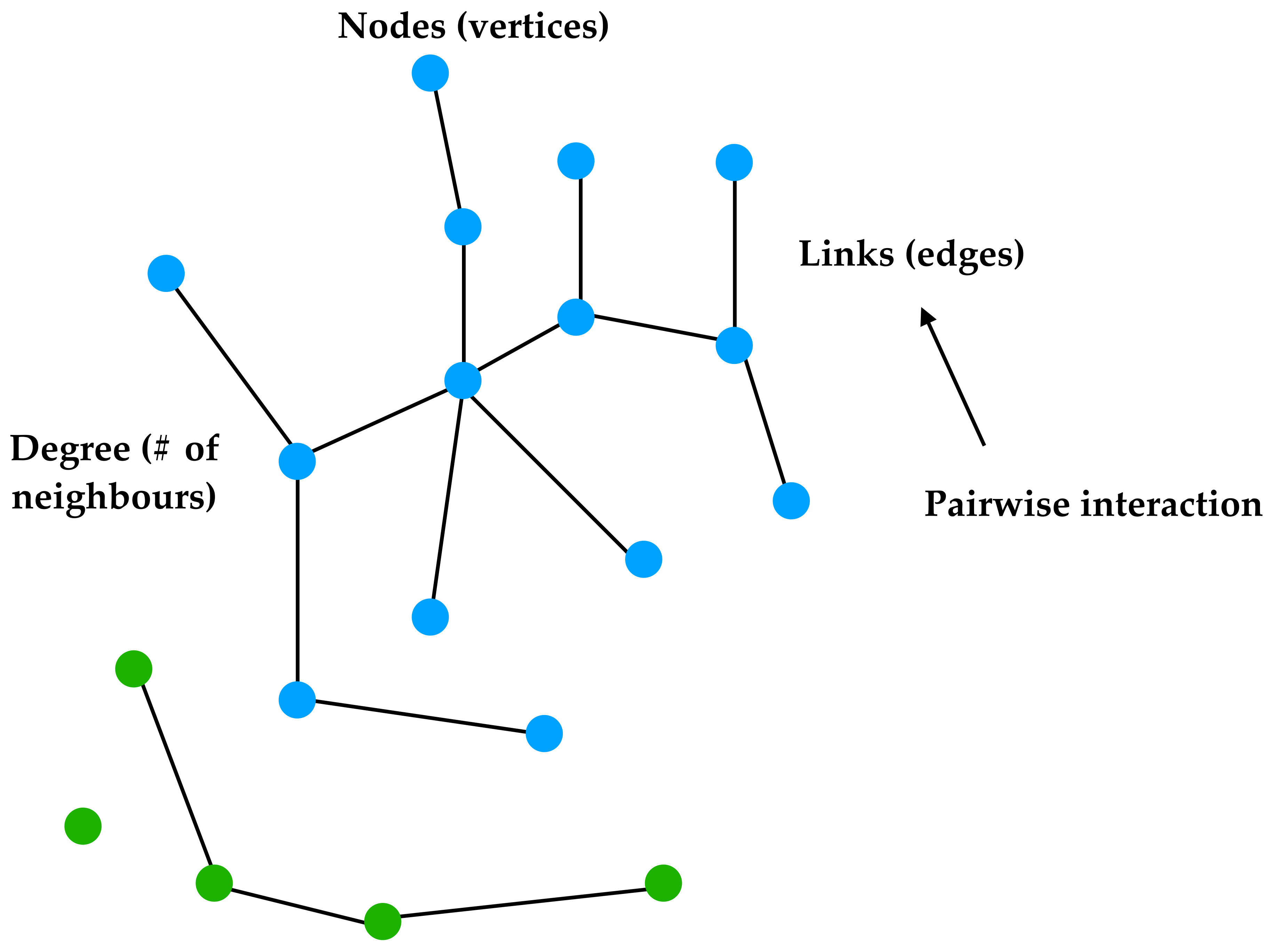
WINQ Research fellow

NORDITA

WINQ | Wallenberg Initiative on Networks and Quantum information

NORDITA
Nordic Institute for Theoretical Physics

Networks / Graphs



Networked data

Biological networks

Ecological networks

(Epidemic) Contact networks

Social networks

Urban networks

Financial network

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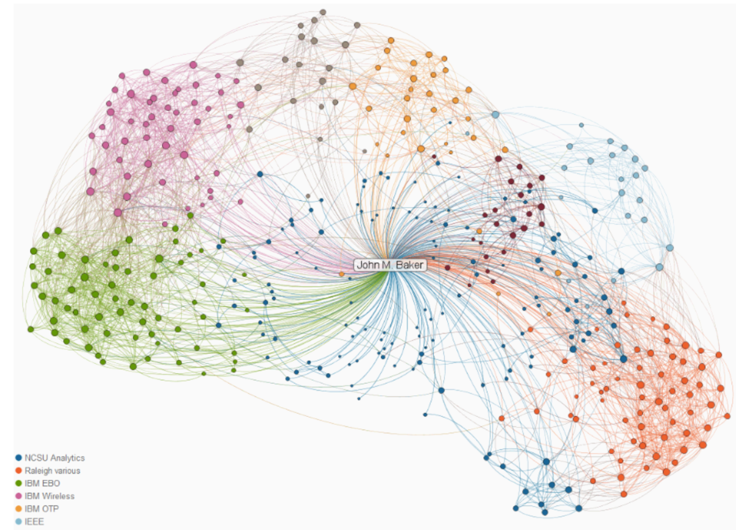
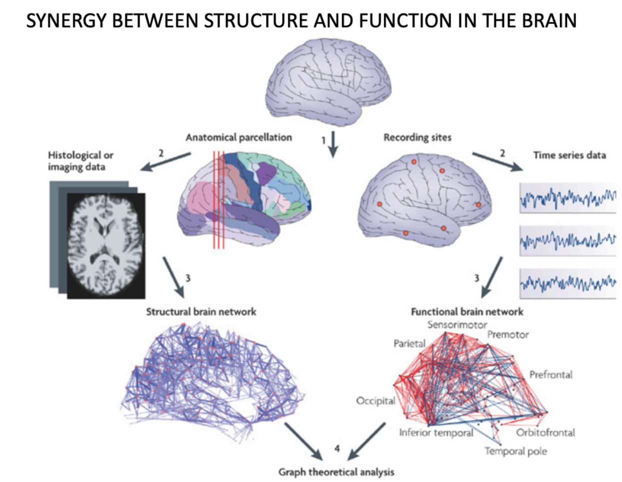
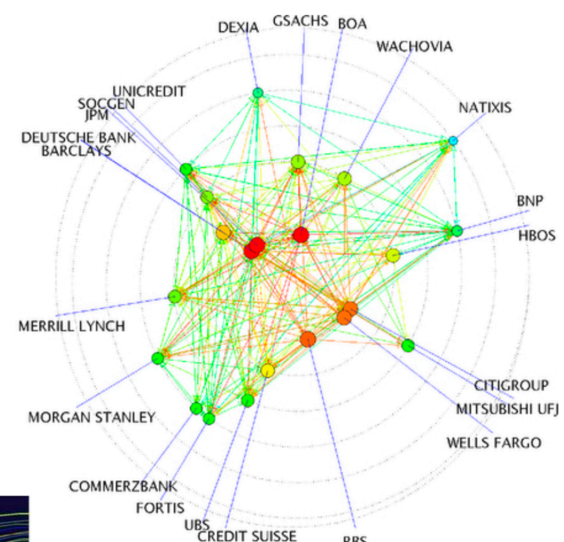
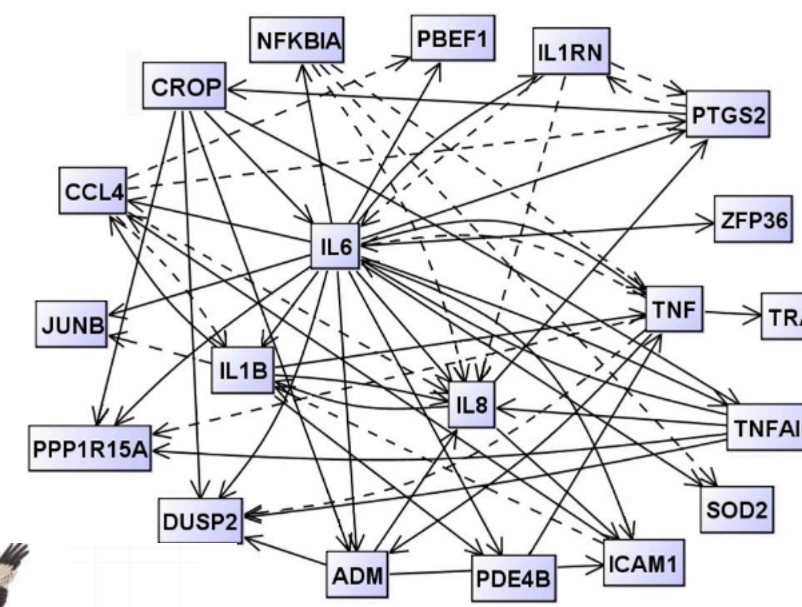
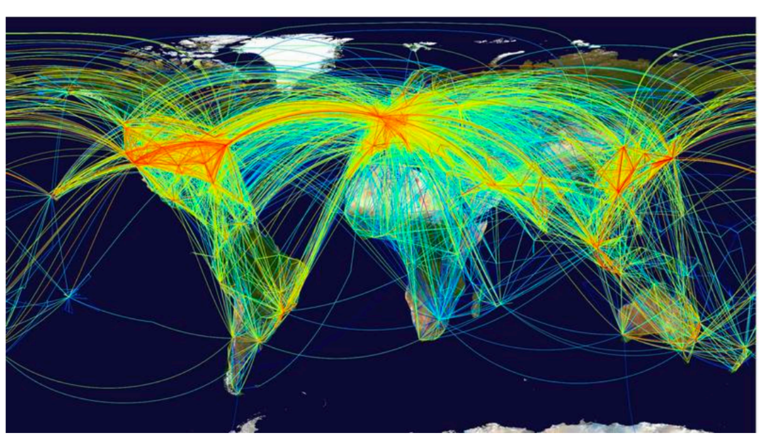
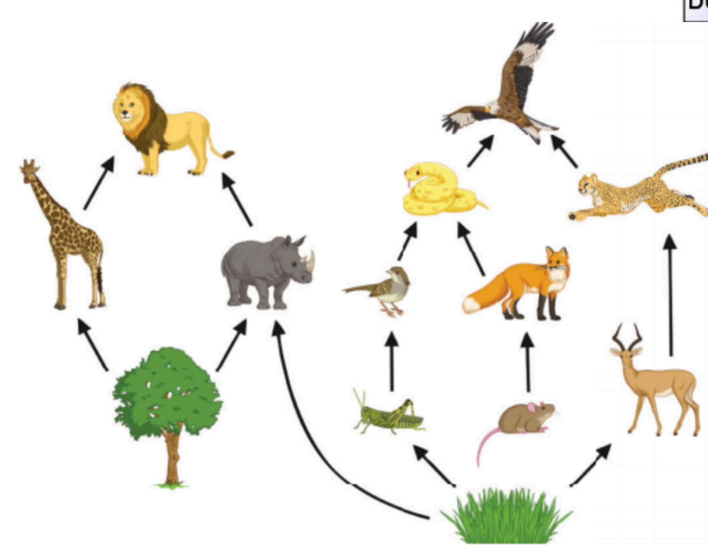








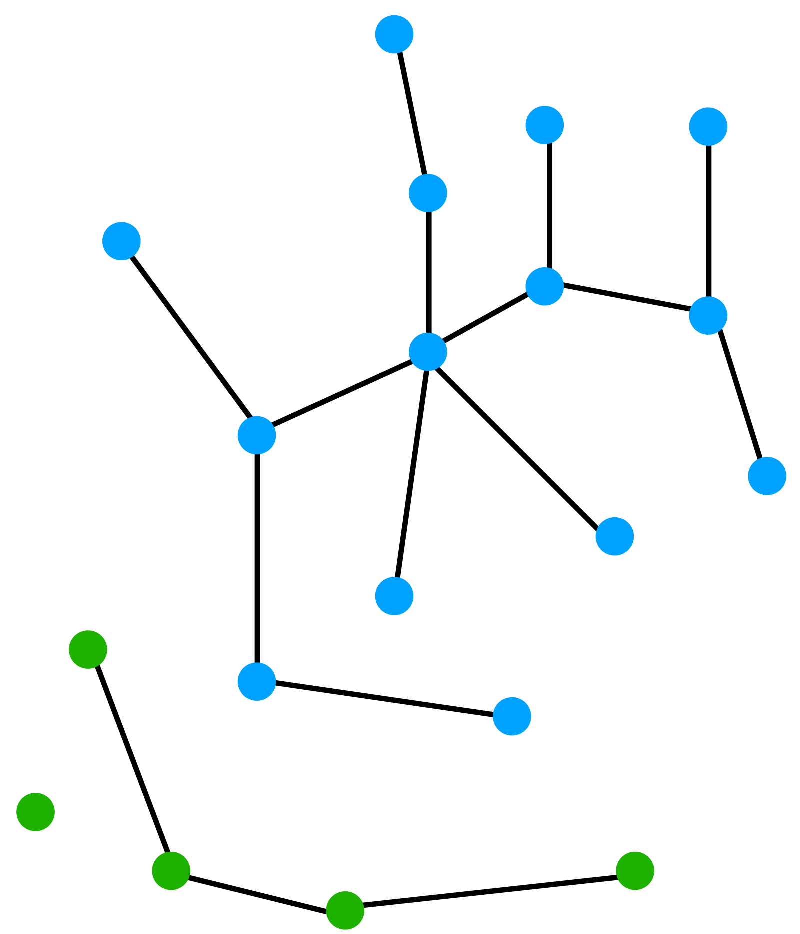




Random graph theory to network theory

Random graph theory, extensive study on Erdös-Renyi graph

- **ER network:** $G(N, p)$ **or** $G(N, m)$
- **Poisson degree distribution,** $\langle k \rangle = \sigma^2$
- **Robustness**
- **……**



Real network?

- **Very robust (WWW)**
- **Huge fluctuation of degrees (scale-free)**
- **Power-law degree distribution** $(2 < \gamma < 3)$
- **……**

Network structure

How to characterise the network structure?

Size: number of nodes/links

Degree: Max, min, average, distribution etc

Diameter: small world?

Centrality: How important is a node? Many measures

Correlation: Preference of connection (High degree-low degree or high degree-high degree?)

Modularity: are there communities that are more densely connected?

……

Higher-order interaction: Co-authorship as an example

Multilayer structure: Transport network between cities

Network function

Pre-requisite of functioning of the network: macroscopic connectivity

Dynamical processes taking place on the network

Diffusion (transportation)

Spreading of virus (Epidemiology)

Synchronisation (brain network, power system)

Social processes (political opinion, innovation)

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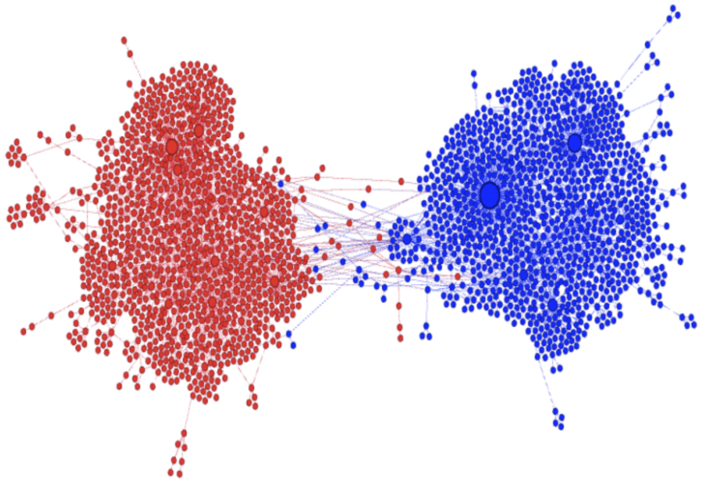
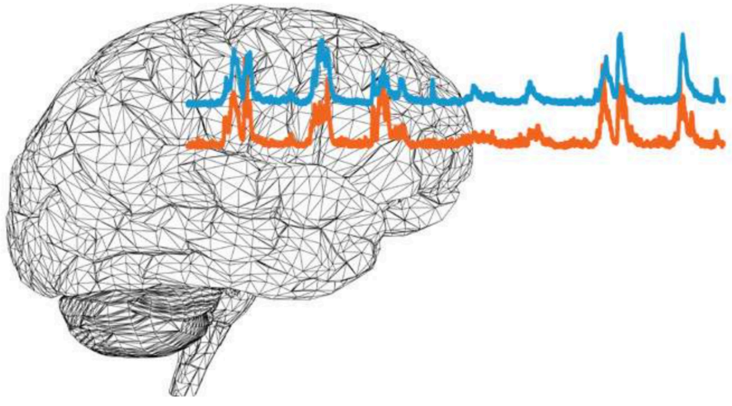




Network structure and network function

The interplay between network structure and network function?

What type of network is more risky in a pandemic?

Which node (individual) is more risky in a epidemic spreading process?

How long it takes to diffuse?

The robustness of a network under random/targeted attack?

What is the character of an epidemic outbreak on a network? Gradual/Abrupt?

……

Network robustness

Imagine a network is experiencing random failure on nodes/links

Internet: servers are failed/connection between servers breaks down

Transport network: Road is blocked/Airport is closed

……

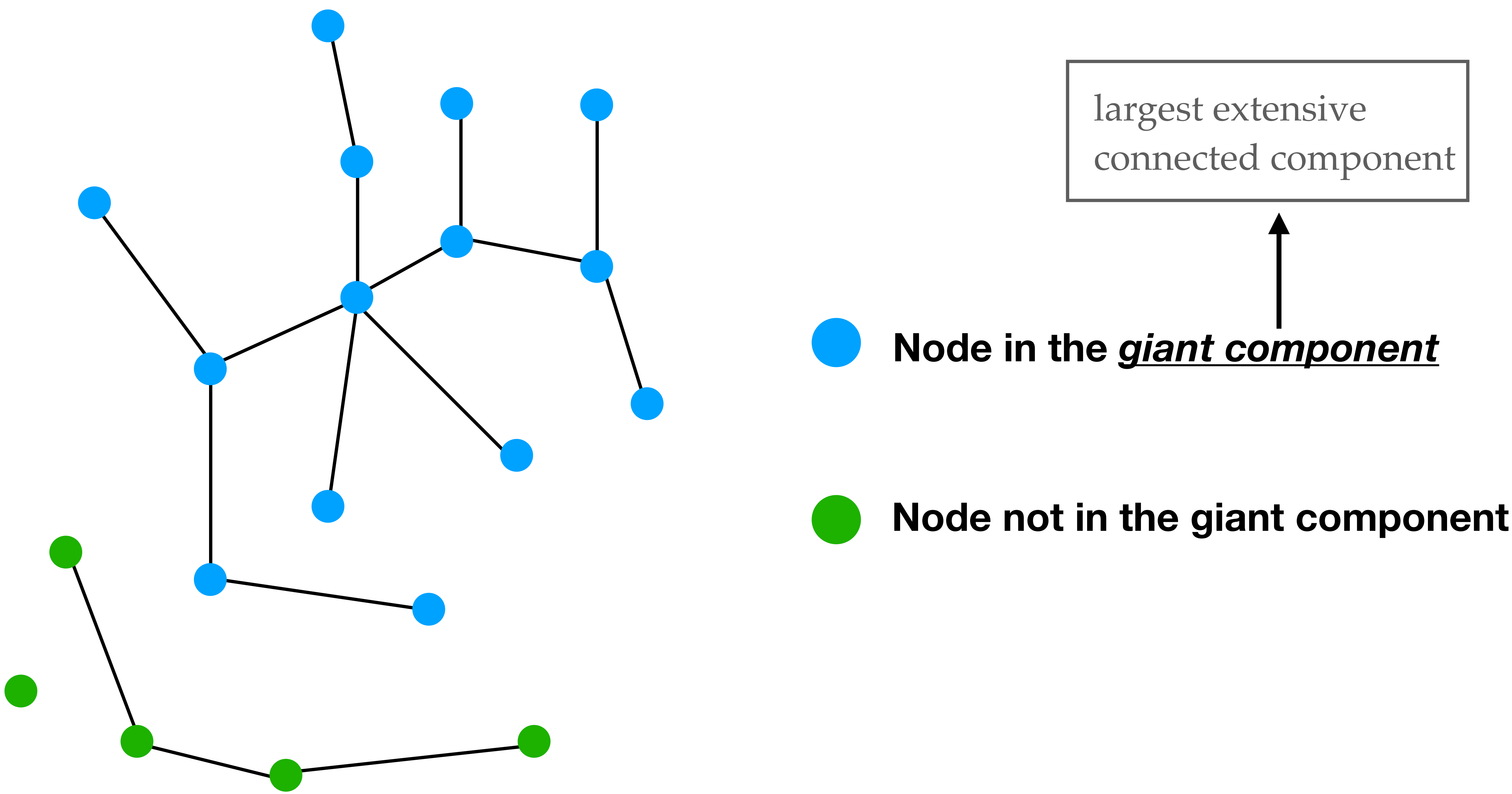
Can this network remain functioning?

A network is more robust if it remains functioning at a higher level of random failure

A network is functioning if it is "largely" connected. The size of largest connected part is comparable to the entire network.

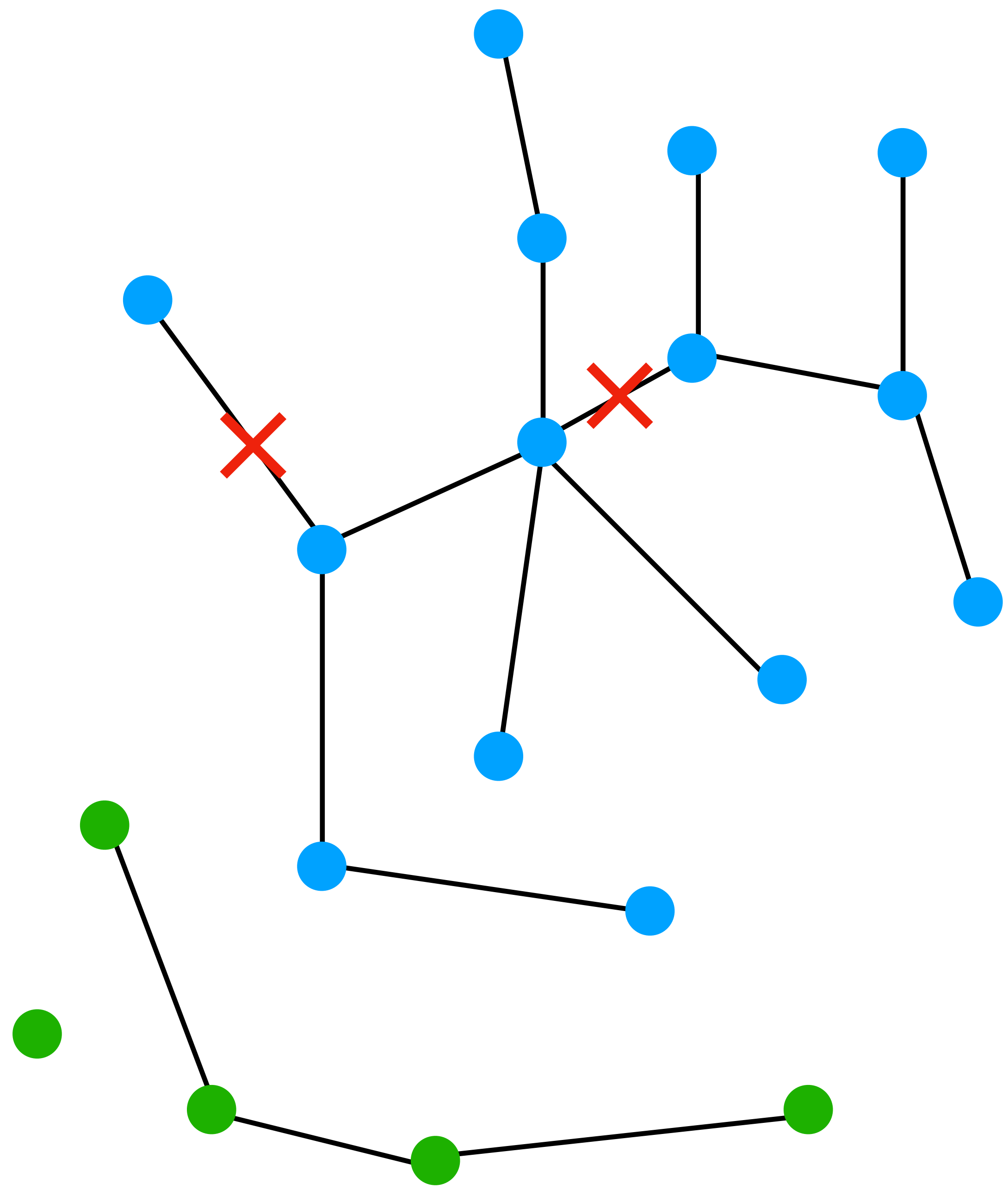
Percolation on (sparse) networks

Percolation theory: evaluate the robustness of a network against random failure



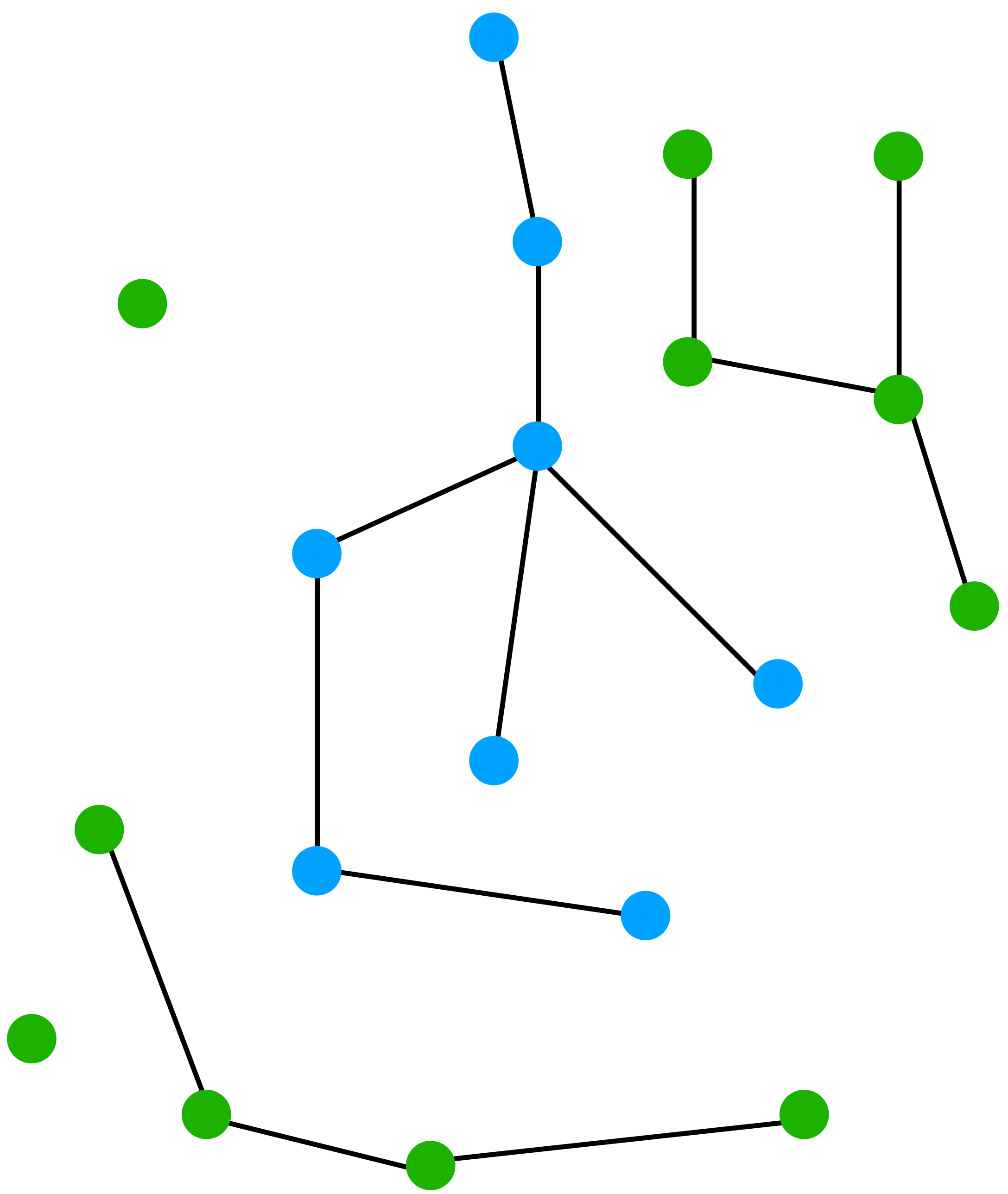
Percolation on networks

Percolation theory: evaluate the robustness of a network against random failure

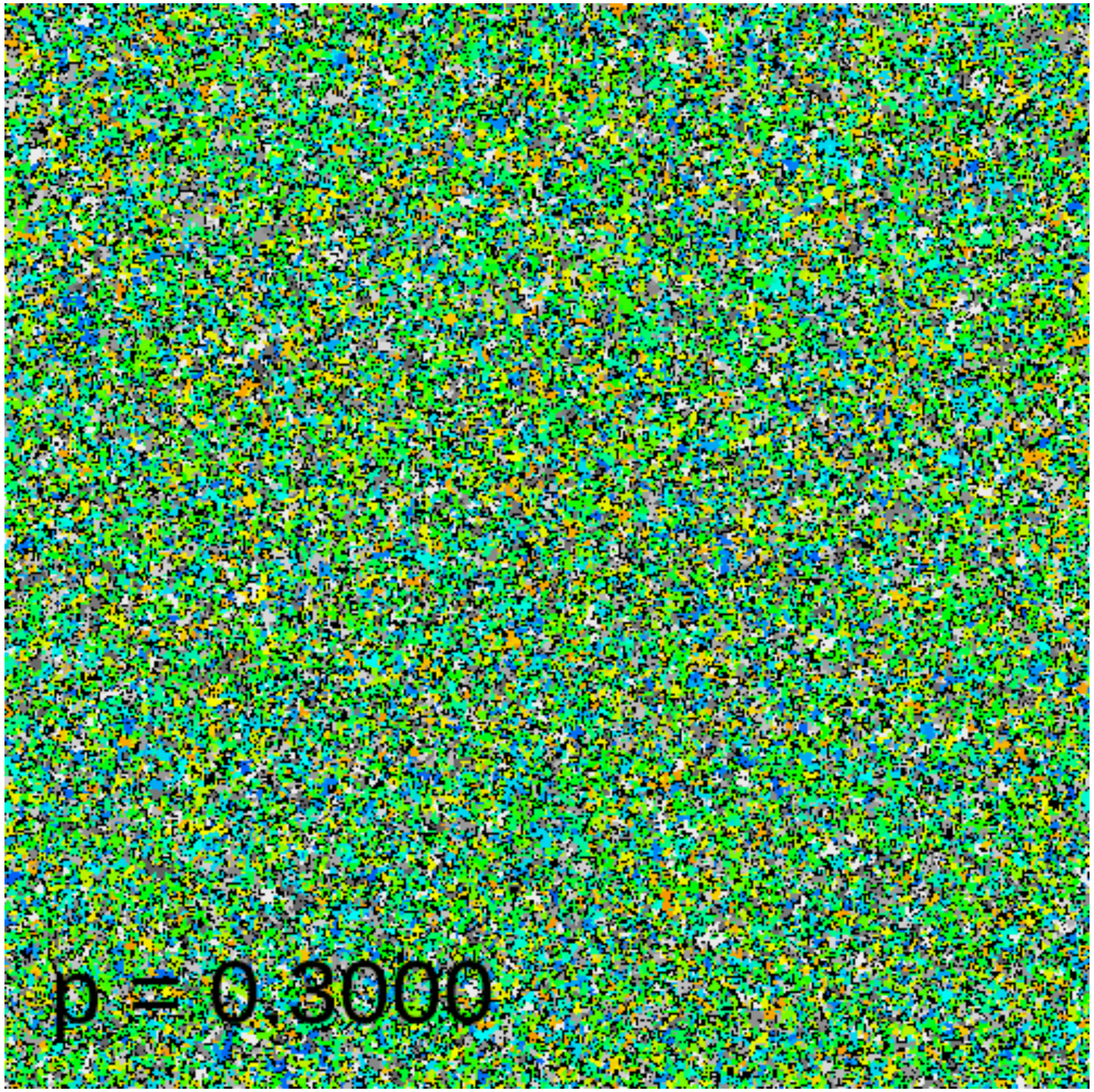


Percolation on networks

Percolation theory: evaluate the robustness of a network against random failure



Percolation (on lattices)



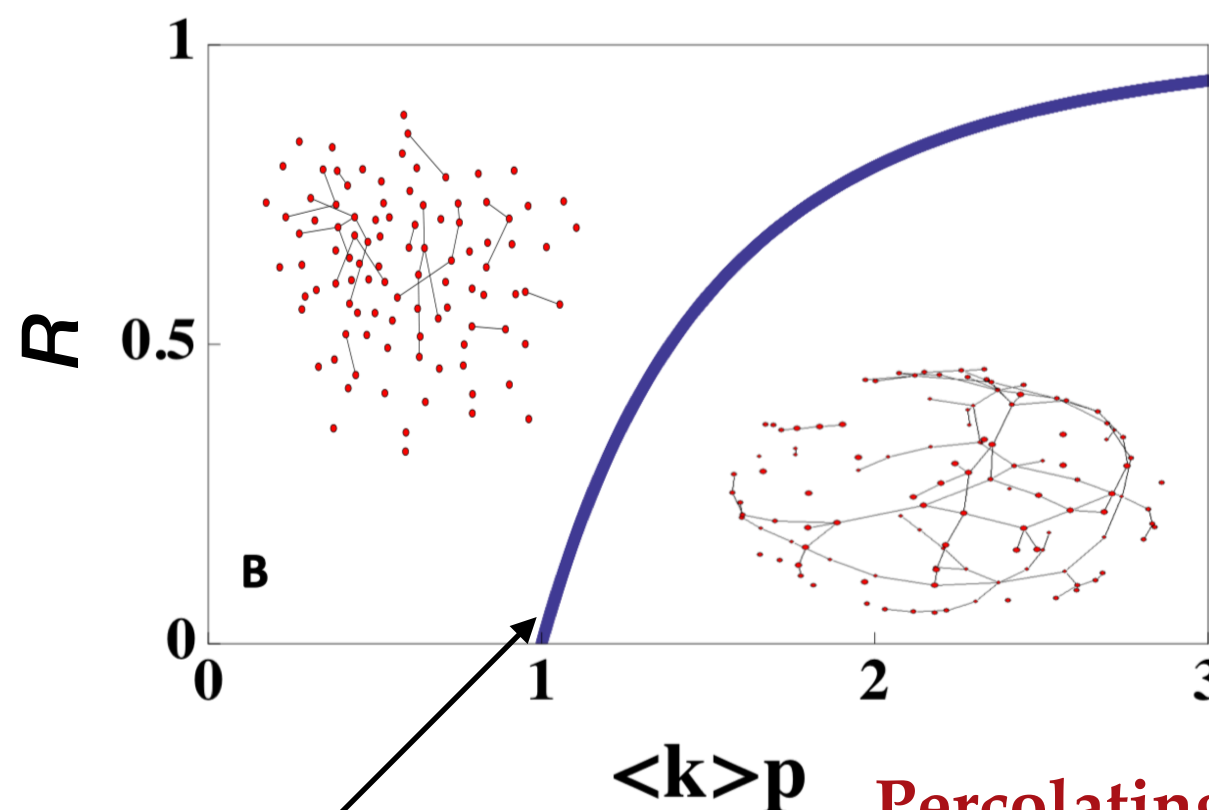
- 2-dimensional lattice
- Links are occupied (connected) with probability *p*

Percolation on networks

Mathematically,

- Nodes/Links are removed with probability $q = 1 - p$
- R: fraction of nodes in the giant component $R = \lim_{N \to \infty} \frac{N_{GC}}{N}$
- Phase transition: Percolating phase and non-percolating phase
- Nodes in the giant components are regarded functioning

Non-percolating phase



Percolating phase

Continuous phase transition

Percolation and Epidemic spreading

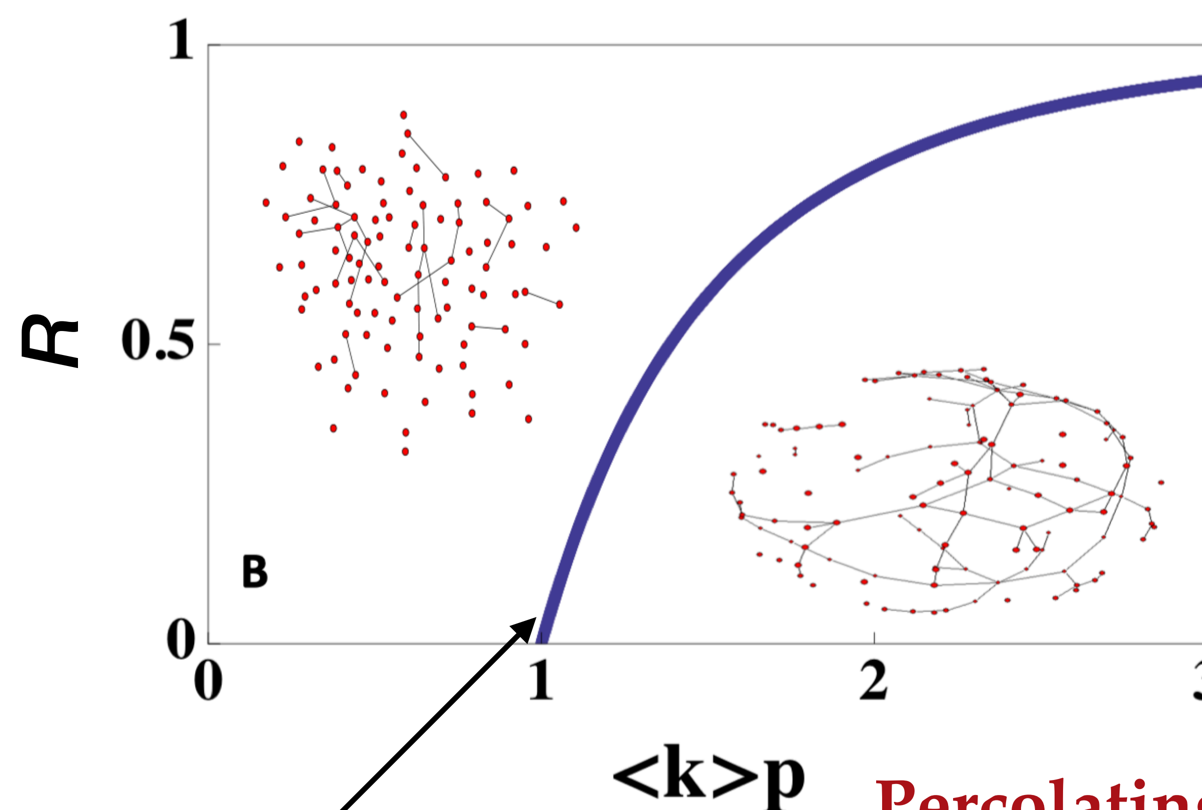
Mathematically,

- Nodes/Links are removed with probability $q = 1 - p$

- R: fraction of nodes in the giant component $R = \lim_{N \to \infty} \frac{N_{GC}}{N}$

- Phase transition: Percolating phase and non-percolating phase

Non-percolating phase



Percolating phase

Continuous phase transition

SIR model → $T = p$ → Link percolation

T: transmissibility, i.e., probability that an infection signal is sent via a link

Percolation and Epidemic spreading

SIR model

S Susceptible

I Infected

R Recovered

$$S_i + I_j \xrightarrow{\beta} I_i + I_j$$

$$I_i \xrightarrow{\mu} R_i$$

SIR model $\xrightarrow{T = p}$ Link percolation

T: transmissibility, i.e., probability that an infection signal is sent via a link

Link percolation on Poisson network

$$S = \sum_k \frac{kP(k)}{\langle k \rangle}\left[1 - \left(1 - pS\right)^{k-1}\right] = 1 - G_1(1 - pS)$$

$$R = \sum_k P(k)\left[1 - \left(1 - pS\right)^{k}\right] = 1 - G_0(1 - pS)$$

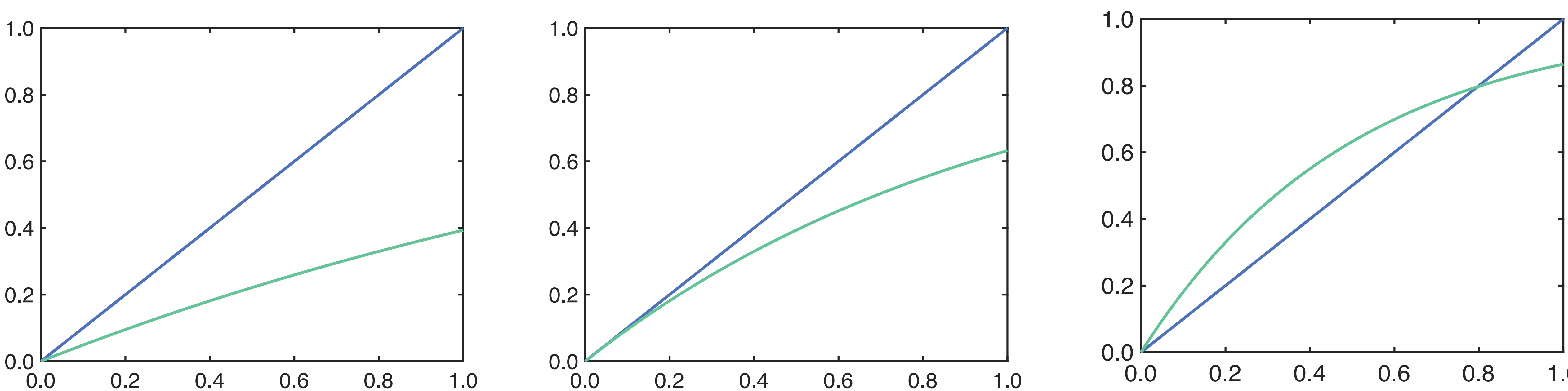
Poisson network: $P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$$S = 1 - G_1(1 - pS) = 1 - e^{-\lambda p S}$$

Critical condition

$$\frac{\mathrm{d}}{\mathrm{d}S} S \,\Big|_{S=0} = \frac{\mathrm{d}}{\mathrm{d}S}(1 - e^{-\lambda p_c S})\,\Big|_{S=0}$$

$$p_c = 1/\lambda = 1/\langle k \rangle$$



$p < p_c$ $\quad$ $p = p_c$ $\quad$ $p > p_c$

- **The phase transition is continuous**

Characterisation of the phase transition

Critical point $p = p_c$

Critical exponent

Order parameter R

$$R - R_c \sim (p - p_c)^{\beta}$$

Average (finite) cluster size $\langle s \rangle$

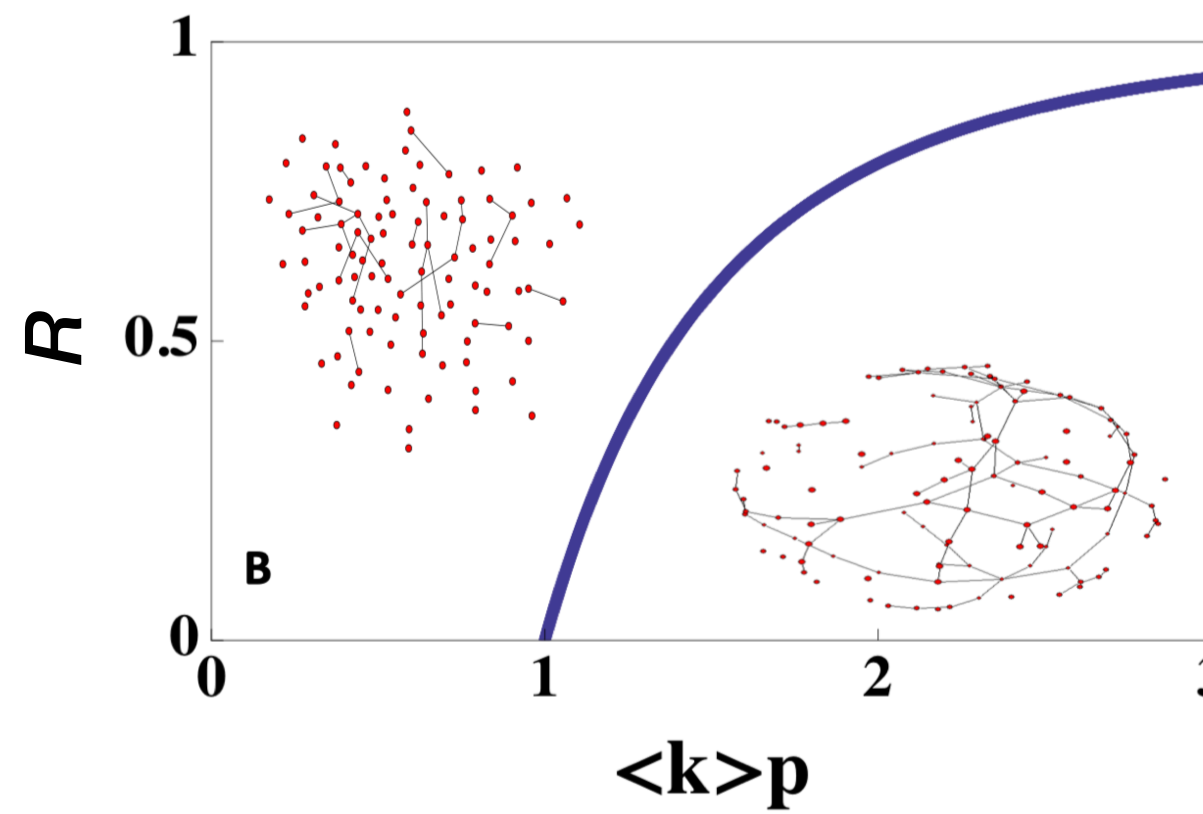
$$\langle s \rangle \sim \quad p - p_c \quad ^{-\gamma}$$

Correlation length ξ **(mean distance between nodes in the same cluster)**

$$\langle \xi \rangle \sim \quad p - p_c \quad ^{-\nu}$$

Size distribution

$$n_s \sim s^{-\tau} e^{-s/s^*}$$



$\beta = 1$ Mean-field class

$\beta = 1/2$ Discontinuous hybrid transition

$\beta = \dfrac{1}{3 - \alpha}$ Power-law degree distribution $2 < \alpha < 3$

$\beta = \dfrac{1}{\alpha - 3}$ Power-law degree distribution $3 < \alpha < 4$

Current research on percolation

The research of percolation today?

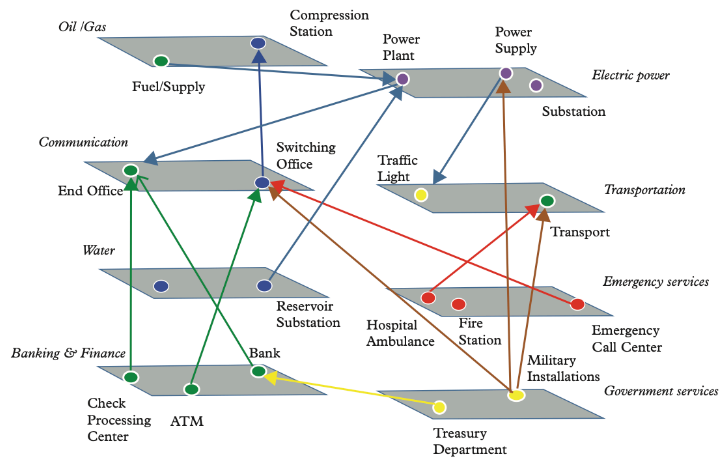
Percolation on Multilayer networks

Percolation on Higher-order networks

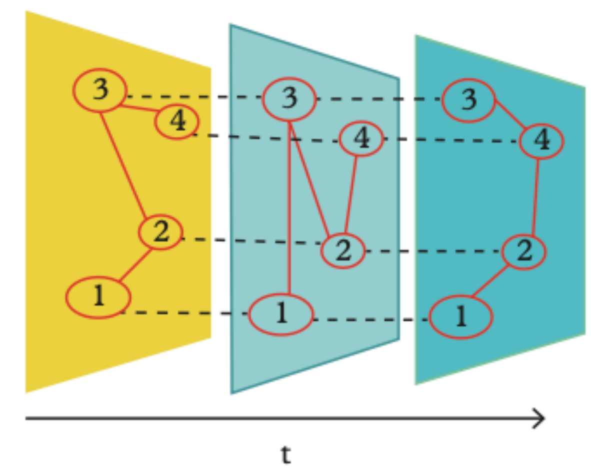
……

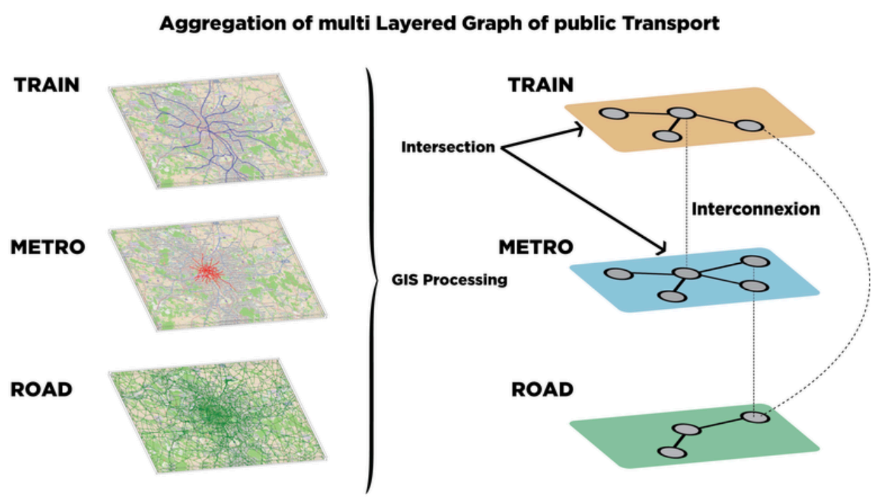
K-core percolation	*Dorogovtsev et al, PRL, 2006*
Explosive percolation	*Achlioptas et al, Science, 2009*
Weak percolation	*Baxter et al, PRE, 2014*
Homological percolation	*Bobrowski et al, PRE, 2020*

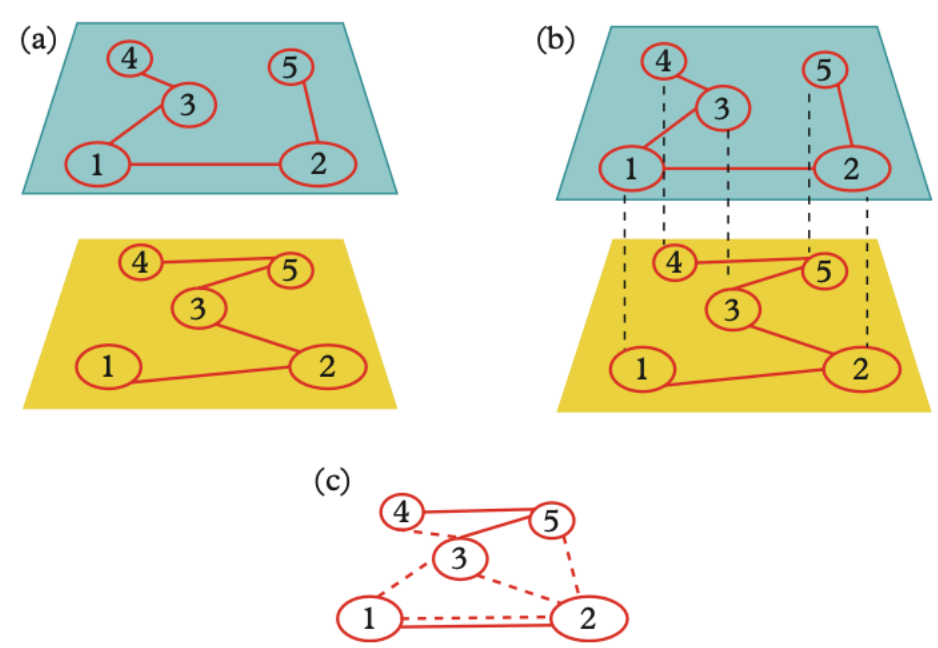
Multilayer Networks

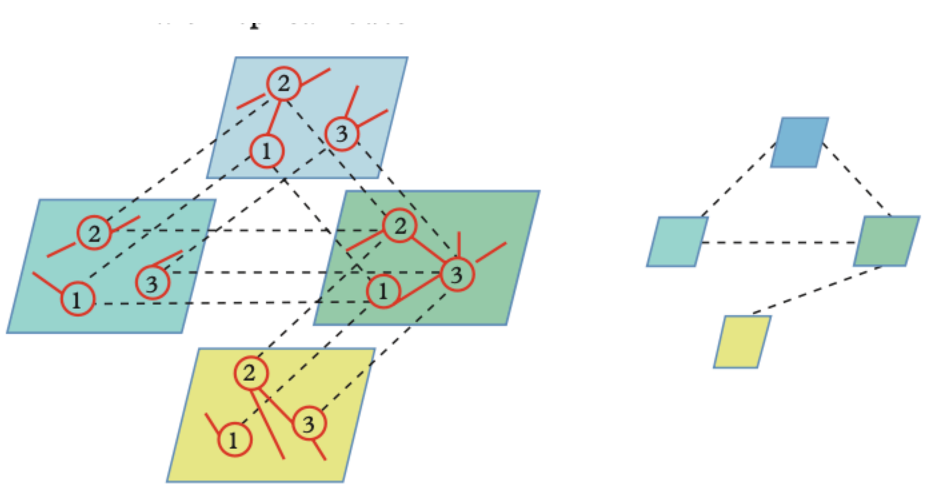


- General multilayer network
- Multiplex network
- Multi-slice network
- Network of network

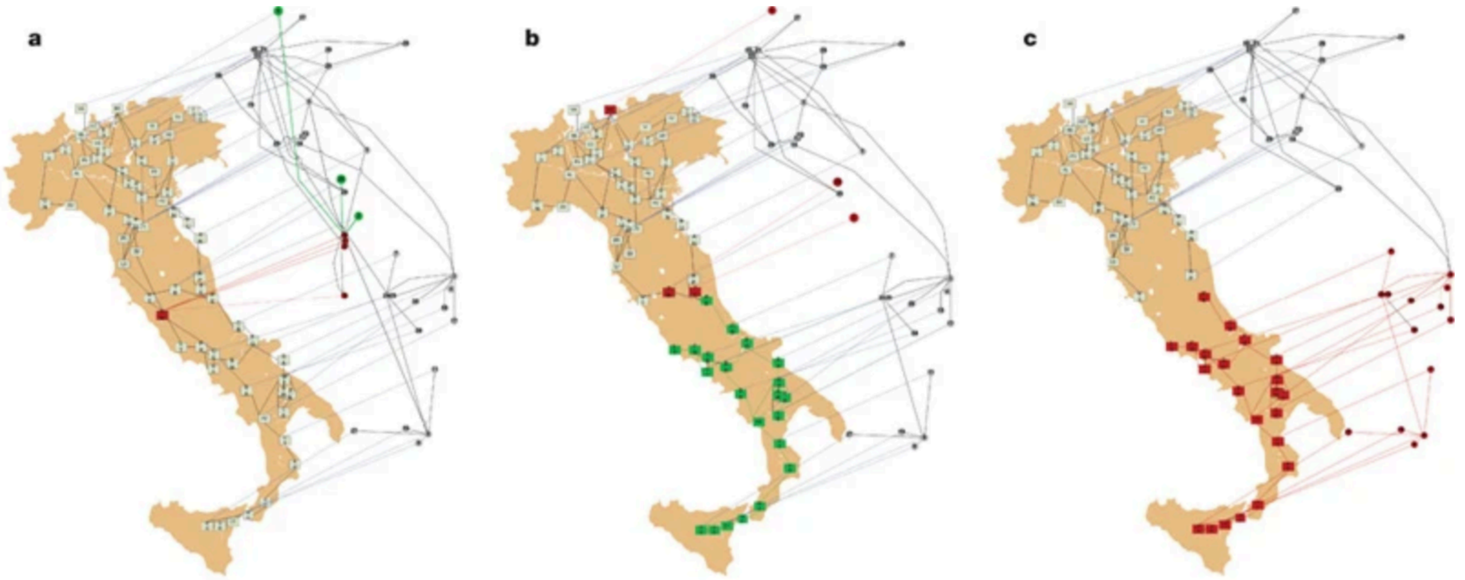








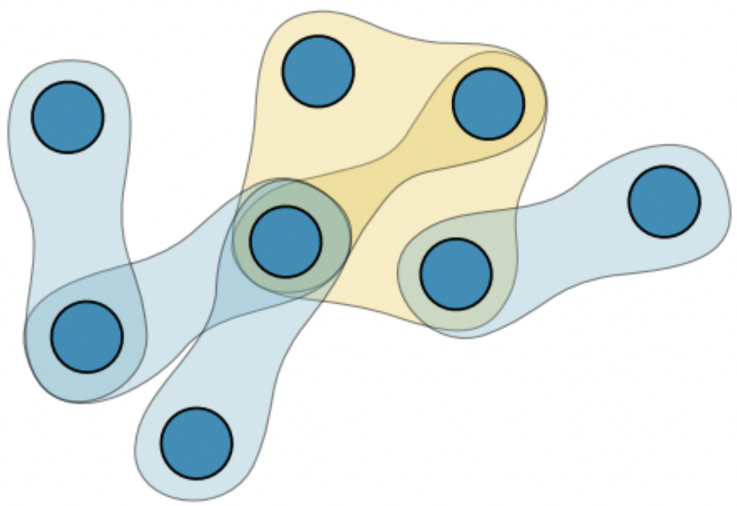
Percolation on multilayer networks



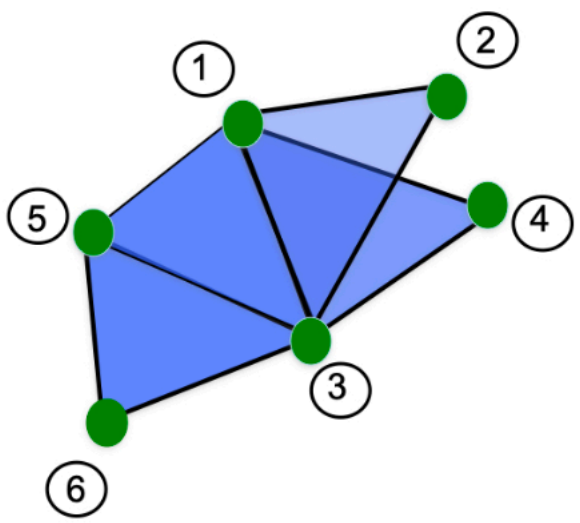
- **Multiplex networks: replica nodes**
- **Interdependent (node) percolation: A node is active if it is in the giant component in all layers**
- **Cascading failure: failure of a node in one layer will cause the failure in other layers**
- **Discontinuous phase transition**

Higher-order Networks

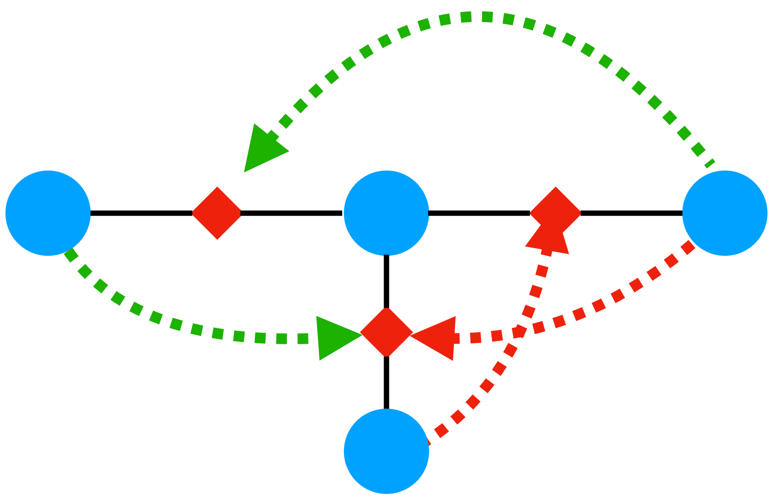
Beyond pairwise interactions



Hypergraphs

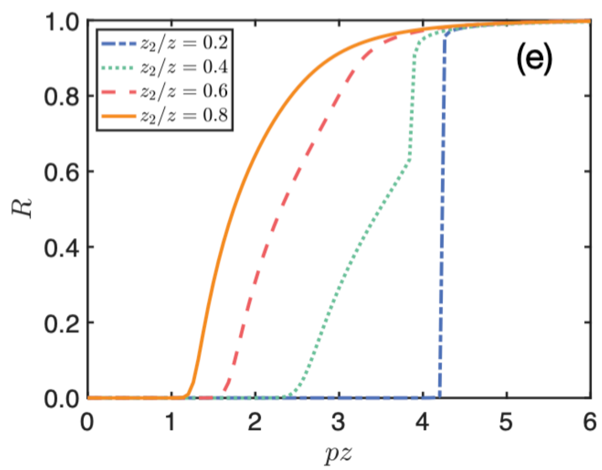


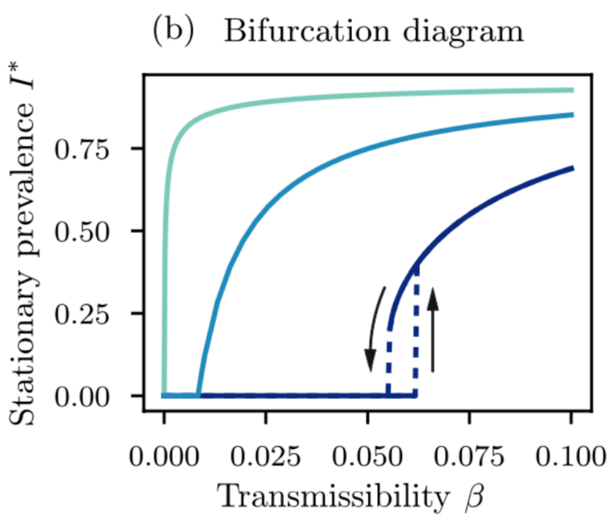
Simplicial complexes

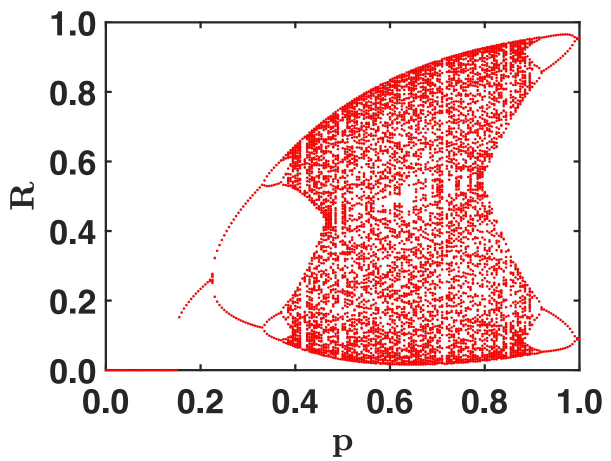


Network with triadic interactions

Percolation on higher-order networks







- Discontinuous hybrid transitions
- Multiple transitions
- Bi-stability (epidemic spreading)
- Orbit diagram
- Unusual critical component (on networks with hierarchical structures)
- ……

Contact, collaboration, news

Thank you!

For more information / collaboration / discussion, contact me:

hanlin.sun@su.se

Room 5208, Floor 5, NORDITA

Join NetPLACE for interesting talk and discussion!