We unravel how **ETH** breaks down when one approaches the boundaries of ergodicity and introduce a scenario in many-body quantum systems, dubbed the **fading ergodicity** regime, that links the breakdown to the non-ergodic behavior.

arXiv:2407.16773

Fading ergodicity

Maksymilian Kliczkowski[1, 2, 3], Rafał Świętek[1, 2], Miroslav Hopjan[2], and Lev Vidmar[1, 2]

[1] *Department of Physics, Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia*
[2] *Department of Theoretical Physics, J. Stefan Institute, SI-1000 Ljubljana, Slovenia*
[3] *Institute of Theoretical Physics, Wrocław University of Science and Technology, 50-370 Wrocław, Poland*
E-mail: maksymilian.kliczkowski@pwr.edu.pl, lev.vidmar@ijs.si

1 Introduction and ETH ansatz

- The quantum chaos conjecture links the emergence of RMT statistics in quantum many-body systems to chaotic dynamics in their classical limit [1,2],
- RMT predictions also apply to the (spectral) statistics of systems without classical counterparts [3-5].

Experiments on nonequilibrium dynamics of isolated systems typically cannot access spectral properties but can measure local observables [6].

The central role is played by the eigenstate thermalization hypothesis (ETH) [7-9], which:

- simply explains the agreement between the *observable* expectation values and the predictions of statistical ensembles [10],
- originates from their analysis in *random pure states* [10] – suggesting thermalization on a level of *eigenstates*.

Hamiltonian eigenstates $\hat{H}|n\rangle = E_n|n\rangle$ are not random pure states and hence (for eigenstates)

- the ETH of an observable $\hat{O}$ contains non-trivial refinements beyond the RMT
- represent the structure function $O(\bar{E})$ of the diagonals (at mean energy $\bar{E} = (E_n + E_m)/2$), and
- the envelope function $f(\bar{E}, \omega)$, where $\omega = E_n - E_m$ is the energy difference ($\hbar \equiv 1$).

These, combined with the random fluctuations R_{nm}, give rise to the conventional ETH ansatz [11],

$$\langle n|\hat{O}|m\rangle = O(\bar{E})\delta_{m,n} + \rho(\bar{E})^{-1/2} f(\bar{E}, \omega) R_{nm}\,. \qquad (1)$$

The many-body *density of states* $\rho(\bar{E})$ exponentially suppresses fluctuations with lattice size L.

2 Motivation and ETH breakdown

For counterexamples, the ETH ansatz likely fails: (a) fluctuations may decay polynomially with L, and (b) some eigenstates (*outliers*) may not match *microcanonical averages*. – weaker forms of ETH that may apply to: (i) integrable systems, (ii) single particle chaos, (iii) many-body scars, (iv) Hilbert space fragmentation.
– **all incompatible** with *ergodicity* and *thermalization*

Here, we answer the fundamental questions:

- How and when does conventional ETH evolve into weaker forms as ergodicity fades?
- How it relates to breakdown of RMT-like short-range statistics?
- What happens to fluctuations near the ergodicity boundary?

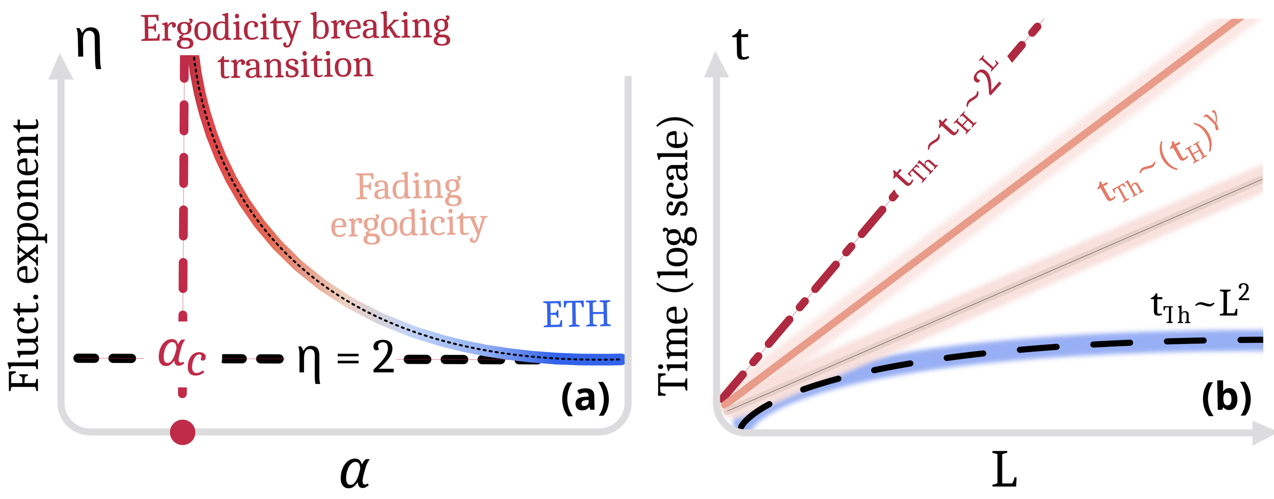

Fig. 1 Fading ergodicity scenario. **(a)** Divergence of the *fluctuation exponent* η as a function of the control parameter α, when approaching EBT at $\alpha = \alpha_c$. **(b)** While the *Thouless time* $t_{\rm Th}$ is proportional to the *Heisenberg time* t_H at the transition point ($t_{\rm Th} \sim t_H$), and it is much smaller than t_H in the conventional ETH regime (e.g., $t_{\rm Th} \sim L^2$), it scales as $t_{\rm Th} \sim (t_H)^\gamma$, with $0 < \gamma < 1$, when the boundary of ergodicity is approached. $\eta = 2$ when conventional ETH applies.

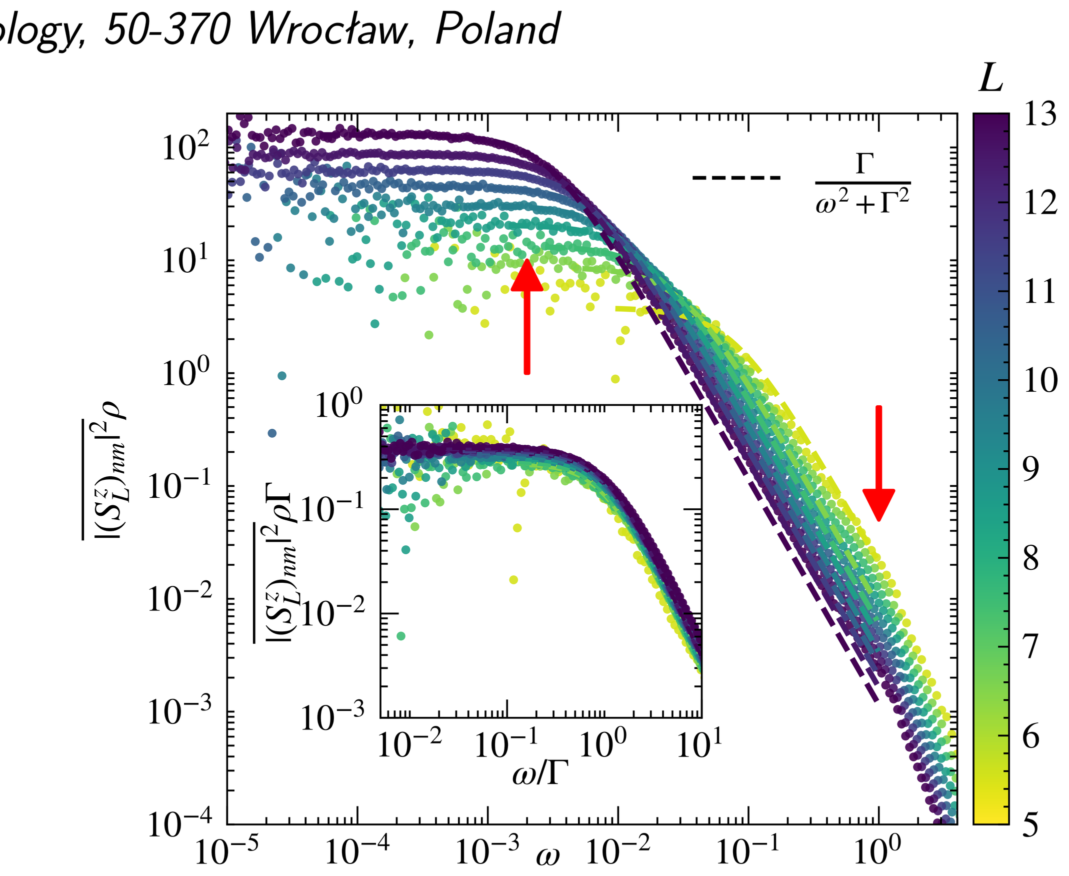

Fig. 2 Coarse-grained *off-diagonal* matrix elements of $\overline{|(\hat{S}^z_L)_{nm}|^2}$ at $\alpha = 0.86$ and different L. Main panel: $\overline{|(\hat{S}^z_L)_{nm}|^2}\rho$ vs ω, where $\rho \propto 2^L$. Dashed lines are fits to the Lorentzian function [Eq. (3)], from which we extract Γ. The red arrows highlight the weight accumulation (depletion) at low (high) ω. Inset: Data collapse – $\overline{|(\hat{S}^z_L)_{nm}|^2}\rho\Gamma$ vs ω/Γ.

We establish the fading ergodicity regime, intermediate between the completely non-ergodic and conventional ETH limits and propose an observable-based precursor of the ergodicity breaking phase transition (EBT).

The approach to EBT in physical systems is better understood through *spectral properties*. Here, one can define:

- the *Thouless energy* Γ – distinguishes short- from long-range spectral statistics.
- the *mean level spacing* Δ (*Heisenberg energy*) – short-range statistics follow RMT predictions, while long-range do not, with Γ shrinking to Δ at EBT.
- *Thouless time* $t_{\rm Th} \propto 1/\Gamma$ and *Heisenberg time* $t_H \propto 1/\Delta$.

In the fading ergodicity regime, **the system remains *ergodic* beyond conventional ETH**, while fluctuations of the *diagonal* and low-ω *off-diagonal* matrix elements **soften**.

3 Softening of ETH at small ω

- In the non-ergodic regime, matrix element weight is expected to accumulate in the diagonal elements, c.f., sum rule

$$\frac{1}{\mathcal{D}} \sum_{n,m=1}^{\mathcal{D}} |O_{nm}|^2 = 1\,, \qquad (2)$$

- Deviations from Eq. (1) occur despite the system being *ergodic* and short-range level statistics following RMT predictions.
- The fluctuating part in Eq. (1) acquires ω-dependence,

$$\rho(\bar{E})^{-1/2} \;\to\; \Sigma(\bar{E}, \omega \to 0, L) \;\to\; \rho(\bar{E})^{-1/\eta},$$

with $2 < \eta < \infty$ in the fading ergodicity regime.

- We consider a Lorentzian functional form of low-ω ($\Delta < \omega < \Gamma$) matrix elements, **with characteristic energy scale Γ, c.f., *Thouless energy.***

$$\overline{|O_{nm}|^2}\rho = \frac{\Gamma}{\Gamma^2 + \omega^2} = \frac{1}{1 + (\omega/\Gamma)^2} \cdot \frac{1}{\Gamma}\,. \qquad (3)$$

and, as a main result of our study,

$$\overline{|O_{nm}|^2} \propto \frac{1}{\rho\Gamma} \approx \frac{\Delta}{\Gamma}\,. \qquad (4)$$

Whenever *Thouless energy* increases as $\Gamma \propto \Delta^\zeta$, with $0 < \zeta < 1$, the system is still *ergodic* but the ETH does not hold in the conventional way and $\eta = 2/(1-\zeta) > 2$ and

η **diverges** at EBT, at which $\Gamma \propto \Delta$.

4 Quantum Sun Model

We provide both *analytical* and *numerical* evidence in the Quantum Sun Model [12-15], that hosts EBT in the thermodynamic limit.

$$\hat{H} = \hat{H}_{\rm dot} + \sum_{j=1}^{L} \alpha^{u_j} \hat{S}^x_{n(j)} \hat{S}^x_j + \sum_{j=1}^{L} h_j \hat{S}^z_j\,, \qquad (5)$$

- $\hat{H}_{\rm dot}$ is a $2^N \times 2^N$ matrix drawn from the Gaussian orthogonal ensemble (GOE) – all-to-all interaction within an *ergodic* dot ($N = 3$).
- 2nd term – coupling between a spin j outside the dot ($j = 1, ..., L$) and a random spin $n(j)$ within the dot – α tunes the EBT, $u_j \propto j$. 3rd term is represented by the disorder h_j,
- has $\tilde{\alpha}_c = 1/\sqrt{2}$ – critical point derived within the avalanche theory,
- allows for using close form expression for the Thouless time $\Gamma \propto \exp\left(-\ln\left(\frac{1}{\alpha^2}\right)L\right)$,
- leading to [see Eq. (4)]

$$\overline{|O_{nm}|^2} \propto e^{-\ln\left(\frac{\alpha^2}{\tilde{\alpha}_c^2}\right)L} \to \eta = 2\left(1 - \frac{\ln\alpha}{\ln\tilde{\alpha}_c}\right)^{-1}\,. \qquad (6)$$

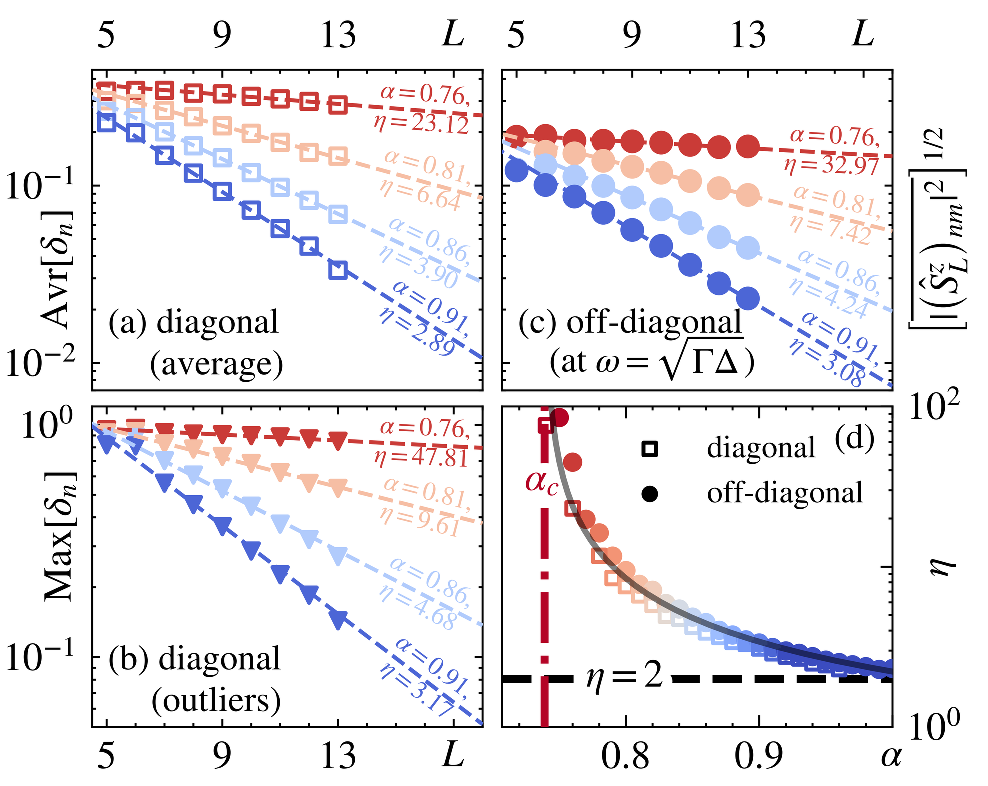

Fig. 3 Scaling of fluctuations of matrix elements. **(a), (b)** Eigenstate-to-eigenstate fluctuations of the *diagonal* matrix elements, $\delta_n \equiv |(\hat{S}^z_L)_{n+1} - (\hat{S}^z_L)_n|$. **(c)** Low-$\omega$ *off-diagonal* matrix elements. Dashed lines in (a)-(c) are fits of $a_0 2^{-L/\eta}$. **(d)** Fluctuation exponents η from Eq. (6) as a function of α. The solid line is a fit of $b_0\eta^*$ to the results for the *off-diagonal* matrix elements.

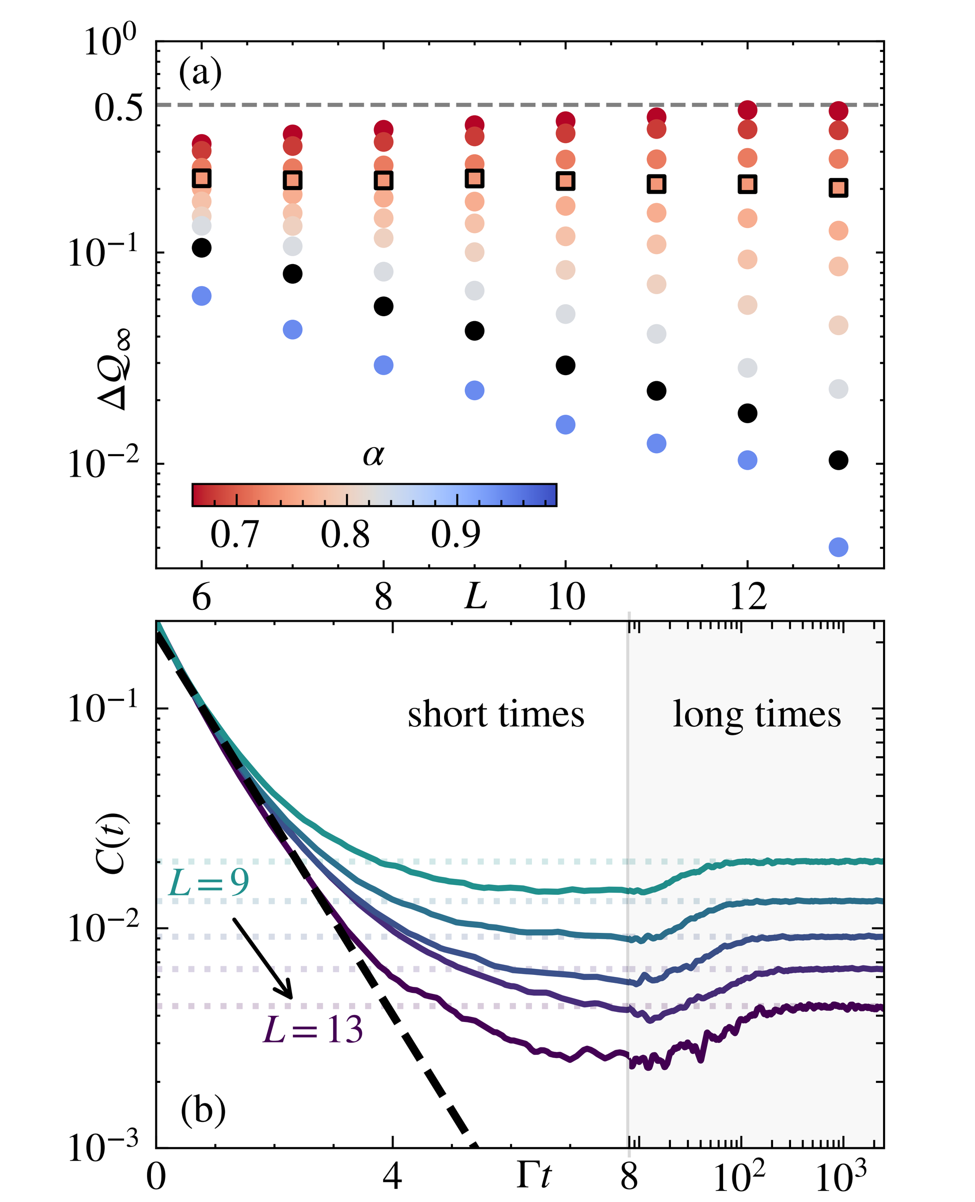

Fig. 4 Quantum dynamics. **(a)** Difference ΔQ_∞, between the *microcanonical ensemble* and the *diagonal ensemble* prediction after a quantum quench $Q(t)^{(\mu)} \equiv \langle\psi_0|\hat{S}^z_L(t)|\psi_0\rangle$, vs L; Squares: $\alpha = 0.74 \approx \alpha_c$, at which η in Fig. 3(d) diverges, $\eta \to \infty$. Black circles: $\alpha = 0.86$ (also studied in Fig. 2). **(b)** Autocorrelation function $C(t) \equiv \langle\hat{S}^z_L(t)\hat{S}^z_L(0)\rangle_\mu$ (brackets $\langle\cdots\rangle_\mu$ denote both quantum expectation value and the disorder average) at $\alpha = 0.86$ and different L, as function of Γt.

[1] O. Bohigas, M. J. Giannoni, C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[2] G. Casati, F. Valz-Gris, I. Guarnieri, Lettere al Nuovo Cimento **28**, 279 (1980)
[3] G. Montambaux, D. Poilblanc, J. Bellissard, C. Sire, Phys. Rev. Lett. **70**, 497 (1993)
[4] L. F. Santos, J. Phys. A. **37**, 4723 (2004).
[5] M. Rigol, L. F. Santos, Phys. Rev. A **82**, 011604 (2010).
[6] I. Bloch, J. Dalibard, S. Nascimbène, Nature Physics **8**, 267 (2012).
[7] M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
[10] L. D'Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, Adv. Phys. **65**, 239 (2016).
[11] M. Srednicki, J. Phys. A. **32**, 1163 (1999).
[12] J. Šuntajs, L. Vidmar, Phys. Rev. Lett. **129**, 060602 (2022).
[13] W. De Roeck, F. Huveneers, Phys. Rev. B **95**, 155129 (2017).
[14] D. J. Luitz, F. Huveneers, W. De Roeck, Phys. Rev. Lett. **119**, 150602 (2017).
[15] J. Šuntajs, M. Hopjan, W. De Roeck, L. Vidmar, Phys. Rev. Res. **6**, 023030 (2024)

Wrocław University of Science and Technology
University of Ljubljana
Jožef Stefan Institute