Quantum vs. classical P-divisibility

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Classical from quantum maps

Classical P-divisibility (stochastic matrices on probability vectors)

$$\boldsymbol{p}_t = T(t)\boldsymbol{p}\,, \qquad \dot T(t) = L(t)T(t)$$
$$T(t) = T(t,s)\,T(s)\,, \qquad t \geq s \geq 0\,,$$
$$T_{ij}(t,s) \geq 0 \quad \sum_i T_{ij}(t,s) = 1,$$

if and only if $L_{ij}(t) \geq 0 \;\; i \neq j, \quad \sum_i L_{ij}(t) = 0$

Quantum P-divisibility (*CPTP maps*)

$$\Lambda_t = \Lambda_{t,s}\,\Lambda_s\,, \qquad \Lambda_{t,s} \text{ PTP}$$

Fix a maximally Abelian subalgebra

$$\mathcal{P} = \{P_i\}_{i=1}^d,\;\; P_iP_j = \delta_{ij}P_j \quad \sum_i P_i = \mathbb{1}.$$

One can define a classical stochastic process out of the quantum one in two-ways:

1. Classical reduction of the **generator**
$$\dot\Lambda_t = \mathcal{L}_t\Lambda_t \;\rightsquigarrow\; L_{ij}(t) = \mathrm{Tr}(P_i\,\mathcal{L}_t[P_j])$$
Theorem (Kossakowski) Λ_t is P-divisible iff
$$L_{ij}(t) \geq 0 \quad i \neq j\,, \qquad \sum_i L_{ij}(t) = 0\,, \quad \forall\,\mathcal{P}.$$
2. Classical reduction of the **dynamical map**
$$T_{ij}(t) = \mathrm{Tr}(P_i\,\Lambda_t[P_j]) \;\rightsquigarrow\; \dot T(t) = L(t)T(t) \quad (1)$$

Question: If the quantum process Λ_t is P-divisible, will the classical process defined by (1) be P-divisible?

Qubit unital dynamics

- Quantum generator $\mathcal{L}_t[\rho] = -i[H_t,\rho] + \mathcal{D}_t[\rho]$,
$$\mathcal{D}_t[\rho] = \frac{1}{2}\sum_{i,j=1}^{3} K_{ij}(t)\Big(\sigma_i\rho\sigma_j - \frac{1}{2}\{\sigma_j\,\sigma_i,\rho\}\Big)$$
- $\Lambda_t[\mathbb{1}] = \mathbb{1}_2 \iff K(t) = K^T(t) \in M_3(\mathbb{R})$.
- Classically reduce on $\mathcal{P} \subset M_2(\mathbb{C})$
$$T(t) = \begin{pmatrix} T_{00}(t) & 1 - T_{00}(t) \\ 1 - T_{00}(t) & T_{00}(t)\end{pmatrix}, \quad P_1 = 1 - P_0$$
- Classical generator:
$$L(t) = \dot T(t)T(t)^{-1} = \frac{\dot T_{00}(t)}{2T_{00}(t) - 1}\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix},$$
$$T(t) \text{ P-divisible} \iff f_t := \frac{\dot T_{00}(t)}{2T_{00}(t) - 1} \leq 0$$
- Matrix representation $\widetilde\Lambda_{ij}(t) = \mathrm{Tr}(\sigma_i\Lambda_t[\sigma_j])$, $P_0 \equiv \frac{1}{2}(\mathbb{1} + \boldsymbol{n}\cdot\boldsymbol{\sigma}) \rightsquigarrow 2T_{00}(t) - 1 = \langle \boldsymbol{n}|\widetilde\Lambda(t)|\boldsymbol{n}\rangle$

✔ Quantum P-div. $\quad -(\widetilde{\mathcal{L}}(t) + \widetilde{\mathcal{L}}^T(t)) \geq 0\,,$

✔ Classical P-div. $\quad f_t = \frac{1}{2}\frac{\langle \boldsymbol{n}|\widetilde{\mathcal{L}}(t)\widetilde\Lambda(t)|\boldsymbol{n}\rangle}{\langle \boldsymbol{n}|\widetilde\Lambda(t)|\boldsymbol{n}\rangle} \leq 0$

Unitary vs. dissipative

1. **Unitary qubit dynamics** $H = 1/2\boldsymbol{\omega}\cdot\boldsymbol{\sigma}$,
$$\widetilde{\mathcal{L}} = -\widetilde{\mathcal{L}}^T \implies e^{t\widetilde{\mathcal{L}}} = \mathbb{1} + \tfrac{\sin(\omega t)}{\omega}\,\widetilde{\mathcal{L}} + \tfrac{1-\cos(\omega t)}{\omega^2}\,\widetilde{\mathcal{L}}^2\,.$$
where $\omega = \|\boldsymbol{\omega}\|_1$. Let $\hat{\boldsymbol{\omega}}\cdot\boldsymbol{n} = \cos(\theta)$.
$$f_t = -\frac{\omega}{2}\frac{\sin(\omega t)\sin^2(\theta)}{\cos^2(\theta) + \cos(\omega t)\sin^2(\theta)} \nleq 0$$
for all $\theta \in (0,\pi) \implies T(t)$ **is not P-divisible**
2. **P-divisible Pauli dynamics**
$$\widetilde{\mathcal{L}}_{ij}(t)[\rho] = -\Gamma_i(t)\,\delta_{ij}\,,\; \widetilde\Lambda(t) = e^{\int_0^t \mathrm{d}s\widetilde{\mathcal{L}}(s)}$$
$$f_t = -\frac{1}{2}\frac{\langle \boldsymbol{n}|\sqrt{\widetilde\Lambda(t)}(-\widetilde{\mathcal{L}}(t))\sqrt{\widetilde\Lambda(t)}|\boldsymbol{n}\rangle}{\langle \boldsymbol{n}|\widetilde\Lambda(t)|\boldsymbol{n}\rangle} \leq 0\,.$$
$T(t)$ is P-divisible $\iff \Lambda_t$ is P-divisible.

A qubit dynamics is purely dissipative $\mathcal{L}_t = \mathcal{D}_t$ if and only if the generator is self-dual $\mathcal{L}_t = \mathcal{L}_t^\ddagger$.

Proposition. *Let $\Lambda_t = \Lambda_t^\ddagger$, be a self-dual, purely dissipative, invertible qubit dynamics. Then, the associated classical stochastic process $T(t)$ is P-divisible if and only if Λ_t is P-divisible.*

Remark. $\Lambda_t = e^{\int_0^t \mathrm{d}s\mathcal{L}_s}$ Time ordering drops out!

Question: Can a purely dissipative P-divisible dynamics have a non P-divisible classical reduction? **We must consider a non-self dual dynamics** (non-trivial time ordering)

A class of orthogonally covariant qubit dynamics

- $E_{ij} = |i\rangle\langle j|$ on the eigenbasis of σ_z and
$$O = \sum_i o_i E_{ii},\;\; o_i = \pm 1$$
Then $\Phi[O\rho O^T] = O\Phi[\rho]O^T$ has the form
$$\Phi^{(A,\lambda,\mu)}[\rho] := \sum_{i,j=0}^{1} A_{ij}E_{ij}\rho E_{ji} + \lambda E_{00}\rho E_{11}$$
$$+ \overline{\lambda}\,E_{11}\rho E_{00} + \mu\,E_{00}\rho^T E_{11} + \overline{\mu}\,E_{11}\rho^T E_{00}\,,$$
- CPTP dynamics in this class $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$
$$A(t) = \begin{pmatrix} a_t & 1 - b_t \\ 1 - a_t & b_t\end{pmatrix}, \quad a_t, b_t \in [0,1]$$
$$|\lambda_t| \leq \sqrt{a_t b_t}\,, \quad |\mu_t| \leq \sqrt{(1-a_t)(1-b_t)}\,.$$
- $\mathcal{L}_t = \dot\Lambda_t\Lambda_t^{-1} = \Phi^{(B(t),\ell_t,m_t)}$, with
$$B(t) = \dot A(t)A(t)^{-1} = \begin{pmatrix} -\gamma_-(t) & \gamma_+(t) \\ \gamma_-(t) & -\gamma_+(t)\end{pmatrix}$$
- Transversal and longitudinal rates:
$$\Gamma_T(t) := -\mathrm{Re}(\ell_t)\,, \qquad \Gamma_L(t) := \gamma_+(t) + \gamma_-(t)\,.$$
- Λ_t is P-divisible if and only if $\gamma_\pm(t) \geq 0$ and
$$\Gamma_T(t) - \frac{\Gamma_L(t)}{2} + \sqrt{\gamma_+(t)\gamma_-(t)} \geq |m_t| \qquad (2)$$

Non-self-dual dynamics from self-dual generator

Self-duality: $\Phi^{(A,\lambda,\mu)} = \Phi^{(A^T,\overline{\lambda},\mu)}$.

Let $\lambda_t = |\lambda_t|e^{i\varphi_t}$ and $\mu_t = |\mu_t|e^{i\theta_t}$,

Dissipativity ($\mathcal{L}_t = \mathcal{L}_t^\ddagger$) iff $\gamma_+(t) = \gamma_-(t)$ and
$$\ell_t = -\Gamma_T(t) \iff \dot\varphi_t|\lambda_t|^2 = \dot\theta_t|\mu_t|^2\,.$$
Define $\quad g_t := |\lambda_t| + |\mu_t|\,, \qquad h_t := |\lambda_t| - |\mu_t|\,,$ with $0 < h_t \leq g_t$, $g_0 = h_0 = 1$. Then, P-divisibility condition (2) is recast as
$$|\dot\theta_t|^2\left(\frac{g_t - h_t}{g_t + h_t}\right)^2 \leq \frac{\dot g_t \dot h_t}{g_t h_t}.$$

Example

Pick $h_t = e^{-3t}$ and $g_t = e^{-t}$

- Non self-dual map given by
$$|\lambda_t| = e^{-2t}\cosh(t)\,, \qquad \varphi_t = C\tanh^3(t)\,,$$
$$|\mu_t| = e^{-2t}\sinh(t)\,, \qquad \theta_t = 3C\tanh(t)\,,$$
$$a_t = b_t = e^{-t}\;\cosh(t)\,,$$
P-divisible iff $0 \leq C \leq 3/2$.
- Self-dual generator given by
$$\ell_t = -\Gamma_T(t) = -2\,, \qquad m_t = \sqrt{1 + r_t^2}\,e^{i(\theta_t + \varphi_t)}\,,$$
$$\gamma_+(t) = \gamma_-(t) = 1\,,$$
- Classical reduction on $P_{\pm\boldsymbol{n}}$, $\boldsymbol{n} = (\cos(\xi),\sin(\xi),0)$
$$2T_{00}(t) - 1 = |\lambda_t|\cos(\varphi_t) + |\mu_t|\cos(\theta_t + 2\xi)\,.$$
For $\xi = \pi/4$ and $C = 3/2$, **loss of classical P-divisibility for a purely dissipative evolution** (see Figure 1.)
$$2T_{00}(t) - 1 \geq 0, \quad \dot T_{00} \nleq 0 \implies f_t \nleq 0$$

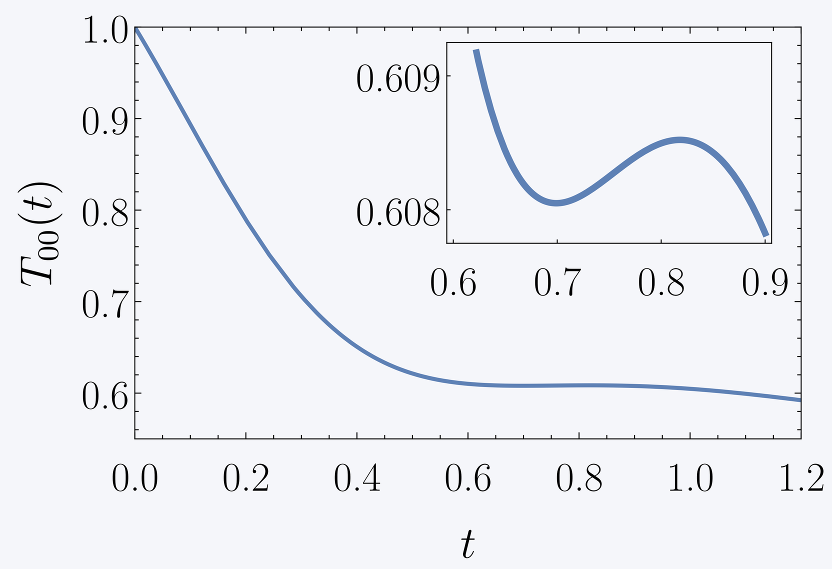

Figure 1. $T_{00}(t)$ for $\boldsymbol{n} = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ and $C = 3/2$

Coherence-assisted backflow of information

- Breuer-Laine-Piilo (*BLP*) approach:

✔ Internal and External information:
$$\mathcal{I}_t^q(\rho,\sigma;\mu) = \|\Lambda_t[\mu\rho - (1-\mu)\sigma]\|_1\,, \quad \mu \in [0,1]$$
$$\mathcal{E}_t^q(\rho,\sigma;\mu) = \mathcal{I}_0^q(\rho,\sigma;\mu) - \mathcal{I}_t^q(\rho,\sigma;\mu)\,,$$
✔ no backflow of information (BFI):
$$\partial_t\mathcal{I}_t^q = -\partial_t\mathcal{E}_t^q \leq 0\,,$$
✔ Invertible Λ_t: P-divisibility $\iff$ no BFI

✔ Information storage $\rho_{SE}(t) = U_t\,\rho_S \otimes \rho_E\,U_t^\dagger$
$$\mathcal{I}_t^q - \mathcal{I}_s^q \leq 2\mu\,D(\rho_{SE}(s),\rho_S(s)\otimes\rho_E(s))$$
$$+\,2(1-\mu)D(\sigma_{SE}(s),\sigma_S(s)\otimes\sigma_E(s)) \quad (3)$$
$$+\,2\min\{\mu,1-\mu\}D(\rho_E(s),\sigma_E(s))$$
- $\rho_{\boldsymbol{r}} = \sum_i r_iP_i \in \mathcal{P}$ (encoding of probability vector). Define the *classical internal information*
$$\mathcal{I}_t^{cl}(\boldsymbol{r},\boldsymbol{s};\mu) := \|\mathbb{D}\Lambda_t[\Delta_\mu(\rho_{\boldsymbol{r}},\sigma_{\boldsymbol{s}})]\|_1$$
$$= \|T(t)(\mu\boldsymbol{r} - (1-\mu)\boldsymbol{s})\|_{\ell_1}$$
where $\mathbb{D}[\rho] = \sum_i P_i\rho P_i$ (full decoherence), and the *coherent internal information*
$$\mathcal{C}_t(\boldsymbol{r},\boldsymbol{s};\mu) := \mathcal{I}_t^q(\rho_{\boldsymbol{r}},\rho_{\boldsymbol{s}};\mu) - \mathcal{I}_t^{cl}(\boldsymbol{r},\boldsymbol{s};\mu) \geq 0.$$
- Quantum P-divisibility yields
$$\mathcal{I}_t^{cl}(\boldsymbol{r},\boldsymbol{s};\mu) + \mathcal{C}_t(\boldsymbol{r},\boldsymbol{s};\mu) \leq \mathcal{I}_s^{cl}(\boldsymbol{r},\boldsymbol{s};\mu) + \mathcal{C}_s(\boldsymbol{r},\boldsymbol{s};\mu)\,.$$
- Upper-bound the difference of $\mathcal{I}_t^{cl}$
$$\mathcal{I}_t^{cl}(\boldsymbol{r},\boldsymbol{s};\mu) - \mathcal{I}_s^{cl}(\boldsymbol{r},\boldsymbol{s};\mu) \leq \mathcal{C}_s(\boldsymbol{r},\boldsymbol{s};\mu)$$
$$\leq \mu\,C_{\ell_1}(\Lambda_s[\rho_{\boldsymbol{r}}]) + (1-\mu)\,C_{\ell_1}(\Lambda_s[\rho_{\boldsymbol{s}}]) \quad (4)$$
($C_{\ell_1}(\rho) := \sum_{i\neq j}|\rho_{ij}|$ ℓ_1-norm of coherence).
- Quantum coherences: information storages acting like the environment in open systems.

Conclusions

- Purely dissipative, self-dual dynamics always have P-divisible classical reductions while purely dissipative, non self-dual dynamics may give rise to non P-divisible classical reductions, similarly to unitary case.
- Classical BFI interpretation: Quantum coherences play an information storing role as the environment does in the quantum scenario (compare (3) and (4)).
- Future directions : dynamics after partial-decoherence and reduction to non-maximally Abelian algebra

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