SCALING THEORY OF MANY-BODY ERGODICITY BREAKING

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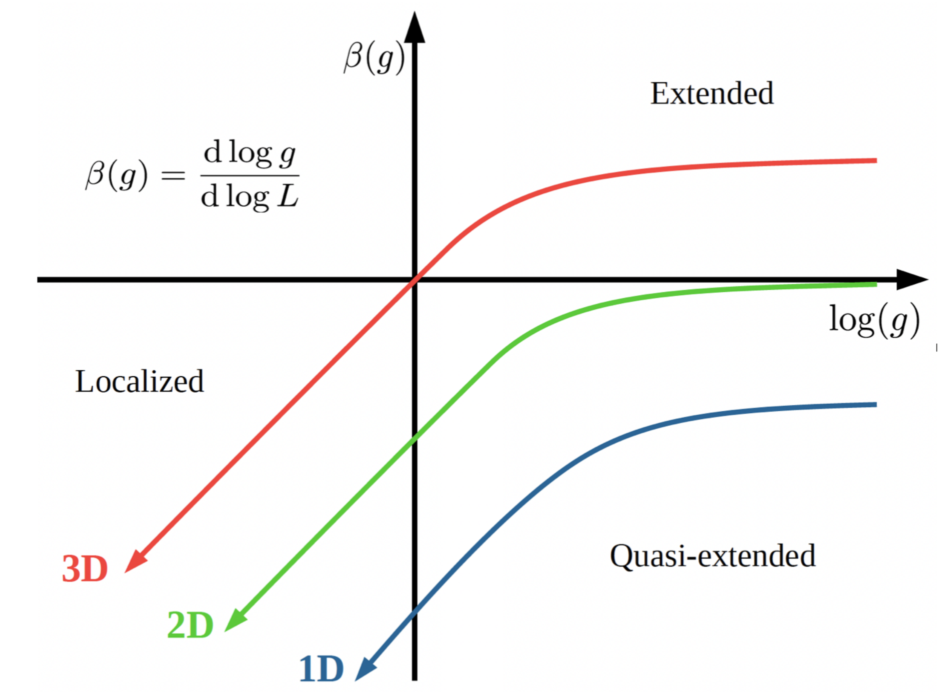
Introduction

- The overwhelming majority of physical systems thermalize, *i.e.*, the long time dynamics match thermal ensembles
- Quantum systems have RMT statistics, when their classical counterpart is chaotic [1].
- Similarly for many-body interacting models → RMT predicts the emergence of ergodicity
- On the level of observables, Eigenstate Thermalization Hypothesis (ETH) is a sufficient condition for thermalization [2]

$$\langle n|\hat{O}|m\rangle = O(\bar{E})\delta_{m,n} + \rho(\bar{E})^{-1/2} f(\bar{E},\omega) R_{nm} \quad (1)$$

Violation of ETH can be manifested in different ways. A paradigmatic example is the absence of transport in the Anderson model. A milestone in the study of Anderson localization in d-dimensions is the analysis of the RG flow for the dimensionless conductance g by the gang of four [3]:

$$g(L) = \begin{cases} \sigma L^{d-2} & \text{metallic regime} \\ \sim \exp\{-L/\xi\} & \text{insulating regime} \end{cases} \quad (2)$$



Recent studies focused then on modern observables, such as the fractal dimension D for both higher-dimensional Anderson model and RRG [4, 5]. **Is this phenomenology applicable to local interacting systems?**

- How is ETH broken when approaching the critical point?
- Is it abrupt or is it a smooth process? Is there an intermediate phase? → see Ref. [6] for details
- Is the single-parameter scaling (SPS) hypothesis at all valid for interacting systems?

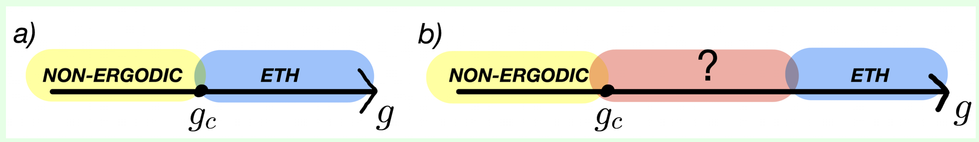


Figure 1: Sketch of scenarios of ergodicity breaking

Phenomenological Theory

For our study we consider the single-site entanglement entropy, *i.e.*, we treat all other spins as bath

$$\hat{\rho}^n = \mathrm{Tr}_{bath}\,|n\rangle\langle n| \;\to\; S_A^n = -\mathrm{Tr}\left(\hat{\rho}^n \ln \hat{\rho}^n\right) \;\to\; s = \frac{\overline{S_A}}{\ln 2}. \quad (3)$$

For $U(1)$ conserving models one can show that the single-site entanglement entropy directly relates to the fluctuations of diagonal matrix elements of the local magnetization $\hat{S}^z_\ell$. For particle-number breaking models this expression is similarly true.

The main quantity of interest is the beta function

$$\beta_s(s,L) = \frac{d\ln s}{d\ln L} = \beta(s) + \beta_1(s,L). \quad (4)$$

Single-parameter part — Corrections

If single-parameter scaling is true, then around the critical point s_c we can define the function $h(s)$

$$\beta(s) = \frac{1}{\nu s} h(s - s_c), \quad (5)$$

where for $h(x) = x$ we have $\nu = (s_c\beta'(s_c))^{-1}$. This function determines the collapse as

$$s(L) = s_c + f\left((L/\xi)^{1/\nu}\right) \quad (6)$$

for some $f(z)$ and localization length $\xi \simeq |\alpha_0 - \alpha_c|^{-\nu}(1 + c_1(\alpha_0 - \alpha_c) + \ldots)$.

We model the entanglement entropy in the entire **ergodic** regime as

$$s = 1 - ce^{-L/\eta} = 1 - c_0 e^{-(L-L_0)/\eta} \quad (7)$$

in the asymptotic limit $L \gg L_0$. The limiting case $\eta \to \infty$ corresponds to the transition point. This ansatz is **consistent with fading ergodicity** scenario developed in Ref. [6].

At the critical point $s = s_c = 1 - c_0$. Next, from Eq. (7) we find

$$\beta_{\rm erg}(s;c) = -\frac{1-s}{s}\ln\frac{1-s}{c}, \quad (8)$$

where c (equivalently L_0) might have parameter dependence. On the other hand if we consider

$$\tilde{\beta}_s = \frac{d\ln s}{d\ln \tilde{L}} = \beta_{\rm erg}(s;c_0) \quad (9)$$

or equivalently

$$\tilde{\beta}_s = \beta_s \cdot \left(1 - \frac{L_0}{L}\right) \quad (10)$$

we find for $L \gg L_0$ the **single-parameter ansatz** with linear corrections

$$\beta_s(s,L) = \beta_{\rm erg}(s;c_0) + \frac{\beta'_{\rm erg}(s;c_0)L_0}{L} \xrightarrow{L\to\infty} \beta_{\rm erg}(s;c_0), \quad (11)$$

provided $L_0 < \infty$, *i.e.*, we can summarize in the following

$$\begin{cases} L_0 \to \infty & \textit{two-parameter scaling} \\ L_0 \sim \text{const} < \infty & \textbf{one-parameter scaling} \end{cases} \quad (12)$$

Coincidentally, this expressions allows for estimating the critical exponent (after linearizing the beta function at the critical point)

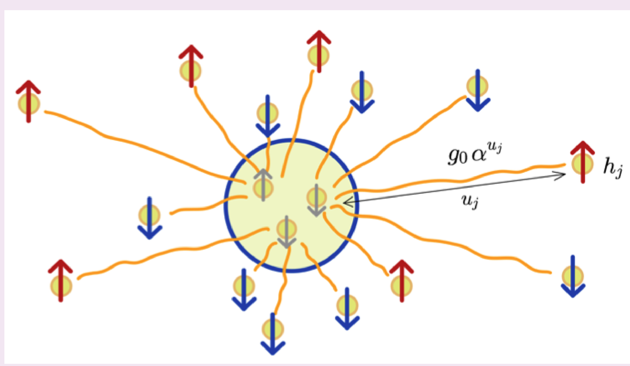
$$\nu = \frac{1}{s_c\beta'_s(s_c)} = 1 \quad (13)$$

On the non-ergodic side the wavefunctions become localized with exponential tails. Hence, the entanglement can be als approximated as exponentially small, *i.e.*, $s \sim \exp\{-L/\xi\}$. This yields the beta function on the localized side

$$\beta_{\rm loc}(s) \sim \ln\frac{s}{s_c} \quad (14)$$

with critical exponent $\nu = 1$ at $s = s_c$.

Quantum Sun Model



We study the toy model for avalanches, the Quantum Sun model [7]

$$\hat{H} = \hat{R} + g_0\sum_{\ell=0}^{L-1}\alpha^{u_\ell}\hat{S}^x_{n_\ell}\hat{S}^x_\ell + \sum_{\ell=0}^{L-1} h_\ell \hat{S}^z_\ell. \quad (15)$$

In this model there exist a transition for $\alpha = \alpha_c \approx 1/\sqrt{2}$ between an ergodic and localized phase.

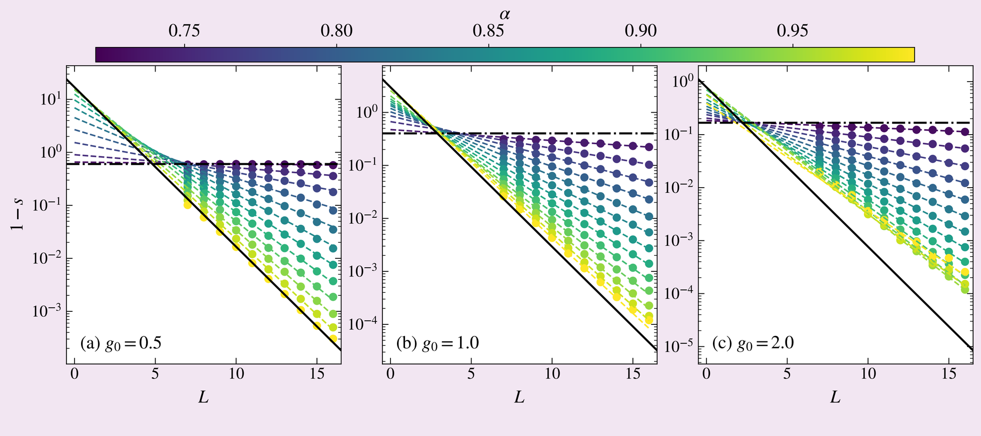


Figure 2: Finite-size scaling of $1-s$ in ergodic phase for (a) $g_0 = 0.5$, (b) $g_0 = 1$ and (c) $g_0 = 2$ with a $F(x) = c\exp\{-L/\eta\}$ fit to the data displayed by the dashed lines. The solid line resembles the RMT-like scaling with $\eta_{\rm RMT} = 1/\ln 2$. The dash-dotted line is the critical value of $c_0 = 1 - s_c$.

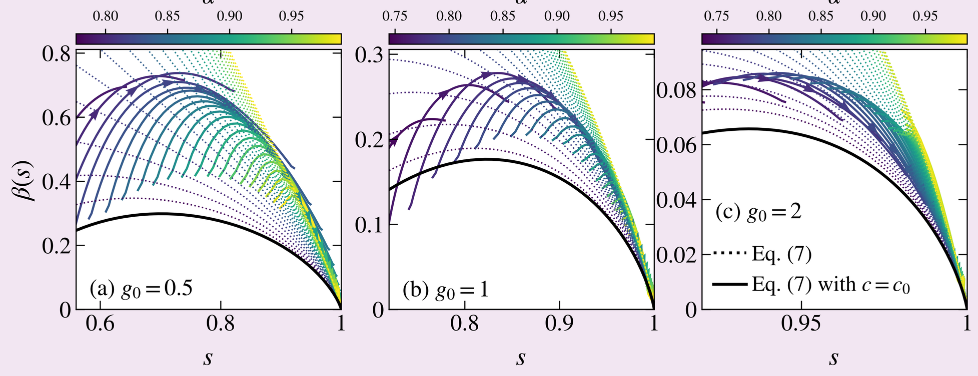


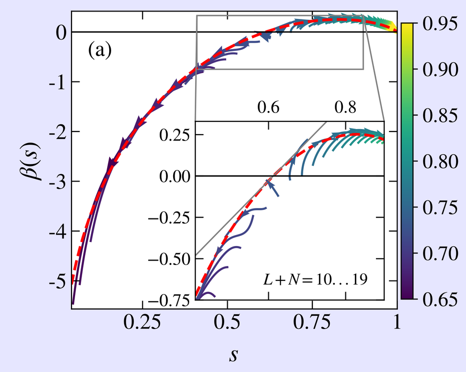
Figure 3: Numerical data for $\beta(s)$ for (a) $g_0 = 0.5$, (b) $g_0 = 1$ and (b) $g_0 = 2.0$ in the ergodic regime. The arrows indicate the increase in system size L and the colors denote the value of interaction α. The dashed lines show Eq. (8) with values of c extracted in Fig. 2, while the solid black line shows Eq. 8 for $c = c_0$.

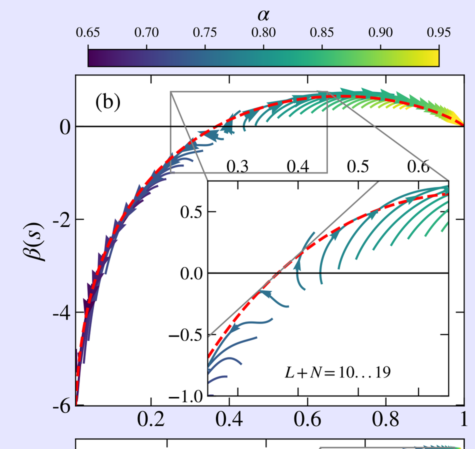
The finite-size corrections shown in Fig. 3 cause the envelope of the beta function to detach strongly from the SPS. We model this behaviour by an additional correction term, which vanish in the ETH and localized limit, *i.e.*, we define

$$\beta_{\rm env}(s) = \ln s - (1-s)\ln\frac{1-s}{a_1} - a_2 s\ln s \quad (16)$$

with some fitting constants a_1 and a_2.

Below in Fig. 4 we test this ansatz to the enevelope of the beta function. Remarkably, the functional dependence in Eq. (16) fits the data in the entire range of α considered.





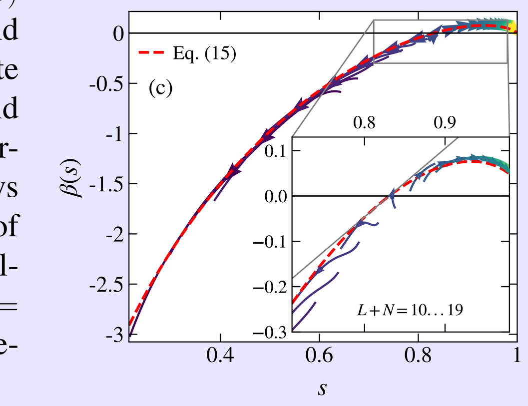


Figure 4: Numerical data for $\beta(s)$ for (a) $g_0 = 1$, (b) $g_0 = 0.5$ and (c) $g_0 = 2$. The arrows indicate the increase in system size L and the colors denote the value of interaction α. The dashed curve shows a fit of Eq. (16) to the envelope of the data with resulting fitting values $A = 3.61, 4.08, 2.69$ and $c = 0.29, 0.08, 0.04$ for $g_0 = 0.5, 1, 2$ respectively.

The decoupling of the envelope function from the SPS cause the critical exponent to deviate from the predicted $\nu = 1$. Fig. 5 shows the data collapse for all values of g_0 considered here.

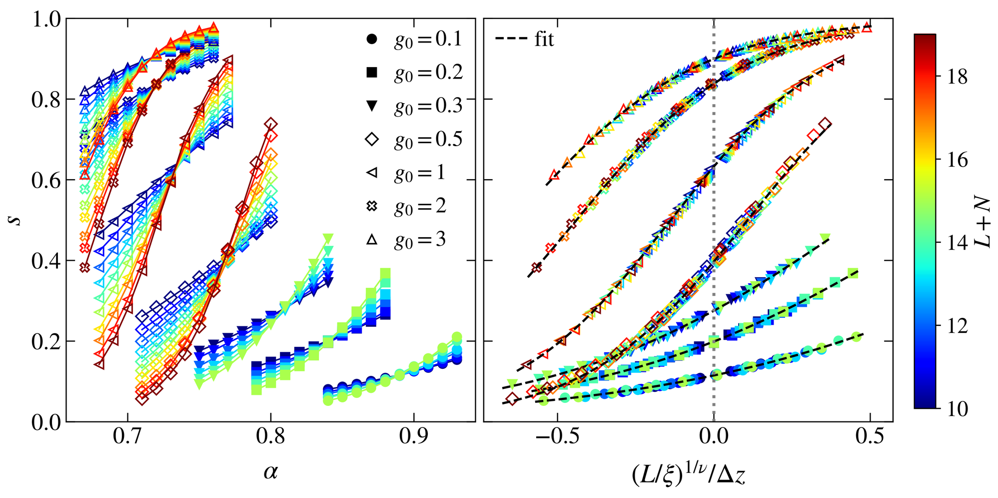


Figure 5: (a) Raw data scaled entanglement entropy s for different system sizes an values of g_0 in the vicinity of the estimated transition. (b) Finite-size data collapse using a cost function minimization for the values of α shown in panel (a). The dashed line denotes a fit of $s = 1/\left(1 + c\exp\{-(L/\xi)^{1/\nu}\}\right)$ to the collapsed data. We normalize the values on the x-xaxis by $\Delta z = \max(L/\xi) - \min(L/\xi)$ to show different g_0 on the same scale.

Lastly let us comment on the critical exponent. We have a set of equations using the beta function in Eq. (16)

$$\begin{cases} \beta(s_c) = 0 \\ s_c\beta'(s_c) = \nu^{-1} \end{cases} \;\to\; \nu^{-1} = 1 + (1-A)s_c\frac{1 - s_c + \ln s_c}{1 - s_c}. \quad (17)$$

Next, we test our prediction to the critical exponent extracted from Fig. 5:

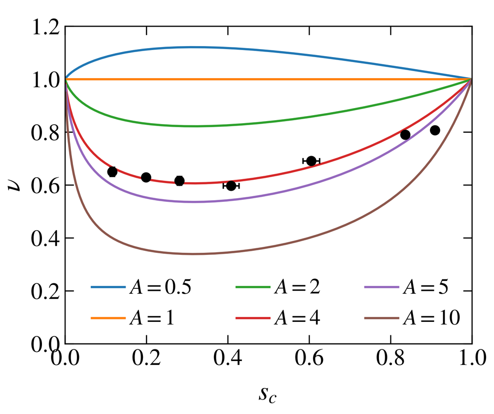


Figure 6: Numerical curves of Eq. (13) for different values of A. Black dots are numerical data for $g_0 = 0.1, 0.2, 0.3, 0.5, 1, 2, 3$.

Conclusions

- At sufficiently large system sizes $L \gg L_0$ there **exists one-parameter scaling** for many-body interacting models
- Many-body ergodicity-breaking transitions are characterized by the critical exponent $\nu = 1$
- There exist finite size corrections, which are irrelevalnt if $L_0 < \infty$
- For finite systems the corrections to the SPS change the critical exponent by decoupling the envelope from the SPS

References

[1] L. D'Alessio *et al.* Adv. Phys. 65 (2022)
[2] M. Srednicki, J. Phys. A. 32, 1163 (1999)
[3] E. Abrahams *et al.* PRL 42, 673 (1979)
[4] B. Altshuler *et al.* arxiv:2403.01974
[5] C. Vanoni *et al.* PNAS 121 (2024)
[6] M. Kliczkowski *et al.* arxiv:2407.16773 (2024)
[7] J. Šuntajs and L. Vidmar, PRL. 129, 060602 (2022)